Supplementary Material to

RESAMPLING-BASED INFERENCE METHODS FOR COMPARING TWO COEFFICIENTS ALPHA

Abstract

In this supporting information we present all theoretical derivations together with additional simulation results and the R-code for applying the novel resampling procedures in practice.

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1. Mathematical Appendix

Let the notation and prerequirements be as in Sections ?? and ??. Also see, e.g., Muirhead (2009) and van der Vaart (1998) for the following multivariate and asymptotic elaborations.

ADF Asymptotics

Let vec() be the usual operator that writes the elements of a symmetric matrix on and below the diagonal into a column vector, see e.g. Muirhead (2009). Due to the assumption of finite eighth order moments (ADF) and $\mathbb{E}(\boldsymbol{X}_1) = \mathbf{0}$ we can write the normalized Cronbach coefficient as

$$\text{vec}\left(\sqrt{n_1}(\widehat{\boldsymbol{\Sigma}}_1 - \boldsymbol{\Sigma}_1)\right) = \frac{1}{\sqrt{n_1}} \sum_{i=1}^{n_1} \text{vec}\left(\boldsymbol{X}_i \boldsymbol{X}_i' - \mathbb{E}(\boldsymbol{X}_i \boldsymbol{X}_i')\right) + o_p(1), \tag{10}$$

where $o_p(1)$ converges in probability to zero as $n_1 \to \infty$. Thus, it follows from the multivariate central limit theorem that $\text{vec}(\sqrt{n_1}(\widehat{\boldsymbol{\Sigma}}_1 - \boldsymbol{\Sigma}_1))$ is asymptotically multivariate normal with mean $\mathbf{0}$ and covariance $\text{cov}(\text{vec}(\boldsymbol{X}_1 \boldsymbol{X}_1'))$. Since $\alpha_{C,1} = \alpha_C(\boldsymbol{\Sigma}_1)$ is a differentiable function of $\boldsymbol{\Sigma}_1$ (or $\text{vec}(\boldsymbol{\Sigma}_1)$ respectively) it follows as in Maydeu-Olivares et al. (2007) that $\sqrt{n_1}\widehat{\alpha}_{C,1}$ is asymptotically normal distributed with mean $\alpha_{C,1}$ and variance $\tilde{\sigma}_1^2$ which depends on moments of fourth order. In particular, the limit variance is given by

$$\tilde{\sigma}_1^2 = \tilde{\sigma}_1^2(\boldsymbol{\Sigma}_1) = \boldsymbol{\delta}(\boldsymbol{\Sigma}_1)' \text{var}(\text{vec}(\boldsymbol{X}_1 \boldsymbol{X}_1')) \boldsymbol{\delta}(\boldsymbol{\Sigma}_1),$$

which can be obtained from the delta method, see Maydeu-Olivares et al. (2007) for details. Here the vector $\boldsymbol{\delta}(\boldsymbol{\Sigma}_1)$ is a function of $\boldsymbol{\Sigma}_1$ and is given in Equation (4) in Maydeu-Olivares et al. (2007). However, we even know more. Note, that $\alpha_{C,1}$ (as a function from $\mathbb{R}^{q_1}$ to $\mathbb{R}$, $q_1 = \frac{k_1(k_1+1)}{2}$) is differentiable at $\text{vec}(\boldsymbol{\Sigma}_1)$ with total derivative, i.e. Jacobi matrix, $\alpha'_{\boldsymbol{\Sigma}_1}$, see van der Vaart (1998) for its explicit formula. Hence, it follows from the proof of the multivariate delta method (to be concrete: the multivariate Taylor theorem), see e.g. Theorem 3.1. in van der Vaart (1998), that $\widehat{\alpha}_{C,1} = \alpha_C(\widehat{\boldsymbol{\Sigma}}_1)$ is even asymptotically linear in this case, i.e.

$$\sqrt{n_1}(\widehat{\alpha}_{C,1} - \alpha_{C,1}) = \frac{1}{\sqrt{n_1}} \sum_{i=1}^{n_1} f_{\boldsymbol{\Sigma}_1}(\boldsymbol{X}_i) + o_p(1) \tag{11}$$

holds as $n_1 \to \infty$ with

$$f_{\boldsymbol{\Sigma}_1}(\boldsymbol{X}_i) = \alpha'_{\boldsymbol{\Sigma}_1} \cdot \text{vec}\left(\boldsymbol{X}_i \boldsymbol{X}_i' - \mathbb{E}(\boldsymbol{X}_i \boldsymbol{X}_i')\right).$$

The latter fulfills $\mathbb{E}(f_{\boldsymbol{\Sigma}_1}(\boldsymbol{X}_i)) = 0$ and $\text{var}(f_{\boldsymbol{\Sigma}_1}(\boldsymbol{X}_i)) = \tilde{\sigma}_1^2$.

Since a similar representation holds for $\sqrt{n_2}(\widehat{\alpha}_{C,2} - \alpha_{C,2})$ (with different variance $\tilde{\sigma}_2^2 = \tilde{\sigma}_2^2(\boldsymbol{\Sigma}_2)$) it follows that the statistic M_n is also asymptotically normal under $H_0 : \alpha_{C,1} = \alpha_{C,2}$ with mean zero and variance $\tilde{\sigma}^2 = (1-\kappa)\tilde{\sigma}_1^2 + \kappa\tilde{\sigma}_2^2$, i.e.

$$\begin{aligned} M_n &= \sqrt{\frac{n_2}{N}}\sqrt{n_1}(\widehat{\alpha}_{C,1} - \alpha_{C,1}) - \sqrt{\frac{n_1}{N}}\sqrt{n_2}(\widehat{\alpha}_{C,2} - \alpha_{C,2}) \\ &\xrightarrow{d} \mathcal{N}(0, (1-\kappa)\tilde{\sigma}_1^2 + \kappa\tilde{\sigma}_2^2) \end{aligned}$$

if $n_1/N \to \kappa \in (0,1)$. A consistent estimator for $\tilde{\sigma}^2$ is given by

$$\begin{aligned} \widetilde{\sigma}^2 =& \frac{n_2}{N}\left(\frac{1}{n_1-1}\sum_{i=1}^{n_1}\left(\widehat{\boldsymbol{\delta}}'(\boldsymbol{S}_{i1} - \boldsymbol{S}_1)\right)^2\right) \\ &+ \frac{n_1}{N}\left(\frac{1}{n_2-1}\sum_{i=1}^{n_2}\left(\widehat{\boldsymbol{\delta}}'(\boldsymbol{S}_{i2} - \boldsymbol{S}_2)\right)^2\right), \end{aligned} \tag{12}$$

see Equation (7) in Maydeu-Olivares et al. (2007) for a similar formula in the one-sample case. Here, $\boldsymbol{S}_k = \text{vec}(\widehat{\boldsymbol{\Sigma}}_k)$ for $k = 1, 2$ and $\boldsymbol{S}_{i1} = \text{vec}\left[(\boldsymbol{X}_i - \overline{\boldsymbol{X}}^{(1)})(\boldsymbol{X}_i - \overline{\boldsymbol{X}}^{(1)})'\right]$ for $1 \le i \le n_1$ and $\overline{\boldsymbol{X}}^{(1)} = \frac{1}{n_1}\sum_{i=1}^{n_1}\boldsymbol{X}_i$ and $\boldsymbol{S}_{i2}$ is defined similarly with the random variables of the second sample. Altogether it follows from Slutzky's theorem that the proposed studentized test statistic $T_n = T_n(\mathbb{X}) = \frac{M_n}{\widetilde{\sigma}}$ in (**??**) is asymptotically standard normal under the null hypothesis H_0, i.e. $T_n \xrightarrow{d} \mathcal{N}(0,1)$.

Parametric Bootstrap

To show that the proposed parametric bootstrap test $\psi_n^\star = \mathbb{1}\{T_n > c_n^\star(\alpha)\}$ is of asymptotic level α we have to prove that the critical value $c_n^\star(\alpha)$, i.e. the conditional $(1-\alpha)$-quantile of the parametric bootstrap procedure, converges in probability to the $(1-\alpha)$-quantile $z_{1-\alpha}$ of a standard normal distribution, i.e.

$$c_n^\star(\alpha) \xrightarrow{p} z_{1-\alpha}$$

as $N \to \infty$, see Lemma 1 in Janssen & Pauls (2003). By continuity of the limit distribution, this is fulfilled if the conditional parametric bootstrap distribution function of the test statistic T_n is asymptotically standard normal in probability due to $T_n \xrightarrow{d} \mathcal{N}(0,1)$ under H_0. By assumption we again have

$$\text{vec}\left(\sqrt{n_1}(\widehat{\boldsymbol{\Sigma}}_1^\star - \boldsymbol{\Sigma}_1)\right) = \frac{1}{\sqrt{n_1}}\sum_{i=1}^{n_1}\text{vec}\left(\boldsymbol{X}_i^\star\boldsymbol{X}_i^{\star\prime} - \mathbb{E}(\boldsymbol{X}_i^\star\boldsymbol{X}_i^{\star\prime})\right) + o_p(1). \tag{13}$$

Different to above, however, the family of random variables $\boldsymbol{X}_i^\star, i \le n_1$ now forms an array of row-wise i.i.d. random variables given the observed data. Thus, we cannot work with the classical multivariate CLT but have to employ the multivariate version of Lindeberg's or Lyapunov's theorems conditioned on the data. Due to the existence of finite eighth order moments and the consistency of $\widehat{\mathbf{\Sigma}}_1$ Lyapunov's condition is fulfilled and we can obtain that $\text{vec}(\sqrt{n_1}(\widehat{\mathbf{\Sigma}}_1^\star - \mathbf{\Sigma}_1))$ is, given the data, asymptotically multivariate normal with mean $\mathbf{0}$ and covariance matrix $\text{cov}(\text{vec}(\mathbf{Z_1}\mathbf{Z_1}'))$ in probability, where $\mathbf{Z_1} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma_1})$. Given the data, we can now proceed as in the prove above, i.e. we first apply the delta-method, then combine the results for the two independent bootstrap samples and finally show that the given variance estimator is also consistent for the bootstrap (which follows, e.g. from the Tchebyscheff inequality) to show that

$$\sup_{x\in\mathbb{R}} |P(T^\star \leq x | \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n) - \Phi(x)| \xrightarrow{p} 0$$

as $\frac{n_1}{N} \to \kappa \in (0,1)$ and the result follows. Here, Φ is the distribution function of $\mathcal{N}(0,1)$. Due to the duality between statistical tests and confidence intervals, this also shows the asymptotic correctness of the latter. Moreover, the same argumentation also shows the lacking proof of the validity of Padilla et al. (2012) one-sample confidence interval for Cronbach's α coefficient.

Permutation Distribution

Now suppose that $k_1 = k_2$. In order to prove that the permutation test is of asymptotic level α we again have to show convergence of the corresponding critical value $c_n^\pi(\alpha)$, i.e. the conditional $(1-\alpha)$-quantile of the permutation distribution function, converge in probability to the $(1-\alpha)$-quantile $z_{1-\alpha}$ of a standard normal distribution, i.e.

$$c_n^\pi(\alpha) \xrightarrow{p} z_{1-\alpha}.$$

In order to prove this, we apply Theorem 2.2 in Chung & Romano (2013) together with a conditional Slutzky-type argument.

As in the beginning it holds that the normalized Cronbach coefficients are asymptotically linear in both groups, i.e. (11) as well as

$$\sqrt{n_2}(\widehat{\alpha}_{C,2} - \alpha_{C,2}) = \frac{1}{\sqrt{n_2}} \sum_{i=n_1+1}^{N} f_{\mathbf{\Sigma}_2}(\boldsymbol{X}_i) + o_p(1), \tag{14}$$

holds, where again $o_p(1)$ stands for a random variable that converges in probability to 0 as $n_2 \to \infty$. Since by assumption $\tilde{\sigma}_1^2 \in (0,\infty)$ all ingredients for applying Theorem 2.2 in Chung & Romano (2013) are fulfilled and it follows by Slutzky that

$$\frac{1}{N!} \sum_{\pi} \mathbb{1}\{T_n(\mathbb{X}^\pi) \leq x\}$$

converges in probability to $\Phi(x)$. Altogether this proves that ψ_n is an asymptotically exact level α testing procedure in the general ADF model.

Derivations for other reliability measures

In the following the derivatives of the different reliability measures $\lambda_\ell,\ \ell = 1,2,4,5,5+,6$ $(\omega_m,\ m = h,t)$ summarised in Section 4 of the main manuscript are given. Let

$\boldsymbol{\sigma} = \text{vec}(\boldsymbol{\Sigma}) = \text{vec}((\sigma_{ij})_{i,j})$, where vec() is a function stacking the elements of a symmetric matrix on and below the diagonal into a vector. Let $\boldsymbol{\delta}_\ell = \boldsymbol{\delta}_\ell(\boldsymbol{\Sigma}) = \lambda'_\ell = \frac{d\lambda_\ell}{d\boldsymbol{\sigma}}$ ($\boldsymbol{\delta}_m = \boldsymbol{\delta}_m(\boldsymbol{\Sigma}) = \omega'_m = \frac{d\omega_m}{d\boldsymbol{\sigma}}$) be the derivative of $\lambda_\ell,\ \ell = 1, 2, 4, 5, 5+, 6$ ($\omega_m,\ m = h, t$). Below the entries of $\boldsymbol{\delta}_\ell$ ($\boldsymbol{\delta}_m$) are given:

$$\frac{\partial\lambda_1}{\partial\sigma_{ij}} = \begin{cases} \frac{\text{tr}(\boldsymbol{\Sigma}) - \mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i = j \\ 2 \cdot \frac{\text{tr}(\boldsymbol{\Sigma})}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i \neq j \end{cases}$$

$$\frac{\partial\lambda_2}{\partial\sigma_{ij}} = \begin{cases} \frac{\text{tr}(\boldsymbol{\Sigma}) - \mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k - \sqrt{\frac{k}{k-1}}C_2^{1/2}}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i = j \\ 2 \cdot \frac{1 - \sqrt{\frac{k}{(k-1)}}\left(C_2^{1/2} - \sigma_{ij}C_2^{-1/2}\right) + \text{tr}(\boldsymbol{\Sigma}) - \mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i \neq j \end{cases}$$

$$\frac{\partial\lambda_4}{\partial\sigma_{ij}} = \begin{cases} (-2) \cdot \frac{\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k - \mathbf{1}_k'\boldsymbol{\Sigma}_A\mathbf{1}_k - \mathbf{1}_k'\boldsymbol{\Sigma}_B\mathbf{1}_k}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i = j \\ (-4) \cdot \frac{\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k - \mathbf{1}_k'\boldsymbol{\Sigma}_A\mathbf{1}_k - \mathbf{1}_k'\boldsymbol{\Sigma}_B\mathbf{1}_k}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i \neq j \end{cases}$$

$$\frac{\partial\lambda_5}{\partial\sigma_{ij}} = \begin{cases} \frac{\text{tr}(\boldsymbol{\Sigma}) - \mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k + 2\bar{C}_2^{1/2}}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i = j \\ 2 \cdot \frac{\text{tr}(\boldsymbol{\Sigma}) - 2\sigma_{ij}\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)\bar{C}_2^{-1/2} + 2\bar{C}_2^{1/2}}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i \neq j \end{cases}$$

$$\frac{\partial\lambda_{5+}}{\partial\sigma_{ij}} = \begin{cases} \frac{\text{tr}(\boldsymbol{\Sigma}) - \mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k + \frac{2k}{k-1}\bar{C}_2^{1/2}}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i = j \\ \frac{2\text{tr}(\boldsymbol{\Sigma}) - \frac{4k}{k-1}\left(\sigma_{ij}\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)\bar{C}_2^{-1/2} + \bar{C}_2^{1/2}\right)}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i \neq j \end{cases}$$

$$\frac{\partial\lambda_6}{\partial\sigma_{ij}} = \begin{cases} \frac{\sum_{t=1}^k e_t^2}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i = j \\ 2 \cdot \frac{\sum_{t=1}^k e_t^2}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i \neq j \end{cases}$$

To handle the coefficients ω_t and ω_h, we assume that $\boldsymbol{c}$ and $\boldsymbol{A}$ are differentiable in $\boldsymbol{\Sigma}$ and additionally, we define the two differentiable functions $g(\boldsymbol{c}) = \mathbf{1}_k'\boldsymbol{c}\boldsymbol{c}'\mathbf{1}_k$ and $h(\boldsymbol{A}) = \mathbf{1}_k'\boldsymbol{A}\boldsymbol{A}'\mathbf{1}_k$. Using the chain rule, the derivatives of ω_t and ω_h are given as follows:

$$\frac{\partial\omega_t}{\partial\sigma_{ij}} = \begin{cases} \frac{\left(\sum_{s=1}^k \frac{\partial g}{\partial c_s}\frac{\partial c_s}{\partial\sigma_{ij}} + \sum_{s,t=1}^k \frac{\partial h}{\partial A_{st}}\frac{\partial A_{st}}{\partial\sigma_{ij}}\right)\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right) - \left(\mathbf{1}_k'\boldsymbol{c}\boldsymbol{c}'\mathbf{1}_k + \mathbf{1}_k'\boldsymbol{A}\boldsymbol{A}'\mathbf{1}_k\right)}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i = j \\ \frac{\left(\sum_{s=1}^k \frac{\partial g}{\partial c_s}\frac{\partial c_s}{\partial\sigma_{ij}} + \sum_{s,t=1}^k \frac{\partial h}{\partial A_{st}}\frac{\partial A_{st}}{\partial\sigma_{ij}}\right)\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right) - 2\cdot\left(\mathbf{1}_k'\boldsymbol{c}\boldsymbol{c}'\mathbf{1}_k + \mathbf{1}_k'\boldsymbol{A}\boldsymbol{A}'\mathbf{1}_k\right)}{\left(\mathbf{1}_k'\boldsymbol{\Sigma}\mathbf{1}_k\right)^2}, & i \neq j \end{cases}$$

$$\frac{\partial \omega_h}{\partial \sigma_{ij}} = \begin{cases} \frac{\sum_{s=1}^{k} \frac{\partial g}{\partial c_s} \frac{\partial c_s}{\partial \sigma_{ij}} (\mathbf{1}_k' \mathbf{\Sigma} \mathbf{1}_k) - (\mathbf{1}_k' \boldsymbol{c}\boldsymbol{c}' \mathbf{1}_k)}{(\mathbf{1}_k' \mathbf{\Sigma} \mathbf{1}_k)^2}, & i = j \\ \frac{\sum_{s=1}^{k} \frac{\partial g}{\partial c_s} \frac{\partial c_s}{\partial \sigma_{ij}} (\mathbf{1}_k' \mathbf{\Sigma} \mathbf{1}_k) - 2 \cdot (\mathbf{1}_k' \boldsymbol{c}\boldsymbol{c}' \mathbf{1}_k)}{(\mathbf{1}_k' \mathbf{\Sigma} \mathbf{1}_k)^2}, & i \neq j \end{cases}$$

Since $\text{vec}\left(\sqrt{n}\left(\widehat{\mathbf{\Sigma}} - \mathbf{\Sigma}\right)\right) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \text{var}(\text{vec}(\boldsymbol{X}_1 \boldsymbol{X}_1')))$, where $\widehat{\mathbf{\Sigma}}$ is the sample covariance matrix of independent and identically distributed random vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ with $\mathbf{\Sigma} = \text{cov}(\boldsymbol{X}_1)$ and finite fourth moments, it thus, follows from the multivariate delta method that

$$\sqrt{n}(\lambda_\ell(\widehat{\mathbf{\Sigma}}) - \lambda_\ell(\mathbf{\Sigma})) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \boldsymbol{\delta}_\ell' \text{var}(\text{vec}(\boldsymbol{X}_1 \boldsymbol{X}_1')) \boldsymbol{\delta}_\ell)$$

$$\left(\sqrt{n}(\omega_m(\widehat{\mathbf{\Sigma}}) - \omega_m(\mathbf{\Sigma})) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \boldsymbol{\delta}_m' \text{var}(\text{vec}(\boldsymbol{X}_1 \boldsymbol{X}_1')) \boldsymbol{\delta}_m)\right)$$

for all choices of $\ell = 1, 2, 4, 5, 5+, 6$ $(m = h, t)$. Due to the form of the derivatives given above the unknown variance can be consistently estimated; in case of ω_t and ω_h they depend on the specific forms of $\boldsymbol{c}$ and $\boldsymbol{A}$.

2. More simulation results

Some simulation results for continuous data are presented. Two different scenarios are conducted: t-distributed and lognormally distributed data. In this section, we compare the ADF method to the permutation test presented in Section 3.1 of the paper. The parametric bootstrap procedure has been left out for lucidity since the permutation method performed slightly better. Moreover, recall that the permutation test is finitely exact under exchangeability. Again 10,000 simulation trails with 500 permutation samples were performed.

To check the behavior of the procedures in case of deviations from the underlying moment assumption, we first deal with t-distributed data with four degrees of freedom. Note, that the assumption of finite eight order moment is clearly violated in this case. The data are generated with the help of the R function `rmvt()` which is included in the `mvtnorm` package. Based on the simulation results of the main manuscript, the following results are based on two correlation matrices only. The reason is that there are matrices following the true-score equivalent model and some do not. Another cause is the comparability of the results of the main simulation study. Thus, in the following we only consider correlation matrices $\boldsymbol{P_1}$ and $\boldsymbol{P_4}$ given in Section 5.1.

The results are summarized in Figure 1, where the type I error levels of the permutation and the asymptotic test for two different correlation matrices are shown. In the left plot, same sample sizes in the different groups are considered, whereas the right plot summarizes the results of unequal sample sizes. It is evident that the asymptotic test does not control the type I error rate satisfactorily in all cases. Even for very large balanced sample sizes ($n_i > 350$) the type I errors are still around 7% and even larger in extremely unbalanced cases or smaller sample sizes. In contrast, the novel permutation test controls the type I error rate fairly well in all situations and is always in the range of 4.7 and 5.3%.

Next, we deal with log-normally distributed data. The data are generated by a scale model with $k = 5$ items $\boldsymbol{X}_i = \boldsymbol{I}_k^{1/2} \boldsymbol{\varepsilon}_i,\ i = 1, \ldots, N$, where $\boldsymbol{\varepsilon}_i = \frac{e_i - \text{E}(e_i)}{\sqrt{\text{var}(e_i)}}$ and $e_i \sim LN(0, 1)$ are equal_samplesize_tdistr.pdf

(a) equal sample sizes

unequal_samplesize_tdistr.pdf

(b) unequal sample sizes

Figure 1: Type I error level ($\alpha = 5\%$) simulation results (y-axis) for t-distributed data of the permutation test ψ_n (——) and the asymptotic test φ_n (········) for different sample sizes (x-axis) and two different correlation matrices $\boldsymbol{P_1}$ (black) and $\boldsymbol{P_4}$ (grey).

equal_samplesize_lognormdistr.pdf

(a) equal sample sizes

unequal_samplesize_lognormdistr.pdf

(b) unequal sample sizes

Figure 2: Type I error level ($\alpha = 5\%$) simulation results (y-axis) for lognormal distributed data of the permutation test ψ_n (——) and the asymptotic test φ_n (········) for different sample sizes (x-axis).

independent standardized log-normally distributed error terms. Figure 2 shows the results of the log-normal distribution. Contrary to the situation with the t-distributed data before, observations simulated under this scenario fulfill the postulated moment assumption. However, the observations are rather similar. For smaller or strongly unbalanced sample sizes the true type I error is around 10% (or even larger) and decrease with increasing n_i. However, even for larger sample sizes the type I error control is not very satisfactory. The asymptotic test exhibits some issues in controlling the type I error rate, whereas the permutation test works quite perfect.

3. R code

In the following, we present the R code of our new permutation and parametric bootstrap procedures. First, we present the different functions for the two resampling methods (`pval.perm()` and `pval.boot()`). In a third part, a function for calculating the test statistic (`tstat()`) and another function which writes the elements of a symmetric matrix on and below the diagonal into a column vector (`vecs()`) are given.

3.1. R code of the permutation test

```
  pval.perm <- function(data, n1 = NULL, n2 = NULL, p = NULL, B = 1000){
    library(MASS)
    perm.results <- matrix(rep(0, (4 * B)), ncol = 4)
    n <- n1 +n2

    # original data estimates of alpha and T statistics
    orig.results <- tstat(data1, data2, n1, n2, p1, p2)

    # permuted data estimates of alpha and T statistics
    for (i in 1:B){
      dat_temp <- data[sample(1:nrow(data)),]
      perm.results[i, ] <- tstat(dat_temp, n1, n2, p)
    }
  perm.p.values = perm.p.values_nonorm <- numeric(3)

  # permutation p-values
  perm.p.values[1] <- (sum(orig.results[1] <= perm.results[, 1]) / B)  # right-sided
  perm.p.values[2] <- (sum(orig.results[1] >= perm.results[, 1]) / B)  # left-sided
  perm.p.values[3] <- (2 * min(perm.p.values[1:2]))  # two-sided
  names(perm.p.values) <- c("right.sided", "left.sided", "two-sided")
  perm.p.values_nonorm[1] <- (sum(orig.results[2] <= perm.results[, 2]) / B)  # right-sided
  perm.p.values_nonorm[2] <- (sum(orig.results[2] >= perm.results[, 2]) / B)  # left-sided
  perm.p.values_nonorm[3] <- (2 * min(perm.p.values_nonorm[1:2]))  # two-sided
  names(perm.p.values_nonorm) <- c("right.sided", "left.sided", "two-sided")

  return(list(perm.p.values=perm.p.values, perm.p.values_nonorm=perm.p.values_nonorm,
      alpha1=orig.results[3], alpha2=orig.results[4]))
}
```

3.2. R code of the parametric bootstrap test

```
pval.boot <- function(data, n1 = NULL, n2 = NULL, p = NULL, B = 1000){
  library(mvtnorm)
  boot.results <- matrix(rep(0, (2 * B)), ncol = 2)
```

```
  n <- n1+n2

  # original data estimates of alpha and T statistics
  orig.results <- tstat(data, n1, n2, p)

  # bootstraped data estimates of alpha and T statistics
  Sigma1 <- cov(data[1:n1, 1:p])
  Sigma2 <- cov(data[(n1 + 1):n, 1:p])
  for (i in 1:B){
    dat_temp <- rbind(mvrnorm(n1, rep(0, p), Sigma1), mvrnorm(n2, rep(0, p), Sigma2))
    boot.results[i, ] <- tstat(dat_temp, n1, n2, p)
  }

   # bootstrap p-values
   boot.p.values[1] <- (sum(orig.results[3] <= boot.results[, 3]) / B)  # right-sided
   boot.p.values[2] <- (sum(orig.results[3] >= boot.results[, 3]) / B)  # left-sided
   boot.p.values[3] <- (2 * min(boot.p.values[1:2]))  # two-sided
   names(boot.p.values) <- c("right.sided", "left.sided", "two-sided")
   boot.p.values_nonorm[1] <- (sum(orig.results_nonorm[3] <= boot.results_nonorm[, 3])
        / B)  # right-sided
   boot.p.values_nonorm[2] <- (sum(orig.results_nonorm[3] >= boot.results_nonorm[, 3])
        / B)  # left-sided
   boot.p.values_nonorm[3] <- (2 * min(boot.p.values_nonorm[1:2]))  # two-sided
   names(boot.p.values_nonorm) <- c("right.sided", "left.sided", "two-sided")


  return(list(boot.p.values=boot.p.values, boot.p.values_nonorm=boot.p.values_nonorm,
      alpha1=orig.results[3], alpha2=orig.results[4]))
}
```

3.3. *R code of the test statistic and the vecs-function*

```
### function vecs
vecs <- function(data){
  upna <- data
  upna[upper.tri(data)] <- NA
  upna_vec <- as.vector(upna)[!is.na(as.vector(upna))]
  return(as.matrix(upna_vec))
}
```

```
### calculates the test statistics of both tests
tstat <- function(data, n1 = NULL, n2 = NULL, p = NULL){
  n <- (n1 + n2)
  Sigma1 <- cov(data[1:n1, 1:p])
  Sigma2 <- cov(data[(n1 + 1):n, 1:p])
  col.mean1 <- matrix(colMeans(data[1:n1, 1:p]), nrow = 1)
  col.mean2 <- matrix(colMeans(data[(n1 + 1):n, 1:p]), nrow = 1)
  trSigma1 <- sum(diag(Sigma1))
  trSigma2 <- sum(diag(Sigma2))
  sSigma1 <- sum(Sigma1)
  sSigma2 <- sum(Sigma2)

  # variances, separately
  sigma1q <- ((2 * p^2 * (sSigma1 * (sum(diag(Sigma1 %*% Sigma1)) + trSigma1^2) - 2 *
      trSigma1 * sum(Sigma1 %*% Sigma1))) / ((p - 1)^2 * sSigma1^3))
  sigma2q <- ((2 * p^2 * (sSigma2 * (sum(diag(Sigma2 %*% Sigma2)) + trSigma2^2) - 2 *
      trSigma2 * sum(Sigma2 %*% Sigma2))) / ((p - 1)^2 * sSigma2^3))

  # Welch-type variance, pooled
  sigma <- sqrt((n2 / n) * sigma1q + (n1 / n) * sigma2q)

```

```
  # variances nonorm, separately
  helpdelta1 <- 2*p/(p-1)*(trSigma1/(sSigma1)^2)
  helpdeltatr1 <- -p/(p-1)*((sSigma1-trSigma1)/(sSigma1)^2)
  delta_1 <- matrix(rep(helpdelta1, p^2), nrow = p)
  diag(delta_1) <- helpdeltatr1

  helpdelta2 <- 2*p/(p-1)*(trSigma2/(sSigma2)^2)
  helpdeltatr2 <- -p/(p-1)*((sSigma2-trSigma2)/(sSigma2)^2)
  delta_2 <- matrix(rep(helpdelta2, p^2), nrow = p)
  diag(delta_2) <- helpdeltatr2

  sigma1q.non <- 0
  wcv <- 0
  v <-0
  tmp <- 0
  for (i in 1:n1){
    v <- (as.matrix(data[i,1:p, drop = FALSE]) - col.mean1)
    wcv <- (t(vecs(delta_1))%*%(vecs((t(v) %*%v))-vecs(Sigma1)))^2
    sigma1q.non <- (sigma1q.non + wcv)
  }

  sigma2q.non <- 0
  wcv <- 0
  v <-0
  tmp <- 0
  for (i in 1:n2){
    v <- (as.matrix(data[i+n1,1:p, drop = FALSE]) - col.mean2)
    wcv <- (t(vecs(delta_2))%*%(vecs((t(v) %*%v))-vecs(Sigma2)))^2
    sigma2q.non <- (sigma2q.non + wcv)
  }

  # variance, pooled
  sigma.non <- sqrt(n2/n*(1/(n1-1)*sigma1q.non)+n1/n*(1/(n2-1)*sigma2q.non))

  # Cronbach alpha
  alpha1 <- (p / (p - 1) * (1 - trSigma1 / sSigma1))
  alpha2 <- (p / (p - 1) * (1 - trSigma2 / sSigma2))

  # test statistic
  Mn <- (sqrt((n1 * n2) / n) * (alpha1 - alpha2))
  tval <- (Mn / sigma)
  tval.nonorm <- (Mn / sigma.non)

  return(c(TSTAT = tval, TSTAT_NONORM = tval.nonorm))
}
```

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