BOOK REVIEWS

THE AXIOM OF DETERMINACY, FORCING AXIOMS, AND THE NONSTATIONARY IDEAL
(de Gruyter Series in Logic and its Applications 1)

By W. HUGH WOODIN: 934 pp., DM298.00, ISBN 3-11-015708-X
(Walter de Gruyter, Berlin, 1999).

The continuum problem, the first of Hilbert's list of problems for the 20th century, has proved durable. The problem of whether there exist uncountable sets of real numbers that are not equinumerous (that is, cannot be put in a one-to-one correspondence) with the whole of the real continuum has eluded solution since Cantor first struggled with these concepts at the end of the last century. The Russian school of analysts at the beginning of the 20th century tried to continue Cantor's work of looking at simple sets (closed, Borel, analytic, and so on) so as to establish some sort of regularity properties for them. In particular, they tried to show that if a set is uncountable, then it contains a perfect (and hence continuum-sized) subset. Nevertheless, it is probably fair to say that no real advance on the problem came until Gödel established the consistency of the continuum hypothesis (in fact, in a generalised form: $2^{\aleph_0} = \aleph_{\alpha+1}$, for all α) with the other axioms of set theory, of Zermelo–Fraenkel, in 1938.

That the continuum could consistently have size greater than $\aleph_1$ was established by Cohen in 1963. In fact, the only restriction on the size of the continuum in ZFC alone is that it should not be a singular cardinal of countable cofinality.

It is Cohen's method of forcing that shows that, whatever large cardinals we may assume to exist in V, the universe of all sets of mathematical discourse, we cannot settle the continuum problem from these additional assumptions. (This had been an expressed hope of Gödel, and was shown to be hopeless by Levy and Solovay in 1967.) Any definitive 'answer' to the continuum problem was as remote as ever. This lack of a solution has led some logicians to question the role of the concept of a 'power set' or, indeed, whether the continuum hypothesis is even a well-formed problem at all. The book under review should, however, be regarded as the most significant—and the most sustained—attack on the continuum problem since Cohen's.

One should bear in mind that large cardinal axioms have been decisive in settling classical questions in *descriptive set theory* where, broadly speaking, definable issues are addressed. Since the early sixties, this has been a fertile region. Work was concentrated on the effects of the so-called 'axiom of determinacy', a principle contradicting that of choice, but one which revealed a rich, and regular, structure theory for the continuum. Research has shown that once sufficient large cardinals are assumed to exist in V, then the axiom of determinacy will hold in the smallest inner model containing all ordinals and the set of all reals, $L(\mathbb{R})$. This model is a natural laboratory in which to study definable properties of the reals. By definition, any question in definable analysis can be settled here.

Since any hereditarily countable set may be considered as being coded by a real, a theory of the reals can be construed as a theory of $H(\omega_1)$, the collection of all such countable sets. Thus the *projective* sets of reals (those obtainable by the processes of projection and complementation from closed subsets in some $\mathbb{R}^n$) correspond to the subsets of $H(\omega_1)$ that are definable over $\langle H(\omega_1), \in \rangle$. *Projective determinacy* (the notion that two-person infinite perfect information games with payoff sets within the projective hierarchy all have winning strategies) settles the classical questions about the projective sets of reals. The only known examples of questions left undecided about the projective sets by projective determinacy are analogous to the Gödel sentences, and to consistency statements from Gödel's work.

However, the continuum hypothesis is concerned with the structure $H(\mathfrak{c}^+)$, where $\mathfrak{c}$ is the size of the continuum. If the continuum hypothesis holds, then this is $H(\omega_2)$. Is there an analogue of projective determinacy that will give us a good theory for $\langle H(\omega_2), \in \rangle$? Or, equivalently, can we do for $\mathscr{P}(\omega_1)$ what has been done by assuming large cardinals for $\mathscr{P}(\omega)$, in other words, $\mathbb{R}$?

It is the contention of this book that there is. It identifies a canonical inner model; moreover, in this model the continuum hypothesis turns out to be false: $2^{\aleph_0} = \aleph_2$. The model can be construed as a method of forcing extension of $L(\mathbb{R})$ by a partial order which is the primary concern of the central chapters of the book, and which is called $\mathbb{P}_{\max}$. This notion is intimately connected with the *non-stationary ideal*, $\mathscr{I}_{NS}$, on ω_1. (This is the ideal of those subsets of ω_1 that have empty intersection with some closed and unbounded subset of ω_1.) The ω_2-saturation of this ideal had been recognised as a strong phenomenon, and is approximately equiconsistent in large cardinal terms to what has become a central concept in set theory in the last decade: the notion of a *Woodin cardinal*.

It is perhaps easiest to explain the book's purpose by speaking about the more recent approach that is outlined in the final chapter. (The author says that if the book were to be rewritten today, the exposition would be couched in the terms of this chapter. The reader is warned that the material becomes significantly denser here, and proofs are often omitted or merely sketched, in order to brake the page counter!) There is a natural hierarchy of sets of reals extending the Borel one, which contains the projective sets if there are arbitrarily large Woodin cardinals. These *universally Baire* sets can be regarded as a transfinite extension of the projective sets. Further, there is an extension of classical first-order logic, extending β-logic, which the author calls Ω-logic, in which (roughly speaking) proofs correspond to universally Baire sets. Ω-logic settles all first-order sentences over $\langle H(\omega_1), \in \rangle$. The author isolates an axiom $(*)$ that, together with ZFC, and always assuming that a proper class of Woodin cardinals exists, is a finite axiomatisation of the theory of $\langle H(\omega_2), \in \rangle$, as it likewise settles in Ω-logic all sentences in this structure. This axiom $(*)$ implies that $\mathfrak{c} = \aleph_2$. Thus, if there is any Π_2 sentence φ which is Ω-consistent, so that '$\langle H(\omega_2), \in \rangle \models \varphi$' settles the continuum hypothesis, then the value of $\mathfrak{c}$ *must* be $\aleph_2$. Further, if there is *any* such finite axiomatisation of the theory of of $\langle H(\omega_2), \in \rangle$, via some formula Ψ in Ω-logic, so that for some κ, $V_\kappa \models ZFC + \Psi$, then the continuum hypothesis must again be false.

It would be invidious to give a definition of Ω-logic here, but we can give one for the strongest possible logic in this direction (and the author conjectures they are equivalent under sufficient large cardinal assumptions): if T is any theory extending ZFC, then φ is derivable in Ω^*-logic if, for all ordinals α, and all forcing partial orders $\mathbb{P}$, if $V_\alpha^{\mathbb{P}} \models T$ then $V_\alpha^{\mathbb{P}} \models \varphi$. This derivability notion is invariant in all

forcing generic extensions. In this way, the reader can perhaps get a glimpse of how the baleful effects of forcing can somehow be 'factored' out, or away from the issue.

The book is intended for the professional set-theorist and the (necessarily) ambitious graduate student. It is the flagship book in a welcome new series of high-level monographs in logic, which should be on the shelves of any serious research library. It is a classic text in modern set theory.

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AN INTRODUCTION TO THE THEORY OF LOCAL ZETA FUNCTIONS (AMS/IP Studies in Advanced Mathematics 14)

By JUN-ICHI IGUSA: 232 pp., US$45.00 ISBN 0-8218-2015-X
(American Mathematical Society, Providence, RI, 2000).

'Local zeta functions are relatively new mathematical objects,' declares Igusa at the beginning of his book. The aim of this volume is to bring together the current lay of the land accessible for exploration by first-year graduate students. The theory spans a wide range of subjects, and represents a great way to learn a lot of important mathematical techniques—from the theory of manifolds, to hands-on applications of Hironaka's resolution of singularities, and from techniques of analysis to structures of groups and algebras.

Local zeta functions are defined as integrals of polynomials with respect to Haar measures on a local field. There are two sorts of local fields: archimedean (for example, the real numbers) and non-archimedean (for example, the p-adic numbers). The theory of local zeta functions has its origins in the archimedean setting after questions posed by Gel'fand in the 50s. The non-archimedean local zeta functions in characteristic zero took longer to bloom, but thanks to work of Igusa, Denef, Loeser and Meuser this theory is flourishing, and is currently more active than its archimedean counterpart. The characteristic p theory has yet to emerge, and is not discussed at all. Here the absence of a theory of resolution of singularities has limited the growth of the subject.

The narrative of the book follows this historical development, with a split between the archimedean and p-adic theory. Indeed, those interested in just one side of the story can reasonably concentrate on that section of the book, although the theories are intimately related.

The first major results concern the meromorphic continuation of these local zeta functions. This was proved by Bernstein and Gel'fand, and independently by Atiyah in the archimedean case, and by Igusa in the p-adic case. The fundamental tool is Hironaka's resolution of singularities. This tool is not essential. The theory of Bernstein polynomials can be used to bypass this tool in the archimedean case, and Denef has shown in the p-adic case that cell-decomposition theorems can be used to evaluate the p-adic integrals.

The motivation for Atiyah, Bernstein and Gel'fand's interest in these local zeta functions arose from connections to solutions of differential equations. Igusa's motivation in the 60s in the p-adic setting was the generalisation of Weil's observation that Siegel's main theorem on quadratic forms is a Poisson formula.

More generally, these p-adic zeta functions count the number of solutions of a polynomial modulo powers of the prime p. Borevich and Shafarevich proposed encoding these numbers into a Poincaré series. It was Igusa's analysis of the associated local zeta function of the polynomial which established the rationality of these series as conjectured by Borevich and Shafarevich. Hensel's lemma explains how solutions mod p can lift to solutions modulo higher powers of p. It is the singular points for which Hensel's lemma fails to apply which provide the content to the p-adic theory.

One can see some analogue with the Weil zeta function, which counts the number of solutions of a polynomial as one ascends a tower of finite fields, as opposed to descending through layers of the p-adic integers. Indeed, these two theories meet in the final chapter, where Igusa explains a functional equation for these p-adic local zeta functions established by Denef and Meuser, which relies on the functional equation of the zeta function of Weil.

This application of local zeta functions as a tool for counting has been extended beyond the realm of counting solutions to polynomials. There is a rich land beyond this book which awaits those who master Igusa's tour. Local zeta functions are a fundamental tool in my own corner of the mathematical world, in understanding the subgroup structure of infinite groups. Without this tool, the discovery of connections between nilpotent groups and elliptic curves would not have been possible (see [**3**]). p-adic integrals are also the key to the theory of p-adic string theory as developed by Brekke, Freund, Olson and Witten [**1**]. More recently, the theory of motivic integration, initiated by Kontsevich and developed by Denef and Loeser, has provided a powerful setting for understanding the uniformity in p of these zeta functions (see [**2**]).

Some have criticised the theory of local zeta functions for its lack of a well-developed global theory. When one looks up 'zeta function' in the *Encyclopaedia of Mathematics*, a function earns the prized title of zeta function only after passing stringent conditions: the existence of an Euler product decomposition, meromorphic continuation and a functional equation. As this book demonstrates, the local theory passes these tests well. But the failure of the Euler product of all these local zeta functions (which can be interpreted as an adelic integral) to demonstrate such good behaviour has deterred some from recognising the importance of this theory. I hope this book will help to demonstrate the strengths of this local theory. As the book illustrates perfectly, the theory has developed far enough to deserve the treatment of such a volume, but nonetheless abounds in fascinating open problems, ranging over a wide spectrum of mathematics.

References

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UNIFORM CENTRAL LIMIT THEOREMS
(Cambridge Studies in Advanced Mathematics 63)

By R. M. DUDLEY: 436pp., £55.00, ISBN 0-521-46102-2
(Cambridge University Press, 1999).

A central concept in statistics and probability theory is that of empirical distribution functions and empirical measures. The book under review deals in a general framework with convergence properties of empirical processes based on independent and identically distributed random variables. Its theme belongs to the borderland between probability theory in Banach spaces, and statistics. Compared to the earlier related monograph by van der Vaart and Wellner [**7**], which has a stronger emphasis on statistical applications, the author provides a more complete and detailed exposition of the underlying mathematical theory.

To become more specific, let $X_1, X_2, \ldots$ be independent and identically distributed random variables with distribution μ, taking values in a measurable space $(S, \mathscr{A})$. Then the nth empirical measure corresponding to this sequence is defined by $\mu_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{X_k}$, where δ_x stands for the Dirac measure concentrated at $x \in S$. Let $\mathscr{F}$ be a (suitable) class of measurable real-valued functions on S. By analogy with the classical Glivenko–Cantelli theorem for the empirical distribution function, and Donsker's invariance principle for the empirical process (which can also be regarded as a central limit theorem in a linear space), the following problems arise.

(G) Under which conditions is the following statement true?

$$\sup_{f \in \mathscr{F}} \left| \int f \, d\mu_n - \int f \, d\mu \right| \to 0 \quad \text{almost surely as } n \to \infty \tag{1}$$

(D) When, and in which sense, is the following statement true for the family $\left(\sqrt{n}\left(\int f \, d\mu_n - \int f \, d\mu\right)\right)_{f \in \mathscr{F}}$, regarded as a 'random element' in the normed linear space $\ell^\infty(\mathscr{F})$ of bounded real-valued functions on $\mathscr{F}$ equipped with the supremum norm?

$$\left(\sqrt{n}\left(\int f \, d\mu_n - \int f \, d\mu\right)\right)_{f \in \mathscr{F}} \quad \text{'converges weakly' as } n \to \infty \text{ to a Gaussian process indexed by } \mathscr{F}. \tag{2}$$

(R) Problem (D) is in turn closely connected with the regularity properties (sample-boundedness and continuity) of general Gaussian processes.

In the course of the last 35 years, the attempts to answer these questions have led to an impressive theory, to which the author himself has contributed decisively. In 1966 he published a paper [**1**] that laid the foundation-stone for a theory of weak convergence of probability measures on non-separable metric spaces, that was needed before rigorous work could start with regard to problem (D). Two articles [**2**, **3**] published in 1967 and 1973 provide a modern framework for analyzing the regularity properties of abstract Gaussian processes. His 1978 paper [**4**] on weak convergence of empirical processes indexed by general classes of sets (indicator functions of sets in the notation used above) initiated a great deal of further fruitful research concerning question (D) and related problems.

A key point in connection with problems (G) and (D) is that of giving a precise meaning to the vague feeling that the class $\mathscr{F}$ should not be 'too large', in order that statements (1) or (2) be true. There are two major concepts available to handle this aspect: metric entropy (already appearing in a 1955 paper by Kolmogorov [**6**]) and Vapnik–Červonenkis classes, which were introduced in 1968 [**8**].

The metric entropy conditions rest upon the assumption that $\mathscr{F}$ be totally bounded with respect to a suitable metric. Metric entropy then controls the minimal cardinality (which should not go to ∞ too fast as $\varepsilon \to 0$) of a set of ε-balls covering $\mathscr{F}$. (The notions actually used, for instance metric entropy with inclusion or with bracketing, are slightly more complicated, but the idea is the same.)

The original definition of Vapnik–Červonenkis classes deals with the special case in which $\mathscr{F}$ is a set of indicator functions of a class $\mathscr{C}$ of subsets of S. The definition then takes the following form.

Let S be any set. Given a collection $\mathscr{C}$ of subsets of S and a set $A \subset S$, consider the trace $\mathscr{C}_A := \{C \cap A : C \in \mathscr{C}\}$ of $\mathscr{C}$ in A. If

$$\max\{\operatorname{card} \mathscr{C}_A : A \subset S, \operatorname{card} A = n\} < 2^n$$

for some natural number n, then $\mathscr{C}$ is said to be a Vapnik–Červonenkis class.

The importance of this definition is illustrated by the following fact. In order for statement (2) to hold for the system of indicator functions $\mathscr{F} = \{1_A : A \in \mathscr{C}\}$, and irrespective of the underlying measure μ, it is necessary and (apart from a measurability condition) also sufficient that $\mathscr{C}$ be a Vapnik–Červonenkis class.

The contents of the book have been arranged around the questions indicated above.

Chapter 2 deals with Gaussian measures in a more-general setting, a highlight being the Fernique–Talagrand theorem concerning the characterization of sample-boundedness of Gaussian processes via majorizing measures. Chapters 3–8 and 10 are in one form or another connected with problems (G) and (D). The author adopts the Hoffmann–Jørgensen definition of convergence in law of not necessarily measurable functions taking values in a possibly non-separable metric space. This notion, the concept of Vapnik–Červonenkis classes, measurability questions and the underlying metric entropy conditions (Koltchinskii–Pollard entropy and metric entropy with inclusion and bracketing) are thoroughly discussed (partly within a chapter of their own). The probabilistic results presented include the Vapnik–Červonenkis–Steele law of large numbers, Pollard's and Ossiander's central limit theorem, and necessary conditions for the central limit theorem.

Chapter 9 is based on an article by Dudley and Philipp [**5**]. It is concerned with the explicit construction of empirical processes and Gaussian processes that are pathwise close to each other (invariance principles). The ideas developed there make it possible to embed the theory of empirical processes into the theory of partial sums of independent random variables taking values in infinite-dimensional vector spaces. They also provide an alternative to the Hoffmann–Jørgensen definition of weak convergence.

Chapter 11 is devoted to statistical applications (the two-sample case, the bootstrap and confidence sets). Finally, Chapter 12 contains deep examples of natural classes $\mathscr{F}$ of indicator functions of sets (such as convex sets in $\mathbb{R}^3$) that are 'too large' for statement (2) to hold.

The material of the book is very well organized, and incorporates recent developments in this area. The book also includes many mathematical results of non-probabilistic character that have been somewhat neglected in the pertinent literature, but which are nevertheless important to put things into a proper perspective. It should therefore become an invaluable reference to researchers in this field. At the same time, it has been written in a style that makes it accessible to students; proofs have been carried out with meticulous care, and definitions are well motivated.

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METRIC SPACES OF NON-POSITIVE CURVATURE
(Grundlehren der Mathematischen Wissenschaften 319)

By Martin R. Bridson and André Haefliger: 643 pp., £58.50,
ISBN 3-540-64324-9 (Springer, Berlin, 1999).

In his seminal article on the uses of negative curvature in group theory [**3**], Gromov introduced several different notions of non-positive curvature which could be applied to study infinite discrete groups. The ideas have their roots in the work of, among others, Alexandrov and Topogonov, and of Busemann, who used the techniques to study the global geometry of non-positively curved metric spaces. The idea has turned out to be extremely fruitful, and this book sets out to introduce and develop many of the key ideas in the subject. I would recommend it highly as a reference work for graduate students or professionals working in the field of geometric group theory. More importantly, it will provide a valuable resource for those working outside geometric group theory whose work brings them in contact with it, including differential geometers, number theorists and group-representation theorists. (For example, Davis showed that buildings are in some sense non-positively curved, and recent progress on the Baum–Connes conjecture has used ideas from the subject.)

The book is divided into three parts (*Geodesic metric spaces*, CAT(χ) *spaces*, and *Aspects of the geometry of group actions*), and it contains a rich bibliography. Perhaps the title of the third section betrays a certain resignation on the part of the authors, who, having set out to write a comprehensive text on the subject, found that the material, like Topsy, 'just growed and growed'. The subject has itself undergone an explosive development, with active research groups around the world mining the rich seam of ideas uncovered by Gromov. Of course, this is an indication of the health of the subject, but it has proved impossible for anyone to write a complete and comprehensive account of these developments. Nonetheless, the authors make an interesting selection in the choice of the 'aspects' that they cover, varying from the introductory (an account of Gromov's theory of δ-hyperbolic groups), to the

advanced, including a full account of the theory of complexes of groups. Perhaps the most challenging fundamental issue facing geometric group theory is to find and fully exploit the correct notions of semi-hyperbolicity; the section on semihyperbolic groups, though short, makes a good starting point.

Some of the material in the book can be found in other sources. In particular, Bridson's article on polyhedral CAT(0) spaces, and Haefliger's article on complexes of groups (which can both be found in [**1**]) have long been cited as standard references, and the material in them is subsumed by this book. The introduction to δ-hyperbolic spaces is covered in Gromov's original article, and in many other expository articles since then (see [**2**], for example). Nonetheless, the authors provide a valuable service by examining this material with a fresh eye, ironing out many of the difficulties that appeared in earlier accounts, and peppering the new accounts with more-detailed examples. Of particular interest to this reviewer is the section on symmetric spaces, which deserves to be read by all graduate students with an interest in group theory, geometry or number theory.

Whether you read this book as an introductory text, adopt it as a graduate course text, or refer to it as a reference work, you are unlikely to be disappointed. It is not, and cannot be, comprehensive, and perhaps the only weakness of the book lies in the attempt to make it so. Nonetheless the authors' taste is impeccable (for which, read 'coincides with mine') and the style strikes just the right balance between formality and discursiveness. There will be other books to write about this important and growing subject. All those wishing to write one should read this one first.

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QUASICONFORMAL TEICHMÜLLER THEORY
(Mathematical Surveys and Monographs 76)

By Frederick P. Gardiner and Nikola Lakic: 372 pp., US$89.00,
isbn 0-8218-1983-6 (American Mathematical Society, Providence, RI, 2000).

The goal of this book is stated in the excellent preface: 'to provide background for applications of Teichmüller theory to dynamical systems', with the natural emphasis on holomorphic dynamical systems including rational maps and Kleinian groups, surface homeomorphisms and one-dimensional maps (since there is a history of studying these via holomorphic extensions). It is made clear that there is no systematic exposition of the dynamical applications in the text—although one or two explicit examples are given, and some chapters, such as Chapter 16, seem to be influenced by existing work on dynamics. The extensive bibliography is instructive. As regards the substantial proportion that might be regarded as applications of

Teichmüller theory, I am fairly sure that most items are not referenced explicitly in the body of the text, but this is legitimate, and it is, of course extremely useful to have the list, even with the necessary disclaimer that the list is incomplete—were it ten times as long, that would still be the case. The spectrum includes: hyperbolic geometry, the topology of three-manifolds, the mapping class group, surface diffeomorphisms, billiards, measured foliations (for Teichmüller theory itself generates dynamical examples), and geodesic flows.

However, this is a book about Teichmüller theory, starting from the basics of quasiconformal maps and Riemann surfaces. The fundamental results of Teichmüller theory were first proved for the Teichmüller space of a compact surface, or of a surface which is of *finite analytical type*; that is, a compact surface minus finitely many punctures. One of the aims of this book is to prove results for a general Riemann surface, as far as possible. This has been a strand in the work of both co-authors over the years, and much of this work is presented in the book, as is their work with collaborators such as Earle, and work in this direction by others such as Krushkal. Some of the exposition in a general framework appears to be new, such as the generalisation to all finite genus Riemann surfaces in Chapter 8, of Royden's theorem characterizing isometries of Teichmüller space (originally proved for compact surfaces), and a generalization of Thurston's earthquake theorem in Chapter 18.

The exposition does start from base-level. One gets a sense of personal accumulation of knowledge, and of consolidation, as the work progresses. Each chapter is accompanied by exercises, a fair proportion of which are linked. With these, too, one has a feeling of exercises experienced by the authors as they worked. I think the exercises are helpful. They seem to be of varying difficulty; I think this is partly a function of how closely one is following the setter's train of thought.

Teichmüller space $T(R)$ is the space of complex structures on a surface R which is fixed up to quasi-conformal homeomorphism equivalence. By the fundamental Ahlfors–Bers Theorem of 1961, $T(R)$ can be described as a quotient either of a space of quasi-conformal homeomorphisms, or of the space of *Beltrami differentials*, the open unit ball in $L^\infty(\mathbb{R})$. In the case of a compact surface R, the adjective 'quasiconformal' can be omitted. This is also virtually the case when R is analytically finite. Classical Teichmüller space comes equipped with extra structure: the Teichmüller metric (which, while not Riemannian, is complete and such that there is a unique geodesic segment between any two points, determined by quadratic differentials), a tangent space (which can be defined as a quotient space of $L^\infty(\mathbb{R})$), a cotangent space of quadratic differentials, and a complex structure (given by the *Bers embedding*).

All these results are proved in as full generality as possible. The definition of the metric, at least, is straightforward for a general surface. The distance between points $[g]$ and $[f \circ g]$ is measured by the infimum of the *distortion* of f. If R is of finite topological type, then the infimum is attained for a unique f. Maps realising the infimum are called *extremal* in this book, and the study of them and infinitesimal analogues occupies several chapters (4, 9, 14–15, 17). There are several aspects to this study. For example, there are characterizations (Chapter 4) of the points in Teichmüller space (or, more accurately, pairs of points) for which unique extremals exist, and an analysis of their properties. Work of Lakic on *asymptotic Teichmüller space* $AT(R)$ is exposed (Chapters 14–15). This space is, in general, a quotient of $T(R)$, which behaves more like the Teichmüller space of a compact

surface. In particular, unique extremals exist. Teichmüller spaces of planar domains receive special attention (Chapter 17), and work (starting from work of Fehlmann) on relating distortion to distortion at the boundary is described. The space $A(R)$ of holomorphic quadratic differentials on R is of crucial importance in Teichmüller theory, since $A(R)$ is the cotangent space of $T(R)$. Unique extremal mappings are described in terms of quadratic differentials. A number of chapters of this book (11–13) are devoted to problems concerning quadratic differentials. For example, the 'Heights theorem', in which quadratic differentials are found with corresponding foliations with annuli foliated by circles of prescribed widths (the *Jenkins Strebel differentials*), is proved in some generality.

This is a very interesting book. The authors have some knowledge of the sort of properties of Teichmüller space that workers in dynamics might want (sometimes in vain). The treatment is clear and methodical. It would probably be possible to read this as an introductory text, yet there is much that is relatively new, innovatory, and perhaps percipient.

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SEMIMODULAR LATTICES: THEORY AND APPLICATIONS
(Encyclopedia of Mathematics and its Applications 73)

By MANFRED STERN: 370 pp., £50.00 (US$80.00), ISBN 0-521-46105-7
(Cambridge University Press, 1999).

The book under review is based on a previous version, published by Teubner-Verlag in 1991. The current edition is significantly revised, enlarged and updated. It is worth mentioning that the project was suggested by the late Gian-Carlo Rota. He wrote an enthusiastic introduction to the first edition, from which I quote.

> Contemporary lattice theory largely ranges between two equally attractive poles. At one end are ideas that originated with the distributive law, and which have filtered through logic, topos theory and sundry other subjects. At the other end we find the heirs of the old 'synthetic geometry', who have met an unexpected customer in combinatorics. The semimodular property, first observed in the theory of matroids, and more recently reappearing in a host of handsome theories now in full swing of development, has proved to be a central unifying concept, with no end of applications in sight. Theories that seemed to be the preserve of geometric lattices are gradually, and not without substantial effort, being carried over to the more difficult but far more inclusive class of semimodular lattices, to everyone's rejoicing.

The book consists of nine chapters, in which the various aspects of the subject are carefully delineated. The first chapter, *From Boolean algebra to semimodular lattices*, places the topic in a historical framework of 'some milestones in the general development of lattice theory', by a detailed mathematical explication, containing several instructive examples and diagrams reflecting hierarchies of lattice properties. The second chapter, on *M-symmetric lattices*, views semimodularity from modular pairs and forbidden pentagons. Furthermore, an extension of the isomorphism theorem for modular lattices is presented. In continuation, Chapter 3 introduces MacLane's condition, which for upper continuous strongly atomic lattices turns out to be equivalent to both semimodularity and M-symmetry.

Chapter 4, on *Supersolvable and admissible lattices; consistent and strong lattices*, relates semimodularity to combinatorics on finite partially ordered sets, such as the Möbius function, order complexes, and shellability. In the case of a finite semimodular lattice, it is shown as a highlight that the concepts of being balanced, consistent or strong all coincide, and characterize the distinguished class of glued geometric lattices. Chapter 5 introduces the concept of the covering graph of a finite partially ordered set, and gives solutions to the following problem raised by G. Birkhoff: find necessary and sufficient conditions on a lattice L so that every lattice with covering graph isomorphic to that of L is already lattice isomorphic to L. Chapter 6 focuses on *Semimodular lattices of finite length*. The first part of the chapter is on rank and covering inequalities restricted to the finite case. It contains Kung's general approach to the matching problem for finite lattices, including applications to the semimodular case. In the second part, pseudorank-preserving embeddings of finite lattices into finite semimodular lattices and finite geometric lattices are considered. The relationship between semimodular lattices of finite length and geometric closure operations is studied, as is the representation of these lattices by selectors. The final section concerns consistent semimodular lattices.

Chapter 7, on *Local distributivity*, mentions firstly characterizations of strongly atomic algebraic lattices having unique irredundant meet decompositions, and then derives Avann's latticial characterization of finite meet-distributive (that is, lower locally distributive) lattices. The latter yields that a finite lattice is locally distributive if and only if it is semimodular such that each of its lower semimodular sublattices is distributive. In the sequel, more combinatorial approaches to finite meet-distributive lattices are presented in connection with an abstract concept of convexity. *Local modularity* is the topic of Chapter 8. The reader is introduced to the subject mainly in the form of an overview, and is made familiar with key results such as the following one: a semimodular strongly atomic algebraic lattice has the Kurosh–Ore replacement property if and only if it is locally modular. Afterwards, lattices of subnormal subgroups are investigated with respect to lower semimodularity. This chapter ends with more combinatorial aspects of the subject. The final chapter is devoted to congruence semimodularity. In essence, (lower) semimodularity is imposed on or derived for the congruence lattices of certain algebras like (pseudocomplemented) semilattices and (inverse) semigroups.

I recommend the book highly to all interested readers, both experts and non-experts. It is equipped with an extensive bibliography, a useful table of notations, and an accurate index. Throughout the book, the author's use of examples and diagrams guides the reader through the material. I cannot do better than to conclude with a further excerpt from Gian-Carlo Rota's introduction to the first version of the book. His assessment is still valid for the new edition.

> Professor Stern's book is the first to include the latest developments in the theory of semimodular lattices. His clear presentation, and his readable mathematical style, will make his treatise an indispensable vade-mecum for lattice theorists, and for all discrete mathematicians everywhere.

New Mexico State University STEFAN E. SCHMIDT

CYCLES, TRANSFERS, AND MOTIVIC HOMOLOGY THEORIES
(Annals of Mathematics Studies 143)

By VLADIMIR VOEVODSKY, ANDREI SUSLIN and ERIC M. FRIEDLANDER: 254 pp., £15.95, ISBN 0-691-04815-0 (Princeton University Press, 2000).

Some years ago, A. Beilinson made an extremely important conjecture concerning the existence of complexes of sheaves, denoted by $\mathbf{Z}(n)$, in the Zariski topology on the category, Sm/k, of smooth schemes over a field, k. This volume, consisting of five related papers, delineates how the ideas of Suslin and Voevodsky, together with the techniques of Friedlander–Lawson, which arose in the context of Lawson homology theory, essentially establish the conjectures. The importance of the conjectures derives from the fact that Beilinson's axioms for $\mathbf{Z}(n)$ explain how its Zariski hypercohomology is connected to: (i) the Chow groups of algebraic cycles on a smooth scheme over k modulo rational equivalence, (ii) Milnor K-theory, (iii) Quillen algebraic K-theory, and (iv) étale cohomology, $H^*(X;\mu_l^{\otimes n})$.

The connection with Chow groups, one of the fundamental objects of modern algebraic geometry, demonstrates forcefully that the culmination of this book—the construction of various triangulated categories of motives over a field—is not just abstraction for its own sake. For example, by way of applications, this material led to Voevodsky's proof of the Milnor conjecture, relating Milnor K-theory to mod 2 Galois cohomology. It has given rise to new techniques of moving families of algebraic cycles, to important new Grothendieck topologies and related comparison theorems.

Perhaps the simplest way to appreciate the flavour of the techniques introduced in this volume is to remind ourselves of Suslin's singular homology groups, $h_*(X)$, of a scheme of finite type over k. This theory, while not central to the material in this book, is a similar type of construction. It is defined to be the homology of the complex whose nth term is the group of finite correspondences from

$$\mathrm{Spec}(k[t_0,\ldots,t_n]/(\sum_i t_i - 1))$$

to X. One may tensor this complex with integers modulo n, and hence obtain $h_*(X;\mathbf{Z}/n)$. If X is an integral scheme of finite type over the complex field, $\mathbf{C}$, a striking result of Suslin–Voevodsky states that there is a natural isomorphism

$$h_*(X;\mathbf{Z}/n) \cong H_*^{\mathrm{sing}}(X(\mathbf{C});\mathbf{Z}/n)$$

between the (modulo n) algebraic and topological singular homology of X. Algebraic singular homology illustrates how algebraic cycles and correspondences pervade the authors' constructions.

A particularly elegant application is Suslin's isomorphism

$$\mathrm{CH}^i(X,n;\mathbf{Z}/m) \cong H^{2i-n}_{\text{ét}}(X;\mathbf{Z}/m(i)),$$

which describes Bloch's higher Chow groups of a smooth integral scheme over an algebraically closed field in terms of étale cohomology.

There are other recent approaches to motivic categories (for example, the work of Masaki Hanamura and Marc Levine) and singular homology of arithmetic schemes (for example that of Alexander Schmidt), which ought to be studied in conjunction with this volume.

University of Southampton VICTOR SNAITH

RIEMANNIAN GEOMETRY DURING THE SECOND HALF OF THE TWENTIETH CENTURY
(University Lecture Series 17)

By MARCEL BERGER: 182 pp., US$34.00, ISBN 0-8218-2052-4
(American Mathematical Society, Providence, RI, 2000).

This is quite an amazing book. In 130 pages of text, 30 pages of bibliography, and (very important) 18 pages of subject and author indices, Marcel Berger surveys a major part of modern Riemannian geometry.

The coverage of the book is quite astonishing, both in breadth and in depth. Berger was an important contributor to some of the developments on which he reports, but he also writes with authority on a wide range of topics. As for timeliness, the book is essentially a reprint of a 1998 volume published by the Deutsche Mathematiker-Vereinigung, but the author has brought it up to date for the new century by adding numerous references to papers up to the year 2000. The reader is left with the feeling that essentially all the topics covered are still the subject of active research. Contrary to what a reader who has not been following the subject might expect, there seem to be no 'dead areas' in Riemannian geometry.

Another outstanding feature of the book is its extensive cross-referencing. Illustrating the hypertextual possibilities of old-fashioned print on paper, this feature reveals the elaborate interconnection of all the areas of Riemannian geometry. It also made the book particularly valuable to my students in a graduate Riemannian geometry course, since they could quickly find out more about a topic which was completely new to them.

Of course, any reader who has worked in this field is bound to check on what Berger has to say about his or her own contributions. In my case, I was amused to find a bit of psychohistory on page 44: 'In a desperate search for a classification of positive curvature manifolds, Weinstein proved that ...'. This quote gives an example of the engaging style which makes the book a page-turner despite its density and thoroughness.

To indicate the content of the book, I can do no better than to list the chapter titles. Following an introduction, a summary of Riemannian geometry up to 1950, and an initiation section of *Comments on the main topics I, II, III, IV, V under consideration*, there follow the chapters: I *Curvature and topology*, II *The geometrical hierarchy of Riemannian manifolds: space forms*, III *The set of Riemannian structures on a given compact manifold: is there a best metric?*, IV *The spectrum, the eigenfunctions*, and V *Periodic geodesics, the geodesic flow*. A last chapter, *TOP. Some other Riemannian geometry topics of interest*, briefly covers ten other topics, including isometric embeddings and harmonic maps. The tenth of these topics, 'Submanifolds', is dispatched in just four pages! This is to my mind the biggest 'gap' in the book, but in the face of such riches, one can hardly complain. Perhaps someone else will endeavor to expand the 'TOP.10' into a survey of size comparable to this whole volume.

Finally, and remarkably for a book of this kind, typographical errors are few and far between.

University of California, Berkeley ALAN WEINSTEIN

MONOIDS, ACTS AND CATEGORIES: WITH APPLICATIONS TO WREATH PRODUCTS AND GRAPHS: A HANDBOOK FOR STUDENTS AND RESEARCHERS
(de Gruyter Expositions in Mathematics 29)

By MATI KILP, ULRICH KNAUER and ALEXANDER V. MIKHALEV: 529 pp., DM298.00, ISBN 3-11-015248-7 (Walter de Gruyter, Berlin, 2000).

This book, as well as providing a sound introduction to the non-additive homological algebra of monoids, acts as a comprehensive reference manual for many current trends in the homological classification of monoids and semigroups.

A monoid is a semigroup with an identity element, and actions of monoids on sets have been studied since the beginnings of semigroup theory. As the authors mention, some areas of semigroup theory have mirrored developments in groups and rings, but in recent years the subject has developed an identity of its own. So it is with monoid acts. Some early work concentrated on generalising results from modules over rings and group actions, but recent applications to epimorphisms, amalgamation, automota theory and decision problems, for example, have increased the status of the theory of monoid acts.

The book is organised into five sections, the first of which provides a basic introduction to semigroups, acts and categories. Section 2 introduces many of the categorical constructions used in non-additive homological algebra such as pullbacks, pushouts, and tensor and wreath products, and provides a host of results that demonstrate some of the differences, as well as some of the similarities, to module theory over rings. In most cases, concepts are introduced with respect to concrete categories first, and then the situation for categories of acts is clarified. Classifying rings by means of properties that their R-modules satisfy has long been an important area of study, and Sections 3 and 4 attempt the same for monoids. Section 3 introduces concepts such as injectivity, projectivity and flatness of acts, and draws important distinctions between these and the analogous concepts in ring theory. For example, an R-module is (right) flat if and only if the tensor functor $\otimes_R$ preserves all embeddings $I \to R$ of (left) ideals of R. It has long been known that this is not so for monoids, and in fact a hierarchy of properties connected with flatness can be constructed. This chapter is a preparation for Chapter 4, where the homological classification of monoids is considered in some detail. The homological classification of monoids has been the work of much research over the last few years, and the authors present many of the most important results in this area. The emphasis is on the projective-flat hierarchy of properties, but it starts with a list of many of the known results relating to injectivity. Chapter 5 gives an account of the Morita theory for monoids, beginning with some preliminary material on adjoint functors in relation to categories of acts.

The book is well written, in a clear and lucid style. It is organised in a 'reference manual' style, with many theorems and lemmas, and a small number of intermediate comments and remarks, but there are plenty of examples and a few exercises in most sections. The bibliography is extensive. This work will prove an indispensable tool for researchers and graduate students working on monoid acts, and for those interested in connections between semigroups and theoretical computer science, graph theory and system theory, among others; I can strongly recommend it.

University of Southampton JAMES RENSHAW

FUNCTIONS OF BOUNDED VARIATION AND FREE DISCONTINUITY PROBLEMS
(Oxford Mathematical Monographs)

By LUIGI AMBROSIO, NICOLO FUCSO AND DIEGO PALLARA: 434 pp., £55.00, ISBN 0-19-850254-1 (Clarendon Press, Oxford, 2000).

Around 1936, L. Tonelli defined, in analogy with the one-dimensional case, a continuous function of two variables to be *of bounded variation* if the surface areas of the projections of the graph of the function onto the vertical coordinate planes (counting multiplicities) are finite. This definition was soon extended by L. Cesari [**1**, **4**] to require only that there exists a Lebesgue representative of the function for which the areas of the projections are finite. This extended definition applies to discontinuous functions, and is independent of coordinates. Note that the variation of a function is related to the area of its graph.

Since then, other variants of the notion of bounded variation have been considered, motivated by various points of view: the theory of distributions, generalized surfaces, variational problems involving surface area, and so forth.

Today, there is an extensive and fascinating theory of functions of bounded variation (BV functions). In the book under review, a BV function is defined as follows. A function $u \in L^1(\Omega)$ is BV in the open set $\Omega \subset \mathbb{R}^n$ if the distributional partial derivatives of u are representable by finite Radon measures; that is, if there exist finite Radon measures $D_i u$ such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega \varphi\, dD_i u$$

for each $\varphi \in C_c^\infty(\Omega)$ and $1 \leqslant i \leqslant n$. The set of all functions of bounded variation on Ω is denoted by $BV(\Omega)$.

It may not be obvious, but the above definition is equivalent to that of Tonelli–Cesari. Moreover, if u is continuous and Ω has finite n-dimensional Lebesgue measure ($\mathscr{L}^n(\Omega) < \infty$), then u is BV if, and only if, the n-dimensional measure of the graph of u is finite. Furthermore, the Lebesgue surface area of the nonparametric surface corresponding to u is equal to the total variation of the $\mathbb{R}^{n+1}$-valued measure $(\mathscr{L}^n, D_1 u, \ldots, D_n u)$; see [**2**].

The first half (Chapters 1–3) of the book under review provides a systematic and largely self-contained presentation (for example, no prior knowledge of Sobolev spaces is required) of the theory of BV functions (and sets of finite perimeter), and a gentle introduction to geometric measure theory. Among many other things presented in Chapter 3 is a precise description of the structure of the distributional derivative of a BV function. Let $\mathscr{H}^m$ denote m-dimensional Hausdorff measure. For each $u \in BV(\Omega)$, we see that S_u, the set of approximate discontinuities of u, is $\mathscr{H}^{n-1}$-rectifiable and $\mathscr{H}^{n-1}$-almost every point $x \in S_u$ is an approximate jump point of u with direction of jump $v_u(x)$ perpendicular to the approximate tangent space to S_u at x. Denote the set of jump points by J_u. Moreover, u is approximately differentiable $\mathscr{L}^n$ almost everywhere. The distributional derivative of u can be written as

$$Du = \nabla u \mathscr{L}^n + Du \llcorner J_u + D^s u \llcorner (\Omega - S_u),$$

where ∇u is the approximate derivative of u and D^s is the singular part of the measure Du with respect to $\mathscr{L}^n$. The three terms on the right-hand side above are

called the *absolutely continuous part*, the *jump part* and the *Cantor part*, respectively, of Du. The jump set J_u is countably rectifiable, and is oriented by the jump vector $v_u(x)$. Thus

$$Du \llcorner J_u(B) = \int_{B \cap J_u} \left(u^+(x) - u^-(x)\right) \otimes v_u(x)\, d\mathscr{H}^{n-1}$$

whenever B is a Borel subset of Ω. Here $u^{\pm}(x)$ denote the one-sided limits at a point $x \in J_u$ relative to the normal $v_u(x)$.

The second half of the book is directed towards the study of certain variational problems modeled on the Mumford–Shah image-segmentation problem. Suppose that Ω is a bounded domain in $\mathbb{R}^n$, that $g \in L^\infty(\Omega)$, and that α and β are positive parameters. Let

$$J(K,u) := \int_{\omega - K} |\nabla u|^2 + \alpha(u-g)\, dx + \beta \mathscr{H}^{n-1}(K \cap \Omega).$$

The Mumford–Shah problem is to minimize J among all admissible pairs

$$\mathscr{A} := \left\{(K,u) \;:\; K \subset \overline{\Omega} \text{ closed},\ u \in W^{1.2}_{\text{loc}}(\Omega - K)\right\}.$$

In two dimensions, this problem was proposed by D. Mumford and J. Shah [**3**] as a variational formulation of the image-segmentation problem in computer vision.

Chapter 4 studies the subspace of BV called SBV (for *special* BV), consisting of those BV functions whose distributional derivatives have zero Cantor part. These functions provide an appropriate setting for certain types of variational problems that involve both volume and surface energies. In particular, important closure and compactness properties hold for minimizing sequences. Chapter 5 establishes lower semicontinuity with respect to weak* convergence in BV for various integral functionals. Chapter 6 surveys in detail the state of the current knowledge regarding the Mumford–Shah image-segmentation problem. The last two chapters present existence and regularity results for the minimization problems modeled on the Mumford–Shah problem. The presentation in the second half of the book benefits in its clarity from the fact that the authors have contributed significantly to the subject.

This book provides an excellent account of the theory of BV functions (apparently for the first time in book form), and a nice introduction to geometric measure theory, as well as a rigorous survey of results for 'free discontinuity' problems modeled on the Mumford–Shah problem. It should serve as a standard reference, especially for the BV theory, for years to come.

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University of Kentucky RONALD F. GARIEPY

TOOLS FOR PDE: PSEUDODIFFERENTIAL OPERATORS, PARADIFFERENTIAL OPERATORS, AND LAYER POTENTIALS
(Mathematical Surveys and Monographs 81)

By MICHAEL E. TAYLOR: 257 pp., US$59.00, ISBN 0-8218-2633-6
(American Mathematical Society, Providence, RI, 2000).

The book deals with pseudodifferential operators, paradifferential operators and applications to linear and nonlinear partial differential equations. The main point is to minimize as much as possible the smoothness assumptions for symbols (pseudodifferential operators), coefficients and boundaries of domains (partial differential operators). The book should be considered as a continuation of the author's previous book [**2**]. This makes it clear that, as a prerequisite, a good working knowledge of the classical (smooth) parts of the theory of pseudodifferential operators is expected. Furthermore, a reader should be familiar with basic facts from the theory of function spaces and harmonic analysis, including the Paley–Littlewood theory. The material presented here is mostly based on recent publications by the author and his co-workers, but related results of other mathematicians are also included in a reasonable way. Occasionally, more detailed links in the text to the papers listed in the bibliography would be desirable.

The book has four chapters. Chapter 1 (*Pseudodifferential operators with mildly regular symbols*) begins with a detailed study of function spaces of generalized Hölder–Zygmund type, the (inhomogeneous) Hardy spaces h^1, the (inhomogeneous) space *bmo* of functions with bounded mean oscillation, and special Besov spaces $B^s_{p,1}$, and their sometimes delicate interrelations. This also includes the recent work of Bourdaud and Marschall. The smoothness of the symbol $p(x,\xi)$ (especially with respect to x) of the pseudodifferential operator

$$p(x,D)u = (2\pi)^{-n/2} \int p(x,\xi)\,\hat{u}(\xi)\,e^{ix\xi}\,d\xi$$

is expressed in terms of these spaces (in particular, Hölder–Zygmund spaces) and generalized Hörmander classes $S^m_{1,\delta}$. The main aim of this chapter is to study the mapping properties of these operators in L_p, h_1, *bmo*, $B^s_{p,1}$, and some types of generalized Sobolev spaces, H^s_p. Commutator estimates are considered in some detail.

Chapter 2 (*Paradifferential operators and nonlinear estimates*) deals with product formulas of the type

$$\|f\,g\,|H^s_p\| \leqslant c\,\left(\|f\,|L_{q_1}\| \cdot \|g\,|H^s_{q_2}\| + \|g\,|L_{r_1}\| \cdot \|f\,|H^s_{r_2}\|\right)$$

where $s > 0$, $1 < p < \infty$, and

$$\tfrac{1}{p} = \tfrac{1}{q_1} + \tfrac{1}{q_2} = \tfrac{1}{r_1} + \tfrac{1}{r_2}\,, \qquad q_2, r_2 \in (1,\infty)\,, \quad q_1, r_1 \in (1,\infty]\,,$$

and, in particular, with composition operators

$$u \mapsto F \circ u\,, \qquad u(u_1, ..., u_k)\,,$$

in diverse function spaces. These assertions are treated on the basis of Bony's application of paramultiplication (around 1980) to nonlinear problems. (In connection with product formulas, paramultiplications had been used before, in the mid-seventies, by J. Peetre and the reviewer, as is also mentioned in the book on page 86.)

Both topics are considered in some detail, but more systematic treatments (both

of product formulas and of composition operators) may be found in [**1**] (this book is listed in the bibliography, but is hardly mentioned in the text).

Chapter 3 (*Applications to PDE*) contains diverse applications of the methods of the two previous chapters to PDEs. The intention is always to minimize the respective smoothness assumptions. Special attention is paid to the Laplacian on Riemannian manifolds where the metric tensor is minimally smooth. Furthermore, the propagation of singularities of linear PDEs (or pseudodifferential operators) is considered (again under minimal assumptions). Of special interest is the study of the method of layer potentials for boundary-value problems on domains Ω with $\partial\Omega \in C^{1,\omega}$, where ω satisfies a Dini condition. Mapping properties of single-layer potentials in $L_p(\partial\Omega)$, $H_p^1(\partial\Omega)$ and $C^\lambda(\partial\Omega)$ are proved, including their consequences for related boundary-value problems. The aim of this chapter is not so much to deal systematically with (linear and nonlinear) PDEs, but to demonstrate the power of the tools developed in the preceding chapters, in application to selected problems related to PDEs.

Chapter 4 (*Layer potentials on Lipschitz surfaces*) covers the most recent research concerning single- and double-layer potentials on Lipschitz surfaces, and applications to the Dirichlet problem on Lipschitz domains. Problems of this type have attracted a lot of attention in recent times. First, the fundamental estimates of Cauchy integrals on Lipschitz curves are established (in both one and higher dimensions). Afterwards, the delicate application of single-layer techniques to Dirichlet problems for the Laplacian in Lipschitz domains in Riemannian manifolds is discussed (including many references).

The book provides an interesting and valuable complement to the existing (book) literature, and will be of great use to specialists working in this field. It is written in a concise but reasonable style.

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Friedrich-Schiller-Universität Jena HANS TRIEBEL

DYNAMICAL SYSTEMS ON HOMOGENEOUS SPACES
(Translations of Mathematical Monographs 190)

By Alexander N. Starkov: 243 pp., US$95.00, isbn 0-8218-1389-7
(American Mathematical Society, Providence, RI, 2000).

The modern study of dynamical systems on homogeneous spaces has roots in classical dynamics and classical number theory. For example, the geodesic flow on a surface of constant negative curvature can be realized as a flow on a homogeneous space of $\mathrm{SL}(2,\mathbb{R})$; this enabled I. M. Gel'fand and S. V. Fomin [**3**] to apply the theory of group representations to the study of geodesic flows. Kronecker's theorem that the integer multiples of an irrational number are dense modulo 1 can be restated as the fact that the orbits of an irrational rotation are dense in the circle. Less obviously, the uniform distribution of orbits of flows on homogeneous spaces

of nilpotent groups implies Weyl's theorem that if $p(x)$ is an irrational polynomial, then $p(\mathbb{Z})$ is dense (and, in fact, uniformly distributed) modulo 1.

Flows on homogeneous spaces are dynamical systems of the following specific form: given a closed subgroup Γ of a connected Lie group G, and a one-parameter subgroup $\{g_t\}$ of G, define a flow ϕ_t on the homogeneous space G/Γ by $\phi_t(x\Gamma) = g_t x\Gamma$. 'A large number of interesting examples of dynamical systems can be realized in this fashion. For that reason, and because of the ease with which computations can be made in this algebraic setting, the [flows on homogeneous spaces] are a natural class of dynamical systems within which to determine criteria for ergodicity, compute spectral type, and so on...' [**2**]. This describes the motivation for much of the work in this area, but, in the past decade, applications to number theory have grown more important.

The shift in emphasis came with G. A. Margulis' [**4**] proof of the Oppenheim conjecture: if $Q(x_1, \ldots, x_n)$ is any nondegenerate, indefinite, irrational quadratic form in $n \geqslant 3$ variables, then $Q(\mathbb{Z}^n)$ is a dense subset of $\mathbb{R}$. This was obtained by studying the closures of the orbits of a certain unipotent flow on the homogeneous space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$. (A one-parameter group $\{g_t\}$ of $n \times n$ matrices is *unipotent* if $(g_t - I)^n = 0$, where I denotes the identity matrix.) Shortly after this, M. Ratner [**5**] completed her long-term program to obtain a complete understanding of the closures of orbits of all unipotent flows (under the assumption that Γ is a lattice in G). This was a major development, which, among other things, led to more accurate versions of the Oppenheim conjecture, establishing that the values of the quadratic form are uniformly distributed. The book includes an essentially complete (but rather demanding) proof of the Oppenheim conjecture. The known proofs of Ratner's theorem are, unfortunately, too complicated to include in detail, but the book provides background material, proves illustrative special cases, and outlines the general proof. This includes an exposition of some quite technical material that will be very helpful for a reader interested either in reading the original papers, or in applying the techniques to other problems.

Chapter 1 studies the general dynamical properties of homogeneous flows. In particular, there is a complete proof of the condition for such a flow to be ergodic, or to be mixing, and the spectrum is determined. Typically, these proofs have three parts: the semisimple case, the solvable case, and a combination of the two previous cases. In the semisimple case, representation theory can be applied, but the book also explains more elementary approaches to some of the problems. For the solvable case, the 'semisimple splitting' is a powerful tool that often allows one to assume that all eigenvalues are real. To combine the other two cases, the book blends representation theory with the structure theory of Lie groups.

Chapter 2 focuses on unipotent flows. The results here, most of which were proved by M. Ratner in the 1980s, show that dynamical properties of the flow are closely connected to the algebraic structure of the underlying homogeneous space. For example, Ratner proved that if μ is an ergodic invariant probability measure for a unipotent flow on G/Γ, then μ is 'homogeneous'; that is, there is a closed subgroup H of G, such that μ is H-invariant, and μ is supported on a single (closed) H-orbit. This leads to the classification of orbit closures, mentioned above.

Chapter 3 explains the situation for non-unipotent flows. In this case, Ratner-type theorems usually fail: invariant measures need not be homogeneous, orbit closures may be badly behaved, and so forth. Much work remains to be done, and this is a promising area for future applications to number theory.

Chapter 4 discusses the applications to number theory. Of course, this includes the Oppenheim conjecture and its generalizations. Other applications are based on the study of flows of subgroups that are diagonalizable, rather than unipotent. For example, this leads to good error estimates for the asymptotic number of lattice points in a dilated polyhedron tP.

The book provides a thorough discussion of many of the main topics in the field. Theorems are stated precisely, references are provided when proofs are omitted, and the historical development of the subject is described, so the book is a very useful reference. It is at a rather high level, assuming familiarity with Lie theory and dynamical systems, so a novice who desires a gentle introduction to some of the elementary ideas of the subject would want to start with the recent text of B. Bekka and M. Mayer [**1**].

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