Supplement of Ice-Flow Perturbation Analysis: A method to estimate ice-sheet bed topography and conditions from surface datasets

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NON-DIMENSIONALISED TRANSFER FUNCTIONS

For ice in a planar slab aligned in the direction of ice flow, the non-dimensionalised Fourier transforms of perturbations in ice surface elevation, ($\hat{S}$), and velocity, ($\hat{U}$, $\hat{V}$), can be calculated from the non-dimensionalised Fourier transforms of perturbations in bed topography, ($\hat{B}$), and basal slipperiness, ($\hat{C}$):

$$\hat{S}(k,l) = T_{SB}(k,l)\ \hat{B}(k,l) + T_{SC}(k,l)\ \hat{C}(k,l), \tag{1}$$

$$\hat{U}(k,l) = T_{UB}(k,l)\ \hat{B}(k,l) + T_{UC}(k,l)\ \hat{C}(k,l), \tag{2}$$

$$\hat{V}(k,l) = T_{VB}(k,l)\ \hat{B}(k,l) + T_{VC}(k,l)\ \hat{C}(k,l), \tag{3}$$

where T_{sb}, T_{sc}, T_{ub}, T_{uc}, T_{vb}, T_{vc} are wavenumber specific non-dimensional transfer functions which describe the amplitude ratio of perturbations in the bed properties relative to the ice surface properties, and which vary with the wavenumbers k and l, angle of slope, α, sliding law parameter, m, and mean non-dimensionalised bed slipperiness, $\bar{C}$.

Depending on whether full-Stokes flow (Gudmundsson, 2003) or the shallow-ice-stream approximation (Gudmundsson, 2008) are considered, these transfer functions take different functional forms, where $j^2 = k^2 + l^2$.

For the shallow-ice-stream approximation (Gudmundsson, 2008)

$$T_{SB}(k,l) = \frac{k\Big(1 + m\big(2j^2\bar{C} + 1\big)\Big)}{k + m\Big(k + 2kj^2\bar{C} + \mathrm{i}j^2\cot\alpha\Big)}, \tag{4}$$

$$T_{UB}(k,l) = \frac{-\mathrm{i}\cot\alpha\Big(l^2 m - k^2\big(1 + 0.5j^2 m\bar{C}\big)\Big)}{\Big(k + m\big(k + 2kj^2\bar{C} + \mathrm{i}j^2\cot\alpha\big)\Big)\Big((m\bar{C})^{-1} + 0.5j^2\Big)}, \tag{5}$$

$$T_{VB}(k,l) = \frac{\mathrm{i}kl\Big(1 + 0.5j^2 mC + m\Big)\ \cot\alpha}{\Big(k + m\big(k + 2kj^2\bar{C} + \mathrm{i}j^2\cot\alpha\big)\Big)\Big((m\bar{C})^{-1} + 0.5j^2\Big)}, \tag{6}$$

$$T_{SC}(k,l) = \frac{k}{k + m\Big(k + 2kj^2\bar{C} + \mathrm{i}j^2\cot\alpha\Big)}, \tag{7}$$

$$T_{UC}(k,l) = \frac{\bar{C}\Big(3kl^2 m\bar{C} + 2k + kj^2 m\bar{C} + 2il^2\cot\alpha\, m\Big)}{\Big(k + m\big(k + 2kj^2\bar{C} + \mathrm{i}j^2\cot\alpha\big)\Big)\Big(2 + j^2 m\bar{C}\Big)}, \tag{8}$$

$$T_{VC}(k,l) = \frac{-klm\bar{C}\Big(2i\cot\alpha + 3k\bar{C}\Big)}{\Big(k + m\big(k + 2kj^2\bar{C} + \mathrm{i}j^2\cot\alpha\big)\Big)\Big(2 + j^2 m\bar{C}\Big)}. \tag{9}$$

For full-Stokes flow (Gudmundsson, 2003)

$$T_{SB}(k,l) = \frac{jk\Big((1+\bar{C})(\bar{C}j\sinh(j)+\cosh(j)) + \cosh(j)(1+\bar{C}+\bar{C}^2j^2)\Big)}{jk(1+\bar{C})\Big(\cosh(j)(\bar{C}j\sinh(j)+\cosh(j)) + 1 + j^2(1+\bar{C})\Big) + i\cot(\alpha)\Big((Cj\sinh(j)+\cosh(j))\sinh(j) - j\Big)}, \tag{10}$$

$$\begin{aligned} T_{UB,num}(k,l) = \Bigg(&\sinh^2(j)\Big(j^2\cot(\alpha)\sinh(j)\cosh(j)\Big(\bar{C}(2-k^2(\bar{C}^2j^2+4+\bar{C}))+4\Big) \\ &+ ik\Big(j^4\bar{C}\Big(3\bar{C}k^2(1+\bar{C})-4\Big) + 2\Big(j^2(k^2(4\bar{C}+2+\bar{C}^2)+4+4\bar{C}) - 2k^2(1+\bar{C})\Big) \\ &\quad + j^3\Big(\bar{C}^3j^2(k^2(1+\bar{C})+2) + \bar{C}k^2(5\bar{C}+4) + 2(3\bar{C}-2)(1+\bar{C})\Big) - 4k^2j\bar{C}(1+\bar{C})\Big) \\ &+ j\cot(\alpha)\Big((-3\bar{C}^2k^2+2\bar{C}(2+\bar{C}))j^2 - k^2(2+\bar{C})^2\Big)\Big) - 2j^3\cot(\alpha)\Big(\bar{C}^2k^2+\bar{C}+2\Big) \\ &+ 2ikj\Big(j^2\Big(k^2(5\bar{C}^2+6\bar{C}+2)-4-4\bar{C}\Big) - j^4\Big(2+5\bar{C}+4\bar{C}^2\Big) + 2k^2\Big(1+\bar{C}\Big)\Big)\Bigg), \end{aligned} \tag{11}$$

$$\begin{aligned} T_{vel,den}(k,l) = j^2\Bigg(&j\cosh^3(j)\Big(ik(1+\bar{C})\Big(\bar{C}^2j^2+2\Big) - 3\bar{C}\cot(\alpha)\Big) \\ &+ j\cosh(j)\Big(\cot(\alpha)\Big(2+3\bar{C}\Big) - ik(1+\bar{C})\Big(j^2(\bar{C}^2-2\bar{C}-2)-2\Big)\Big) \\ &+ \sinh^3(j)\Big(3ikj^2\bar{C}\Big(1+\bar{C}\Big) - \cot(\alpha)\Big(2+\bar{C}^2j^2\Big)\Big) \\ &+ \sinh(j)\Big(\cot(\alpha)\Big(j^2\bar{C}-2\Big) + ikj^2\bar{C}(1+\bar{C})\Big(j^2(1+\bar{C})+4\Big)\Big)\Bigg), \end{aligned} \tag{12}$$

$$T_{UB}(k,l) = T_{UB,num}(k,l) \,/\, T_{vel,den}(k,l), \tag{13}$$

$$\begin{aligned} T_{VB,num}(k,l) = kl\Bigg(&\sinh(j)\cosh(j)\Big(ik\Big(3\bar{C}^2j^4(1+\bar{C}) + 2j^2(2+4\bar{C}+\bar{C}^2) - 4 - 4\bar{C}\Big) \\ &\qquad - j^2\bar{C}\cot(\alpha)\Big(j^2\bar{C}^2+\bar{C}+4\Big)\Big) \\ &+ \sinh^2(j)\Big(ikj\Big(\bar{C}^3j^4(1+\bar{C}) + \bar{C}j^2(5\bar{C}+4) - 4\bar{C}(1+\bar{C})\Big) \\ &\qquad - j\cot(\alpha)\Big(3j^2\bar{C}^2 + (2+\bar{C})^2\Big)\Big) \\ &- 2\bar{C}^2j^3\cot(\alpha) + 2ikj\Big(j^2\Big(5\bar{C}^2+6\bar{C}+2\Big)+2+2\bar{C}\Big)\Bigg), \end{aligned} \tag{14}$$

$$T_{VB}(k,l) = T_{VB,num}(k,l) \,/\, T_{vel,den}(k,l), \tag{15}$$

$$T_{SC}(k,l) = \frac{-\bar{C}kj\cosh(j)}{jk(1+\bar{C})\Big(\cosh(j)(\bar{C}j\sinh(j)+\cosh(j)) + 1 + j^2(1+\bar{C})\Big) + i\cot(\alpha)\Big((Cj\sinh(j)+\cosh(j))\sinh(j) - j\Big)}, \tag{16}$$

(17)

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\begin{aligned}
T_{UC,num}(k,l) =& \bar{C}\Big(\sinh(j)\cosh(j)\Big(j^2\cot(\alpha)\Big(\bar{C}k^2-2\Big) \\
&+ ik\Big(2\bar{C}j^4(1+\bar{C}) - j^2\Big(k^2(2+\bar{C})(1+\bar{C})+4\Big)+2k^2\Big)\Big) \\
&+ j\sinh^2(j)\Big(\cot(\alpha)\Big(k^2(2+\bar{C})-2j^2\bar{C}\Big)+ik\Big(j^2(2-\bar{C}k^2(1+\bar{C}))+\bar{C}k^2\Big)\Big) \\
&+ 2j\Big(j^2\cot(\alpha)+ik\Big(j^4(1+\bar{C})^2+j^2(2+\bar{C}-k^2(1+\bar{C})^2)-k^2\Big)\Big)\Big),
\end{aligned} \tag{18}
$$

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T_{UC}(k,l) = T_{UC,num}(k,l) \,/\, T_{vel,den}(k,l), \tag{19}
$$

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\begin{aligned}
T_{VC,num}(k,l) =& -kl\bar{C}\Big(\sinh(j)\cosh(j)\Big(ik\Big(j^2(2+\bar{C})(1+\bar{C})-2\Big)-j^2\bar{C}\cot(\alpha)\Big) \\
&+ \sinh^2(j)\Big(ikj\bar{C}\Big(j^2(1+\bar{C})-1\Big)-\cot(\alpha)\Big(2+\bar{C}\Big)\Big)+2ijk\Big(j^2(1+\bar{C})^2+1\Big)\Big),
\end{aligned} \tag{20}
$$

$$
T_{VC}(k,l) = T_{VC,num}(k,l) \,/\, T_{vel,den}(k,l). \tag{21}
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