Supplementary information to: A note on the thrust of airfoils

José M. Gordillo*
Área de Mecánica de Fluidos, Departamento de Ingeniería Aeroespacial y Mecánica de Fluidos, Universidad de Sevilla, Avenida de los Descubrimientos s/n 41092, Sevilla, Spain
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Here we provide results that complement those presented in the main text.

* jgordill@us.es

I. SOLUTION OF THE INTEGRAL EQUATION (2.16) IN THE MAIN TEXT

It is explained in the main text that, in order to calculate the pressure jump across the airfoil, namely,

$$\Delta p_a(x,t) = \rho\frac{\partial \Gamma_a}{\partial t} + \rho U_\infty \frac{\partial \Gamma_a}{\partial x} = \rho \frac{\partial}{\partial t}\left(\int_0^x \gamma_a(x_0,t)\, dx_0\right) + \rho U_\infty \gamma_a(x,t) \tag{1}$$

as well as the unsteady lift and the torque,

$$\ell(t) = \int_0^c \Delta p_a(x,t)\, dx \quad \text{and} \quad m(t) = \int_0^c x\, \Delta p_a(x,t)\, dx\,, \tag{2}$$

we first need to solve the integral equation

$$\begin{aligned} w_a'(x, z=0^\pm, t) &= -\frac{dh}{dt} - U_\infty \alpha(t) - \frac{d\alpha}{dt}(x - x_e) = \\ &= \frac{1}{2\pi}\int_0^c \frac{\gamma_a(x_0,t)}{x_0 - x}\, dx_0 - \frac{1}{2\pi}\int_0^t \frac{d\Gamma_e/dt_0}{c + U_\infty(t-t_0) - x}\, dt_0\,, \end{aligned} \tag{3}$$

with

$$\Gamma_e(t) = \int_0^c \gamma_a(x,t)\, dx \tag{4}$$

indicating the circulation around the airfoil, making use of the expansion

$$\frac{\gamma_a(x,t)}{U_\infty} = A_0(t)\sqrt{\frac{1 - x/c}{x/c}} + \sum_{n=1}^{\infty} A_n(t)\sin(n\theta) = A_0(t)\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n(t)\sin(n\theta)\,, \tag{5}$$

where we have introduced the change of variables

$$\frac{x}{c} = \frac{1-\cos\theta}{2} \Rightarrow dx = \frac{c}{2}\sin\theta\, d\theta \tag{6}$$

and, therefore, $\theta = 0$ at $x = 0$ and $\theta = \pi$ at $x = c$.

Before we do so, let us first express the circulation $\Gamma_a(x,t)$, $\ell(t)$ and $m(t)$ in terms of θ and of the coefficients $A_i(t)$ making use of equations (1), (2), (5) and (6), with dots denoting, from now on, time derivatives with respect to the dimensionless time τ, defined as

$$\tau = t\frac{2U_\infty}{c} \quad \text{and} \quad \tau_0 = t_0\frac{2U_\infty}{c} \tag{7}$$

obtaining the following results:

$$\begin{aligned} \Gamma_a(x,t) &= \int_0^x \gamma_a(x_0,t)\, dx_0 = \frac{U_\infty c}{2}\int_0^\theta \left(A_0\frac{1+\cos\theta_0}{\sin\theta_0} + \sum_{n=1}^{\infty} A_n\sin(n\theta_0)\right)\sin\theta_0\, d\theta_0 = \\ &= \frac{U_\infty c}{2}\left[A_0(\theta + \sin\theta) + \sum_{n=1}^{\infty}\frac{A_n}{2}\int_0^\theta \left(\cos[(n-1)\theta_0] - \cos[(n+1)\theta_0]\right) d\theta_0\right] = \\ &= \frac{U_\infty c}{2}\left[A_0(\theta+\sin\theta) + \frac{A_1}{2}\left(\theta - \frac{\sin(2\theta)}{2}\right) + \sum_{n=2}^{\infty}\frac{A_n}{2}\left(\frac{\sin((n-1)\theta)}{n-1} - \frac{\sin((n+1)\theta)}{n+1}\right)\right]\,, \end{aligned} \tag{8}$$

$$\begin{aligned} \ell(t) &= \frac{\rho U_\infty c^2}{4}\frac{d}{dt}\int_0^\pi \sin\theta\left(A_0(\theta + \sin\theta) + \frac{A_1}{2}\theta + \frac{A_2}{2}\sin\theta\right) d\theta + \rho U_\infty \Gamma_e(t) = \\ &= \frac{\rho U_\infty^2 c\pi}{2}\left(\frac{3\dot{A}_0}{2} + \frac{\dot{A}_1}{2} + \frac{\dot{A}_2}{4} + \left(A_0 + \frac{A_1}{2}\right)\right) \quad \text{with} \quad \Gamma_e(t) = \Gamma_a(\theta = \pi, t)\,, \\ m(t) &= \rho\frac{d}{dt}\int_0^c x\,\Gamma_a\, dx + \rho U_\infty \int_0^c x\frac{\partial \Gamma_a}{\partial x}\, dx = \rho\frac{d}{dt}\int_0^c x\,\Gamma_a\, dx + \rho U_\infty \Gamma_e\, c - \rho U_\infty \int_0^c \Gamma_a\, dx \\ &= \frac{\rho U_\infty^2 c^2\pi}{4}\left(\frac{3\dot{A}_0}{2} + \frac{\dot{A}_1}{2} + \frac{\dot{A}_2}{4}\right) + \frac{\rho U_\infty^2 c^2 \pi}{4}\left(\frac{\dot{A}_0}{4} + \frac{3\dot{A}_1}{16} - \frac{\dot{A}_3}{16}\right) - \frac{\rho U_\infty^2 c^2\pi}{4}\left(\frac{3A_0}{2} + \frac{A_1}{2} + \frac{A_2}{4}\right) + \rho U_\infty \Gamma_e(t)\, c\,, \end{aligned} \tag{9}$$

and where we have taken into account that

$$\int_0^\pi \theta \sin\theta \, d\theta = \pi \,, \quad \int_0^\pi \theta \sin(2\theta) \, d\theta = -\frac{\pi}{2} \,, \quad \int_0^\pi \sin^2(n\theta) \, d\theta = \frac{\pi}{2} \,, \quad \int_0^\pi \sin(n\theta)\sin(m\theta) \, d\theta = 0 \quad \text{for} \quad n \neq m \,. \tag{10}$$

Moreover, in order to deduce equation (9), we have also taken into account that the particularization of equation (8) at $\theta = \pi$ yields the following expression for the circulation around the airfoil:

$$\Gamma_e(t) = \Gamma_a(x = c, t) = \Gamma_a(\theta = \pi, t) = \frac{\pi \, U_\infty c}{2} \left(A_0 + \frac{A_1}{2} \right) \,. \tag{11}$$

Hence, the substitution of the results in equations (5) and (11) into equation (3) provides with the following integral equation for the coefficients $A_i(t)$:

$$\begin{aligned} w'(x, z = 0^\pm, t) &= \frac{1}{2\pi} \int_0^c \frac{\gamma_a(x_0, t)}{x_0 - x} \, dx_0 - \frac{\pi \, U_\infty c}{4\pi} \int_0^t \frac{d\left(A_0 + A_1/2\right)/dt_0}{c + U_\infty \left(t - t_0\right) - x} \, dt_0 = \\ \frac{U_\infty}{2\pi} \int_0^\pi \frac{\sin\theta_0}{\cos\theta - \cos\theta_0} & \left(A_0 \frac{1 + \cos\theta_0}{\sin\theta_0} + \sum_{n=1}^\infty A_n \sin(n\theta_0) \right) d\theta_0 - \frac{U_\infty}{2} \int_0^\tau \frac{\dot{A}_0 + \dot{A}_1/2}{1 + (\tau - \tau_0) + \cos\theta} \, d\tau_0 \,, \end{aligned} \tag{12}$$

where we have made use of the definitions of the dimensionless times τ and τ_0 in equation (7).

Before solving equation (12) we first need to calculate the value of integrals of the type

$$\int_0^\pi \frac{\cos(n\theta) \, d\theta}{1 + (\tau - \tau_0) + \cos\theta} = \frac{1}{2} \int_0^{2\pi} \frac{e^{i\,n\theta}}{1 + (\tau - \tau_0) + \cos\theta} \,, \tag{13}$$

which can be easily evaluated using the calculus of residues by applying Cauchy's theorem once the integral in equation (13) is evaluated carrying out the line integral along the unit circle $z = e^{i\theta}$ in the complex plane. Indeed, introducing the change of variables

$$z = e^{i\theta} \Rightarrow dz = i e^{i\theta} \, d\theta \Rightarrow d\theta = \frac{-i \, dz}{z} \,, \quad e^{in\theta} = z^n \,, \quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z} \right) \,, \tag{14}$$

equation (13) can be written as

$$\frac{1}{2} \int_0^{2\pi} \frac{e^{i\,n\theta} \, d\theta}{1 + (\tau - \tau_0) + \cos\theta} = -i \int_0^{2\pi} \frac{z^n \, dz}{z^2 + 2\,z\left(1 + (\tau - \tau_0)\right) + 1} = \int_0^{2\pi} \frac{-i \, z^n \, dz}{(z - z_1)\,(z - z_2)} \,, \tag{15}$$

with

$$z_1 = -\left(1 + (\tau - \tau_0)\right) + \sqrt{(\tau - \tau_0)^2 + 2\,(\tau - \tau_0)} \quad \text{and} \quad z_2 = -\left(1 + (\tau - \tau_0)\right) - \sqrt{(\tau - \tau_0)^2 + 2\,(\tau - \tau_0)} \,. \tag{16}$$

Since $\tau_0 < \tau$, $|z_1| < 1$, i.e., the pole z_1 is included within the unit circle and hence, the calculus of residues yields that

$$\int_0^{2\pi} \frac{-i \, z^n \, dz}{(z - z_1)\,(z - z_2)} = 2\pi \, i \frac{-i \, z_1^n}{z_1 - z_2} = \pi \frac{\left(-B + \sqrt{B^2 - 1}\right)^n}{\sqrt{B^2 - 1}} \quad \text{with} \quad B = 1 + (\tau - \tau_0) \tag{17}$$

and, therefore,

$$\begin{aligned} n = 0 \,, \quad & \int_0^\pi \frac{d\theta}{1 + (\tau - \tau_0) + \cos\theta} = \pi \frac{1}{\sqrt{B^2 - 1}} \,, \\ n = 1 \,, \quad & \int_0^\pi \frac{\cos(\theta) d\theta}{1 + (\tau - \tau_0) + \cos\theta} = \pi \left(1 - \frac{B}{\sqrt{B^2 - 1}} \right) \,, \\ n = 2 \,, \quad & \int_0^\pi \frac{\cos(2\theta) d\theta}{1 + (\tau - \tau_0) + \cos\theta} = \pi \left(-2B + \frac{2B^2 - 1}{\sqrt{B^2 - 1}} \right) \,, \\ n = 3 \,, \quad & \int_0^\pi \frac{\cos(3\theta) d\theta}{1 + (\tau - \tau_0) + \cos\theta} = \pi \left(4B^2 - 1 - 2B\sqrt{B^2 - 1} - B \frac{2B^2 - 1}{\sqrt{B^2 - 1}} \right) \,, \\ & \text{with} \quad B = 1 + (\tau - \tau_0) \,. \end{aligned} \tag{18}$$

Let us also recall here that

$$\sin(n\theta)\sin\theta = \frac{\cos((n-1)\theta) - \cos((n+1)\theta)}{2}\,, \tag{19}$$

and also that the value of the so-called Glauert integral is:

$$I_G(n) = \int_0^\pi \frac{\cos(n\theta_0)\,d\theta_0}{\cos\theta_0 - \cos\theta} = \pi\frac{\sin(n\theta)}{\sin\theta}\,, \tag{20}$$

and, hence,

$$\frac{U_\infty}{2\pi}\int_0^\pi \frac{\sin\theta_0}{\cos\theta - \cos\theta_0}\left(A_0\frac{1+\cos\theta_0}{\sin\theta_0} + \sum_{n=1}^{\infty} A_n\sin(n\theta_0)\right)d\theta_0 = -\frac{U_\infty}{2}\left(A_0 - \sum_{n=1}^{\infty} A_n\cos(n\theta)\right). \tag{21}$$

The expressions for A_i are obtained once the result of equation (21) is introduced into equation (12) and the resulting integral equation is projected in $\cos(m\theta)$, namely:

$$\begin{aligned}\int_0^\pi w'(\theta, z=0^\pm, t)\cos(m\theta)\,d\theta = &-\frac{U_\infty}{2}\left(A_0\frac{\pi}{2}F(0,m) - A_m\frac{\pi}{2}F(n,m)\right) - \\ &-\frac{U_\infty}{2}\int_0^\tau \frac{d}{d\tau_0}\left(A_0 + \frac{A_1}{2}\right)\left(\int_0^\pi \frac{\cos(m\theta)\,d\theta}{1+(\tau-\tau_0)+\cos\theta}\right)d\tau_0\end{aligned} \tag{22}$$

where we have taken into account that

$$\int_0^\pi \cos(n\theta)\cos(m\theta)\,d\theta = \frac{\pi}{2}F(n,m) \quad \text{with} \quad F(0,0)=2\,, \quad F(n,n)=1 \quad \text{or} \quad F(n,m)=0 \quad \text{if} \quad n\neq m\,. \tag{23}$$

The expressions for $\ell(t)$ and $m(t)$ in equation (9) reveal that the aerodynamic force and torque depend on the values of the coefficients A_i, with $0 \le i \le 3$ and then, the particularization of equation (22) for $m = 0,1,2,3$ yields:

$$\begin{aligned}
&m=0\,, \quad \frac{1}{U_\infty}\left(U_\infty\alpha(t) + \frac{dh}{dt} - \left(x_e - \frac{c}{2}\right)\frac{d\alpha}{dt}\right) = \frac{A_0}{2} + \frac{I_1}{2}\\
&m=1\,, \quad \frac{c}{4U_\infty}\frac{d\alpha}{dt} = -\frac{A_0}{2} + \frac{I_1}{2} + \frac{I_2}{2}\\
&m=2\,, \quad 0 = \frac{A_2}{4} + A_0 + \frac{A_1}{2} - \frac{I_1}{2} + \int_0^\tau (\tau-\tau_0)\left(\dot A_0 + \dot A_1/2\right)d\tau_0 - \int_0^\tau \left(\dot A_0 + \dot A_1/2\right)\sqrt{B^2-1}\,d\tau_0\\
&m=3\,, \quad 0 = \frac{A_3}{4} - \frac{3}{2}\left(A_0 + \frac{A_1}{2}\right) + \frac{I_1+I_2}{2} + 2\int_0^\tau \left(\dot A_0 + \dot A_1/2\right)\sqrt{B^2-1}\,d\tau_0 + \\
&+2\int_0^\tau (\tau-\tau_0)\left(\dot A_0 + \dot A_1/2\right)\sqrt{B^2-1}\,d\tau_0 - 2\int_0^\tau (\tau-\tau_0)^2\left(\dot A_0 + \dot A_1/2\right)d\tau_0 - 4\int_0^\tau (\tau-\tau_0)\left(\dot A_0 + \dot A_1/2\right)d\tau_0\\
&\text{with} \quad I_1(\tau) = \int_0^\tau \left(\dot A_0(\tau_0) + \frac{\dot A_1(\tau_0)}{2}\right)\frac{d\tau_0}{\sqrt{B^2-1}}\,, \quad I_2(\tau) = \int_0^\tau \left(\dot A_0(\tau_0) + \frac{\dot A_1(\tau_0)}{2}\right)\frac{\tau-\tau_0}{\sqrt{B^2-1}}d\tau_0\,, \quad B = 1 + (\tau-\tau_0)\,,
\end{aligned} \tag{24}$$

where we have made use of the results in equation (18) and of

$$\begin{aligned}\int_0^\pi w'(\theta, z=0^\pm, t)\cos(m\theta)\,d\theta &= \int_0^\pi \left(-\frac{dh}{dt} - U_\infty\alpha(t) - \frac{d\alpha}{dt}\left(x - \frac{c}{2}\right) + \frac{d\alpha}{dt}\left(x_e - \frac{c}{2}\right)\right)\cos(m\theta)\,d\theta = \\ &= \left(-\frac{dh}{dt} - U_\infty\alpha(t) + \frac{d\alpha}{dt}\left(x_e - \frac{c}{2}\right)\right)\frac{\pi}{2}F(0,m) + \frac{d\alpha}{dt}\frac{c}{2}F(1,m)\frac{\pi}{2}\end{aligned} \tag{25}$$

with the values of $F(n,m)$ given in equation (23), $w'(x, z=0^\pm, t)$ given in equation (3) and where we have taken into account that

$$\int_0^\tau \left(\dot A_0(\tau_0) + \frac{\dot A_1(\tau_0)}{2}\right)d\tau_0 = A_0(\tau) + \frac{A_1}{2}(\tau) \tag{26}$$

because, at $\tau = 0$, $\Gamma_e = 0$, with Γ_e the circulation around the airfoil given in equation (11).

With the purpose of finding the expressions for $\ell(t)$ and $m(t)$ notice first that Leibniz's rule for the derivative of time-dependent integrals yields:

$$
\begin{aligned}
&\frac{d}{d\tau}\left(\int_0^\tau (\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0\right) = \int_0^\tau \left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0 = A_0 + \frac{A_1}{2}\,,\\
&\frac{d}{d\tau}\left(\int_0^\tau (\tau-\tau_0)^2\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0\right) = 2\int_0^\tau (\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0\,,\\
&\frac{d}{d\tau}\left(\int_0^\tau \left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0\right) = I_1(\tau)+I_2(\tau)\,,\\
&\frac{d}{d\tau}\left(\int_0^\tau (\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0\right) = 2\int_0^\tau \left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0 - I_2(\tau).
\end{aligned}
\tag{27}
$$

Notice also that the addition of the first and second of the equations in (24) yields that

$$
\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{3c}{4}\right)\frac{d\alpha}{dt}\right) = I_1(\tau)+\frac{I_2(\tau)}{2} \Rightarrow I_1(\tau)+\frac{I_2(\tau)}{2} = \frac{-w'(x=3c/4, z=0^\pm, t)}{U_\infty}\,,
\tag{28}
$$

where we have made use of the expression for w' given in equation (3). It will be discussed below that the solution of the integral equation (28) provides with the circulation around the airfoil $\Gamma_e(t)$. Moreover, the addition of the first and third equations in equation (24) yields,

$$
\begin{aligned}
&\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{c}{2}\right)\frac{d\alpha}{dt}\right) = \frac{3A_0}{2}+\frac{A_1}{2}+\frac{A_2}{4}+\int_0^\tau (\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0 -\\
&-\int_0^\tau \left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0 \Rightarrow\\
&\frac{c}{2U_\infty}\frac{d}{dt}\left(\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{c}{2}\right)\frac{d\alpha}{dt}\right)\right) = \frac{3\dot{A}_0}{2}+\frac{\dot{A}_1}{2}+\frac{\dot{A}_2}{4}+A_0+\frac{A_1}{2}-(I_1+I_2)
\end{aligned}
\tag{29}
$$

where we have made use of the results in equation (27).

The substitution of the results in equations (28)-(29) into the expression for $\ell(t)$ given in equation (9) provides with the following equation for the unsteady lift:

$$
\ell(t) = \frac{\rho c^2\pi}{4}\frac{d}{dt}\left(U_\infty\alpha(t)+\frac{dh}{dt}+\left(\frac{c}{2}-x_e\right)\frac{d\alpha}{dt}\right)+\frac{\rho U_\infty^2 c\,\pi}{2}\left(I_1(\tau)+I_2(\tau)\right) = \ell_a(t)+\ell_c(t)\,.
\tag{30}
$$

The result for the unsteady lift in equation (30) reveals that $\ell(t)$ results from the addition of two different terms: the first term at the right hand side of equation (30), $\ell_a(t)$, represents the contribution to the lift associated with the acceleration of the airfoil in the vertical direction; it will be termed, in what follows, as added mass term. The second term at the right hand side of equation (30), ℓ_c, is the so-called circulatory lift and represents the contribution of the wake to the lift force. For those cases in which the airfoil does not accelerate, the only lift force experienced by the airfoil is associated with the effect of the wake vortices, quantified through the integrals $I_1(\tau)$ and $I_2(\tau)$.

With the purpose of finding the expression for the time-dependent torque $m(t)$, notice first that the subtraction of the equations corresponding to $m = 1$ and $m = 3$ in (24) yields,

$$
\begin{aligned}
&A_0(\tau)+\frac{3A_1(\tau)}{4}-\frac{A_3(\tau)}{4} = \frac{c}{4U_\infty}\frac{d\alpha}{dt}+2\int_0^\tau \left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0+2\int_0^\tau (\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0 -\\
&-2\int_0^\tau (\tau-\tau_0)^2\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0 - 4\int_0^\tau (\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0 \Rightarrow \dot{A}_0(\tau)+\frac{3\dot{A}_1(\tau)}{4}-\frac{\dot{A}_3(\tau)}{4} = \frac{c^2}{8U_\infty^2}\frac{d^2\alpha}{dt^2}+\\
&+2I_1+4\int_0^\tau \left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0 - 4\int_0^\tau (\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0 - 4\left(A_0+\frac{A_1}{2}\right)\,,
\end{aligned}
\tag{31}
$$

where we have made use of the results in equation (27). In addition, equation (29) expresses that:

$$
\frac{3A_0}{2}+\frac{A_1}{2}+\frac{A_2}{4} = \frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{c}{2}\right)\frac{d\alpha}{dt}\right)-\int_0^\tau (\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0+\int_0^\tau \left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0\,.
\tag{32}
$$

The substitution of equations (31)-(32) into the equation for the torque calculated at $x=0$, $m(t)$, given in equation (9), and using the expression for $\ell(t)$ also given in equation (9) yields:

$$
\begin{aligned}
m(t) = \frac{\rho U_\infty^2 c^2 \pi}{4}\left(\frac{3\,\dot{A}_0}{2}+\frac{\dot{A}_1}{2}+\frac{\dot{A}_2}{4}\right)+\\
+\frac{\rho U_\infty^2 c^2 \pi}{16}\left(\frac{c^2}{8\,U_\infty^2}\frac{d^2\alpha}{d\,t^2}+2\,I_1+4\int_0^\tau\left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0-4\int_0^\tau(\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)\,d\tau_0-4\left(A_0+\frac{A_1}{2}\right)\right)-\\
-\frac{\rho U_\infty^2 c^2 \pi}{4}\left(\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{c}{2}\right)\frac{d\alpha}{dt}\right)-\int_0^\tau(\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)\,d\tau_0+\int_0^\tau\left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0\right)+\\
+\frac{\rho\,U_\infty^2\,c^2}{2}\left(A_0+\frac{A_1}{2}\right)=\frac{\rho U_\infty^2 c^2 \pi}{4}\left(\frac{3\,\dot{A}_0}{2}+\frac{\dot{A}_1}{2}+\frac{\dot{A}_2}{4}+A_0+\frac{A_1}{2}\right)+\frac{\rho c^4\pi}{128}\frac{d^2\alpha}{d\,t^2}+\\
+\frac{\rho U_\infty^2 c^2 \pi}{4}\left(\frac{I_1}{2}-\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{c}{2}\right)\frac{d\alpha}{dt}\right)\right)=\frac{c\,\ell(t)}{2}+\frac{\rho c^4\pi}{128}\frac{d^2\alpha}{d\,t^2}+\frac{\rho U_\infty c^3\pi}{16}\frac{d\alpha}{d\,t}-\frac{\rho U_\infty^2 c^2\pi}{8}\left(I_1+I_2\right)\,,
\end{aligned}
\tag{33}
$$

where the last term in equation (33) has been deduced adding the equations corresponding to $m=0$ and $m=1$ in equation (24). Hence, making use of the result in equation (30) for $\ell(t)$, the torque $m(t)$ can be calculated as,

$$
\begin{aligned}
m(t) &= \frac{c}{2}\times\frac{\rho c^2\,\pi}{4}\frac{d}{dt}\left(U_\infty\alpha(t)+\frac{dh}{dt}+\left(\frac{c}{2}-x_e\right)\frac{d\alpha}{dt}\right)+\frac{\rho c^4\pi}{128}\frac{d^2\alpha}{d\,t^2}+\frac{\rho U_\infty c^3\pi}{16}\frac{d\alpha}{d\,t}+\frac{\rho U_\infty^2 c^2\pi}{8}\left(I_1(\tau)+I_2(\tau)\right)\\
&= \frac{c}{2}\ell_a(t)+\frac{c}{4}\ell_c(t)+\frac{\rho c^4\pi}{128}\frac{d^2\alpha}{d\,t^2}+\frac{\rho U_\infty c^3\pi}{16}\frac{d\alpha}{d\,t}\,.
\end{aligned}
\tag{34}
$$

II. THE CIRCULATORY LIFT $\ell_c(t)$

The values of $\ell(t)$ and $m(t)$ in equations (30) and (34) depend on the value of the circulation around the airfoil $\Gamma_e(\tau)$, given by -see equations (8) and (11):

$$\Gamma_e(\tau)=\frac{\pi\,U_\infty c}{2}\left(A_0(\tau)+\frac{A_1}{2}(\tau)\right)\,. \tag{35}$$

The circulation around the airfoil, $\Gamma_e(\tau)$, is calculated solving the integral equation (28), reproduced here for clarity purposes:

$$
\begin{aligned}
&\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{3c}{4}\right)\frac{d\alpha}{dt}\right)=I_1(\tau)+\frac{I_2(\tau)}{2}\Rightarrow\\
&I_1(\tau)+\frac{I_2(\tau)}{2}=\frac{-w'(x=3c/4,z=0^\pm,t)}{U_\infty}=\frac{-w'_{3/4}}{U_\infty}\quad\text{with}\quad w'(x=3c/4,z=0^\pm,t)=w'_{3/4}\,.
\end{aligned}
\tag{36}
$$

The integrals $I_1(\tau)$ and $I_2(\tau)$ in equation (36), which depend on $\dot{\Gamma}_e(\tau)$, are defined in equation (24). We will firstly solve equation (36) for the case of a general time-dependent function $-w'_{3/4}$ making use of the fact that *any* function $F(t)$ can be expressed as:

$$F(t)=F(0)H(t)+\int_0^t\frac{dF}{d\,t'}\,H(t-t')\,d\,t'\,, \tag{37}$$

with the Heaviside function $H(\tau)$ defined as

$$H(\tau)=1\quad\text{if}\quad\tau\geq 0\quad\text{and}\quad H(\tau)=0\quad\text{if}\quad\tau<0\,, \tag{38}$$

and hence, the right hand side of equation (36) can be expressed as:

$$\frac{-w'_{3/4}}{U_\infty}(\tau)=\frac{-w'_{3/4}}{U_\infty}(0)H(\tau)-\int_0^\tau\frac{d\,w'_{3/4}/U_\infty}{d\tau_0}\,H(\tau-\tau_0)\,d\tau_0\,. \tag{39}$$

Therefore, since the integral equation (36) is linear in the unknown $\dot{A}_0(\tau)+\dot{A}_1(\tau)/2$, the addition of solutions is also a solution of equation (36). Consequently, due to the fact that equation (39) expresses that *any* function $-w'_{3/4}(\tau)$ can be expressed as a linear combination of Heaviside functions $H(\tau)$, the general solution of equation (36) can be expressed as the linear combination of the function $g_e(\tau)$ which results from the solution of equation (36) particularized for the case in which the forcing term is a Heaviside function:

$$I_{1W}+\frac{I_{2W}}{2}=\int_0^{\tau}\dot{g}_e(\tau_0)\frac{1+(\tau-\tau_0)/2}{\sqrt{(\tau-\tau_0)^2+2(\tau-\tau_0)}}\,d\tau_0=H(\tau)=1\,, \tag{40}$$

where we have made use of the definition of the integrals I_1 and I_2 in equation (24).

Once $g_e(\tau)$ with $g_e(\tau<0)=0$ is known from the solution of equation (40), we first notice that the solution of equation (40) when the right hand side is $H(\tau-\tau_0)$ is nothing but $g_e(\tau-\tau_0)$. This said, we can straightforwardly calculate the general expression of the circulatory lift. Indeed, let us first define the so-called Wagner function as:

$$\phi(\tau)=\frac{I_{1W}(\tau)+I_{2W}(\tau)}{2}=\frac{1}{2}\int_0^{\tau}\dot{g}_e(\tau_0)\frac{1+(\tau-\tau_0)}{\sqrt{(\tau-\tau_0)^2+2(\tau-\tau_0)}}\,d\tau_0\,, \tag{41}$$

which is now a known function because $g_e(\tau)$ is determined solving the integral equation (40). Next, notice that, in view of equation (39), the general solution of equation (36) can be expressed in terms of the following linear combination of the known function $g_e(\tau)$ as:

$$\begin{aligned}\dot{A}_0(\tau_1)+\frac{\dot{A}_1(\tau_1)}{2}&=\frac{-w'_{3/4}}{U_\infty}(0)\,\dot{g}_e(\tau_1)-\int_0^{\tau_1}\frac{d\,w'_{3/4}/U_\infty}{d\tau_0}\,\dot{g}_e(\tau_1-\tau_0)\,d\tau_0=\\&=\frac{-w'_{3/4}}{U_\infty}(0)\,\dot{g}_e(\tau_1)-\int_0^{\tau}\frac{d\,w'_{3/4}/U_\infty}{d\tau_0}\,\dot{g}_e(\tau_1-\tau_0)\,d\tau_0\,,\end{aligned} \tag{42}$$

where we have taken into account that, for $\tau>\tau_1$, $\dot{g}_e(\tau_1-\tau_0)=0$ when $\tau_0>\tau_1$. Therefore, since the circulatory lift in equation (30) is given by

$$\ell_c(\tau)=\frac{\rho U_\infty^2 c\pi}{2}\int_0^{\tau}\left(\dot{A}_0(\tau_1)+\frac{\dot{A}_1(\tau_1)}{2}\right)\frac{1+(\tau-\tau_1)}{\sqrt{(\tau-\tau_1)^2+2(\tau-\tau_1)}}\,d\tau_1\,, \tag{43}$$

the substitution of equation (42) into equation (43) yields,

$$\begin{aligned}&\ell_c(\tau)=\frac{\rho U_\infty^2 c\pi}{2}\times\\&\times\int_0^{\tau}\left(\frac{-w'_{3/4}}{U_\infty}(0)\,\dot{g}_e(\tau_1)-\int_0^{\tau}\frac{d\,w'_{3/4}(\tau_0)/U_\infty}{d\tau_0}\,\dot{g}_e(\tau_1-\tau_0)\,d\tau_0\right)\frac{1+\tau-\tau_1}{\sqrt{(\tau-\tau_1)^2+2(\tau-\tau_1)}}\,d\tau_1\\&=\rho\,U_\infty^2 c\pi\left(\frac{-w'_{3/4}(0)}{U_\infty}\phi(\tau)\right)-\\&-\rho U_\infty^2 c\pi\times\int_0^{\tau}\frac{d\,w'_{3/4}(\tau_0)/U_\infty}{d\tau_0}\,d\tau_0\int_0^{\tau}\frac{\dot{g}_e(\tau_1-\tau_0)}{2}\frac{1+(\tau-\tau_0-(\tau_1-\tau_0))}{\sqrt{(\tau-\tau_0-(\tau_1-\tau_0))^2+2(\tau-\tau_0-(\tau_1-\tau_0))}}\,d\tau_1\\&=\rho\,U_\infty^2 c\pi\left(\frac{-w'_{3/4}(0)}{U_\infty}\phi(\tau)\right)-\\&-\rho U_\infty^2 c\pi\times\int_0^{\tau}\frac{d\,w'_{3/4}(\tau_0)/U_\infty}{d\tau_0}\,d\tau_0\int_0^{\tau-\tau_0}\frac{\dot{g}_e(\tau'_1)}{2}\frac{1+(\tau-\tau_0-\tau'_1)}{\sqrt{(\tau-\tau_0-\tau'_1)^2+2(\tau-\tau_0-\tau'_1)}}\,d\tau'_1\,,\end{aligned} \tag{44}$$

where we have exchanged the order of integration in equation (44) by performing first the integration with respect to τ_1 and have made the change of variables $\tau'_1=\tau_1-\tau_0$ taking into account that $\dot{g}_e(\tau_1-\tau_0<0)=0$. Consequently, making use of the definition of the Wagner function in equation (41), the general expression for the circulatory lift is:

$$\ell_c(\tau)=\rho\,U_\infty^2 c\pi\left(\frac{-w'_{3/4}(0)}{U_\infty}\phi(\tau)\right)-\rho\,U_\infty^2 c\pi\int_0^{\tau}\frac{d\,w'_{3/4}(\tau_0)/U_\infty}{d\tau_0}\phi(\tau-\tau_0)\,d\tau_0\,. \tag{45}$$

Equation (45) reveals that the circulatory lift can be calculated as a convolution integral in terms of the Wagner function defined in equation (41).

Notice that, for the particular case in which a symmetric airfoil changes the angle of attack to α with $\dot{h}=0$ namely, when $-w'_{3/4}/U_\infty = \alpha\, H(\tau)$, the particularization of equation (45) yields

$$\ell_{cW}(\tau) = \rho U_\infty^2 c\pi\alpha\phi(\tau) \Rightarrow \phi(\tau) = \frac{\ell_{cW}(\tau)}{1/2\rho U_\infty^2 c2\pi\alpha} = \frac{\ell_{cW}(\tau)}{\ell_{cW}(\tau\to\infty)}\,. \tag{46}$$

Since the added mass lift is zero for $\tau>0$ when $-w'_{3/4}/U_\infty = \alpha\, H(\tau)$, the Wagner function can be calculated as the ratio between the transient lift experienced by the airfoil at the instant τ divided by the lift on the airfoil reached when the flow around the airfoil is steady: this is, in fact, the way Wagner function is calculated numerically using the code provided below. Clearly, in view of equation (46), $\phi(\tau\to\infty)=1$.

Here we will also make use of the well-known, approximate expression given by Jones (1938) [1] of the Wagner function:

$$\phi(\tau) = 1 - 0.165\, e^{-0.0455\,\tau} - 0.335\, e^{-0.3\tau}\,, \tag{47}$$

which, as it can be depicted in figure 1, is an excellent approximation to the numerical solution found by means of the numerical code, based on the vortex-lattice method, detailed below.

In the following, we will calculate the value of the Wagner function in the limit $\tau \ll 1$ in two different ways. Indeed, the so-called Wagner function obtained as the solution of Eq. (40), needs to be calculated numerically but it is also possible to find its analytical expression in the limit $\tau\ll 1$, for which:

$$\begin{aligned} &\int_0^\tau \dot{g}_e(\tau_0)\frac{1+(\tau-\tau_0)/2}{\sqrt{(\tau-\tau_0)^2+2(\tau-\tau_0)}}\,d\tau_0 = \frac{1}{\sqrt{2}}\int_0^\tau \dot{g}_e(\tau_0)\frac{\sqrt{1+(\tau-\tau_0)/2}}{\sqrt{\tau-\tau_0}}\,d\tau_0 = 1 \Rightarrow \\ &\text{for}\quad \tau\ll 1 \quad\Rightarrow\quad \frac{1}{\sqrt{2}}\int_0^\tau \dot{g}_e(\tau_0)\frac{1+(\tau-\tau_0)/4}{\sqrt{\tau-\tau_0}}\,d\tau_0 = 1 \end{aligned} \tag{48}$$

In this case, we seek for solutions of Eq. (48) of the type

$$g_e(\tau) = 2\,g_0\tau^{1/2} + \frac{2}{3}g_1\tau^{3/2} + \ldots \Rightarrow \dot{g}_e(\tau) = g_0\tau^{-1/2} + g_1\,\tau^{1/2} + \ldots\,, \tag{49}$$

and, hence, introducing the expansion for $\dot{g}_e(\tau)$ into equation (48) and taking into account that

$$\begin{aligned} &\int_0^\tau \tau_0^{-1/2}\frac{d\tau_0}{\sqrt{\tau-\tau_0}} = 2\int_0^1 \frac{d\sqrt{\tau_0/\tau}}{\sqrt{1-\tau_0/\tau}} = 2\int_0^{\pi/2}\frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \pi\,, \\ &\int_0^\tau \tau_0^{1/2}\frac{d\tau_0}{\sqrt{\tau-\tau_0}} = \tau\int_0^1 (\tau_0/\tau)^{1/2}\frac{d(\tau_0/\tau)}{\sqrt{1-\tau_0/\tau}} = \tau\int_0^{\pi/2}\sin\theta\frac{2\sin\theta\cos\theta\, d\theta}{\cos\theta} = \frac{\pi}{2}\tau\,, \\ &\int_0^\tau \tau_0^{-1/2}\sqrt{\tau-\tau_0}\,d\tau_0 = \tau\int_0^1 (\tau_0/\tau)^{-1/2}\sqrt{1-\tau_0/\tau}\,d(\tau_0/\tau) = \tau\int_0^{\pi/2}\frac{2\sin\theta\cos^2\theta\, d\theta}{\sin\theta} = \frac{\pi}{2}\tau \\ &\int_0^\tau \tau_0^{1/2}\sqrt{\tau-\tau_0}\,d\tau_0 = \tau^2\int_0^1 (\tau_0/\tau)^{1/2}\sqrt{1-\tau_0/\tau}\,d(\tau_0/\tau) = 2\tau^2\int_0^{\pi/2}\left(\frac{\sin(2\theta)}{2}\right)^2 d\theta = \frac{\pi}{8}\tau^2\,, \end{aligned} \tag{50}$$

where the values of the different integrals in Eq. (50) have been found making the change of variables $\sqrt{\tau_0/\tau}=\sin\theta$ and, then, $d\sqrt{\tau_0/\tau}=\cos\theta\, d\theta$ and $d(\tau_0/\tau) = 2\sin\theta\cos\theta\, d\theta$, we obtain the following equation for the coefficients g_0 and g_1:

$$\frac{g_0}{\sqrt{2}}\pi + \frac{g_1}{\sqrt{2}}\frac{\pi}{2}\tau + \frac{g_0}{4\sqrt{2}}\frac{\pi}{2}\tau + O(\tau^2) = 1 \Rightarrow \frac{g_0}{\sqrt{2}}\pi = 1\,,\quad \frac{g_1}{\sqrt{2}}\frac{\pi}{2}\tau + \frac{g_0}{4\sqrt{2}}\frac{\pi}{2}\tau = 0 \Rightarrow g_1 = -\frac{g_0}{4}\,. \tag{51}$$

By virtue of the definition of the Wagner function in equation (41), the integral equation (48) can be written as:

$$\begin{aligned} &\phi(\tau) + \frac{1}{2}\int_0^\tau \dot{g}_e(\tau_0)\frac{1}{\sqrt{(\tau-\tau_0)^2+2(\tau-\tau_0)}}\,d\tau_0 = 1 \Rightarrow \\ &\text{for}\quad \tau\ll 1 \quad\Rightarrow\quad \phi(\tau) + \frac{1}{2\sqrt{2}}\int_0^\tau \dot{g}_e(\tau_0)\frac{1-(\tau-\tau_0)/4}{\sqrt{\tau-\tau_0}}\,d\tau_0 = 1\,. \end{aligned} \tag{52}$$

The substitution of the values of the integrals (50) and of the values of g_0 and g_1 in equation (51) into equation (52) yields:

$$\phi(\tau \ll 1) = 1 - \frac{1}{2}\left(\frac{g_0}{\sqrt{2}}\pi + \frac{g_1}{\sqrt{2}}\frac{\pi}{2}\tau - \frac{g_0}{4\sqrt{2}}\frac{\pi}{2}\tau + O(\tau^2)\right) \simeq \frac{1}{2} + \frac{1}{8}\tau + O(\tau^2)\,. \tag{53}$$

Clearly, additional terms could be added to the analytical solution in equation (53) by taking additional terms in the expansion of $\dot{g}_e(\tau)$ in equation (49), but we prefer to calculate the value of $\phi(\tau)$ either numerically or using the approximation in equation (85).

Indeed, the comparison between the value of the Wagner function in Eq. (46), which has been calculated numerically using the vortex lattice method detailed below, the value calculated using the well-known approximate expression given by Jones (1938) [1] in Eq. (85) and the value calculated by means of the analytical expression given in Eq. (53), valid for $\tau \ll 1$, is shown in figure 1.

It is interesting to note that the expression of the circulatory lift force experienced by an airfoil whose angle of attack varies suddenly at $t=0$ i.e., $\alpha(\tau) = \alpha H(\tau)$ can be easily calculated at $\tau \to 0^+$ using the alternative procedure described next, which makes use of the result in Appendix A of the main text, reproduced here for clarity purposes. Indeed, the fact that the value of the circulation at $\tau = 0^+$ namely, right after the angle of attack has changed to a value $\alpha \neq 0$, is $\Gamma_e(\tau = 0^+) = 0$, means that the flow around the airfoil is symmetric with respect to $x = c/2$. The potential flow which satisfies this requirement corresponds to the one generated by a distribution of vortices with a circulation per unit length given by

$$\gamma(x) = U_\infty A_0'\left(\sqrt{\frac{1-x/c}{x/c}} - \sqrt{\frac{x/c}{1-x/c}}\right) = U_\infty\, A_0'\frac{1-2x/c}{\sqrt{x/c\,(1-x/c)}},\,. \tag{54}$$

Indeed, notice that Eq. (54) expresses a symmetric distribution of γ around $x = c/2$ and also that the flow turns around both the leading and trailing edges of the airfoil.

The value of A_0' in Eq. (54) is determined by imposing the linearized impenetrability condition namely,

$$w' = -U_\infty\,\alpha = \frac{U_\infty\,A_0'}{2\pi}\int_0^c \frac{1-2x_0/c}{\sqrt{x/c\,(1-x/c)}}\frac{dx_0}{x_0 - x} = \frac{U_\infty\,A_0'}{2\pi}\int_0^\pi \frac{\sin\theta_0}{2}\frac{2\cos\theta_0}{\sin\theta_0}\frac{d\theta_0}{1/2\,(\cos\theta - \cos\theta_0)} = \frac{-2\pi\,U_\infty\,A_0'}{2\pi}\,, \tag{55}$$

where we have made use of the value of the Glauert integral (20) and, consequently, the perturbed potential satisfying the condition that the circulation around the airfoil is zero is given by, see Eq. (54),

$$\gamma(x) = U_\infty\alpha\,\frac{1-2x/c}{\sqrt{x/c\,(1-x/c)}}\,. \tag{56}$$

However, the solution expressed by Eq. (56) does not satisfy the Kutta condition because $\gamma(x = c) \to \infty$ as a consequence of the fact that, as it was explained above, the solution in Eq. (56) the flow is symmetric around $x = c/2$ and, hence, the flow turns around both the leading and trailing edges, a fact implying that the Kutta condition is not fulfilled by the potential flow generated by the distribution of vortices with $\gamma(x)$ given by Eq. (56). Then, how is it possible to fulfill at the same time the following two conditions namely, a zero initial circulation, which implies a symmetric flow around the airfoil and also the Kutta condition? The solution to this apparent paradox is the following: it is possible to comply with both conditions when we seek for a symmetric potential flow around an airfoil with a length increasing in time as $dc/dt = U_\infty$: in this case, the flow turns around the leading edge of the airfoil *but not the trailing edge*. Indeed, whereas the trailing edge is located at any instant of time at $x = c$, the potential flow generated by the distribution of $\gamma(x,t)$ given by

$$\gamma_a(x,t) = U_\infty\alpha\,\frac{1-2x/c(t)}{\sqrt{x/c(t)\,(1-x/c(t))}} \quad \text{with} \quad \frac{dc}{dt} = U_\infty\,. \tag{57}$$

corresponds to a potential flow which turns around $x = c + U_\infty\,dt$ namely, a potential flow which turns around the *stating vortex*, which is located downstream the leading edge and it is transported with a velocity U_∞.

Hence,

$$\ell_c(t=0^+) = \int_0^c \Delta\,p_a(x,t)\,dx \quad \text{with} \quad \Delta\,p_a(x,t) = \rho\frac{\partial\Gamma_a}{\partial t} + \rho U_\infty\frac{\partial\Gamma_a}{\partial x} \Rightarrow \ell(t=0^+) = \rho\frac{d}{dt}\int_0^c \Gamma_a\,dx \tag{58}$$

because

$$\int_0^c \frac{\partial \Gamma_a}{\partial x} dx = \Gamma_a(x=c) - \Gamma_a(x=0) = 0 \tag{59}$$

for the potential flow generated by the symmetric distribution given in Eq. (57). Now, notice that

$$\Gamma_a(x,t) = \int_0^x \gamma_a dx = c(t) \int_0^{x/c(t)} U_\infty \alpha \, \frac{1 - 2x/c(t)}{\sqrt{x/c(t)\,(1 - x/c(t))}} d\left(x/c(t)\right) = 2\, U_\infty \alpha\, c(t) \sqrt{x/c(t) - (x/c(t))^2} \tag{60}$$

with $dc(t)/dt = U_\infty$ and, therefore, the substitution of the result in Eq. (60) in Eq. (58) yields,

$$\begin{aligned}
\ell_c(t=0^+) &= 2\rho\, U_\infty\, \alpha \frac{d}{dt}\left(c^2(t) \int_0^{x/c(t)} \sqrt{x/c(t)\,(1 - x/c(t))}\; d\left(x/c(t)\right)\right) = 2\rho\, U_\infty\, \alpha \frac{d}{dt}\left(c^2(t) \int_0^\pi \sqrt{\frac{1-\cos^2\theta}{4}}\; d\theta \right) = \\
&= 2\rho\, U_\infty\, \alpha \frac{d}{dt}\left(c^2(t) \int_0^\pi \frac{\sin^2\theta}{4} d\theta \right) = 2\rho\, U_\infty\, \alpha \frac{d}{dt}\left(c^2(t) \int_0^\pi \frac{1-\cos(2\theta)}{8} d\theta \right) = 2\rho\, U_\infty\, \alpha \frac{d}{dt}\left(\frac{\pi\, c^2(t)}{8} \right) = \\
&= \frac{\pi}{2}\rho U_\infty \frac{dc}{dt} c(t = 0+)\alpha = \frac{\pi}{2} \alpha \rho U_\infty^2 c \,,
\end{aligned} \tag{61}$$

where we have made use of the change of variables $x/c(t) = \left(1 - \cos\theta\right)/2 \to d(x/c(t)) = 1/2 \sin\theta\, d\theta$. The result in Eq. (61) shows that the initial lift around an airfoil which experiences a sudden change of the angle of attack is, indeed, one-half the lift force corresponding to steady flow.

One of the advantages of using this alternative way of finding the value of $\ell(t=0^+)$, is that it reveals the idea that, in order to satisfy Kutta's condition, the potential flow needs to turn around, not at the trailing edge but at the *starting vortex*, which is convected downstream at a velocity U_∞. Notice also that the local distribution of $\gamma(x,t)$ around the starting vortex is identical to that at the leading edge of the airfoil.

III. EQUATION FOR $A_0(t)$

The subtraction of the first two equations in (24) yields,

$$A_0 - \frac{I_2}{2} = \frac{-w'_{3/4}}{U_\infty} - \frac{c}{2\,U_\infty}\frac{d\alpha}{d\,t} \Rightarrow A_0(t) = \frac{-w'_{3/4}}{U_\infty} - \frac{c}{2\,U_\infty}\frac{d\alpha}{d\,t} + \frac{I_2}{2}\,. \tag{62}$$

In order to determine $I_2/2$ in equation (62), we make use of equation (36),

$$I_1 + \frac{I_2}{2} = \frac{-w'_{3/4}}{U_\infty} \Rightarrow I_1 = \frac{-w'_{3/4}}{U_\infty} - \frac{I_2}{2} \Rightarrow \ell_c(t) = \frac{\rho U_\infty^2 c\pi}{2}\left(\frac{-w'_{3/4}}{U_\infty} + \frac{I_2}{2}\right) = \frac{\rho U_\infty^2 c\pi}{2}\left(A_0(t) + \frac{c}{2U_\infty}\frac{d\alpha}{dt}\right)\,, \tag{63}$$

where we have made use of equation (62) and we have made use of the definition of the circulatory lift, $\ell_c(t)$ in equation (30), namely

$$\ell_c(t) = \frac{\rho U_\infty^2 c\pi}{2}\left(I_1 + I_2\right)\,, \tag{64}$$

with $\ell_c(t)$ calculated by means of equation (45) as a function of $-w'_{3/4}/U_\infty$ and as a function of the Wagner function defined in equation (41), which can be approximated using equation (85). Consequently, making use of the results in equation (63), the equation for A_0 is

$$A_0(t) = \left(\frac{\rho U_\infty^2 c\pi}{2}\right)^{-1} \ell_c(t) - \frac{c}{2U_\infty}\frac{d\alpha}{d\,t} = 2\left(\frac{\ell_c(t)}{1/2\rho U_\infty^2 c 2\pi\alpha(t)}\alpha(t) - \frac{c}{4U_\infty}\frac{d\alpha}{dt}\right) \tag{65}$$

and, therefore, the suction force at the leading edge is given by

$$-\rho U_\infty^2 \pi\, c \frac{A_0^2(t)}{4} = -\rho U_\infty^2 \pi c \left(\frac{\ell_c(t)}{1/2\rho U_\infty^2 c 2\pi\alpha(t)}\alpha(t) - \frac{c}{4U_\infty}\frac{d\alpha}{dt}\right)^2\,, \tag{66}$$

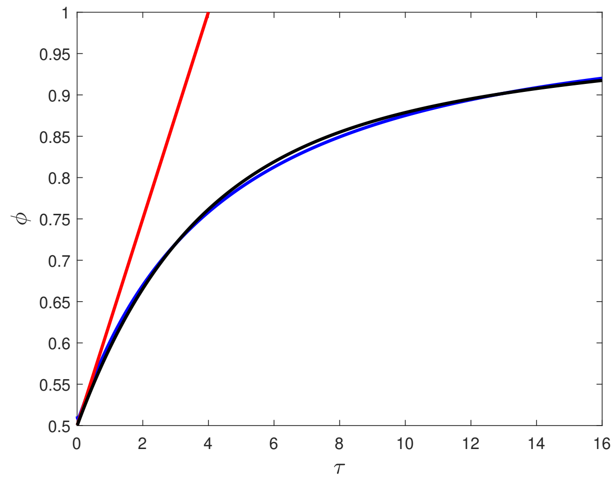

FIG. 1. Wagner function calculated in three different ways: numerically, using the vortex-lattice code detailed below (blue), using the analytical solution, valid for $\tau \ll 1$ given in equation (53) (red) or using the classical approximation by R.T. Jones [1], reproduced in equation (85) (black).

with $\ell_c(t)$ calculated through the Duhamel integral in equation (45).

For the particular case in which $-w'_{3/4}/U_\infty = \alpha\, H(t)$, equation (45) reduces to

$$\ell_c(\tau) = \rho U_\infty^2 c\pi\alpha\phi(\tau) \tag{67}$$

and therefore, the substitution of equation (67) into equations (65) and (66) yields

$$A_0(\tau) = 2\alpha\phi(\tau)\,, \quad d(t) = \ell(t)\alpha(t) - \rho U_\infty^2 \pi c \left(\frac{A_0}{2}\right)^2 = \rho U_\infty^2 c\pi\alpha^2\phi(\tau) - \rho U_\infty^2 \pi\, c\phi^2(\tau)\alpha^2 = \rho U_\infty^2 c\pi\alpha^2 \left(\phi(\tau) - \phi^2(\tau)\right)\,. \tag{68}$$

Due to the fact that $\phi(\tau \to \infty) \to 1$, equation (68) recovers the well known result that the drag force around an airfoil in a potential, steady flow, is zero.

Figure 6 in the main text compares the suction force predicted by equation (68) with the values of the suction force obtained by means of the vortex impulse theory, calculated numerically using the vortex-lattice code detailed below.

IV. MATLAB CODES FOR: WAGNER PROBLEM AND FOR THE CASE OF AN OSCILLATING PLUNGING PLATE, THE MEAN THRUST OF OSCILLATING AIRFOILS CALCULATED USING THE VORTEX LATTICE METHOD AND THE MEAN THRUST CALCULATED USING EQUATIONS (4.20)-(4.22) IN THE MAIN TEXT.

A. Vortex Lattice method for unsteady flows: numerical values of the Wagner function and of the suction force at the leading edge

Equation (3) can be written as

$$\begin{aligned} w(x,t) &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{c+U_\infty t}\frac{\gamma(x_0,t)}{x_0-x+\epsilon}\,dx_0 = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{c+U_\infty t}\frac{\partial}{\partial x_0}\left(\frac{\Gamma(x_0,t)}{x_0-x+\epsilon}\right)+\frac{\Gamma(x_0,t)}{\left(x_0-x+\epsilon\right)^2}\,dx_0 = \\ &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{c+U_\infty t}\frac{\Gamma(x_0,t)}{\left(x_0-x+\epsilon\right)^2}\,dx_0\,, \end{aligned} \tag{69}$$

where we have made use of the fact that $\Gamma(x=0) = \Gamma(x = c + U_\infty t) = 0$. Next, we divide both the airfoil and the wake in N panels of identical width h, bounded by x_{0i} and $x_{0i} + h$ where Γ is constant and equal to the value at the midpoint of the panel, Γ_i. Hence, the integral in (69) can be approximated as

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{c+U_\infty t}\frac{\Gamma(x_0,t)}{\left(x_0-x+\epsilon\right)^2}\,dx_0 = \frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i(t)\left(\frac{1}{x_{0i}-x}-\frac{1}{x_{0i}+h-x}\right)\,. \tag{70}$$

The number of unknowns is the number of panels at the airfoil, N_p. In order to calculate the N_p values of Γ at the airfoil at the instant τ, the values of x in equation (70) are particularized at the midpoints of the N_p panels namely, $x = x_j$, with j varying from 1 to N_p, where the perturbed vertical velocity is known, $w'(x_j, t) = w'_a(x_j, t)$. The values of Γ at the wake panels can be expressed as a function of $\Gamma(x = c, t) = \Gamma_{i=Np}(t)$ because, by virtue of the Euler Bernouilli equation,

$$\frac{\partial \Gamma}{\partial t} + U_\infty \frac{\partial \Gamma}{\partial x} = 0 \Rightarrow \Gamma(x = c + U_\infty (t - t_0), t) = \Gamma(x = c, t = t_0) = \Gamma_{i=Np}(t_0) . \tag{71}$$

The numerical code detailed below compares the values of

$$\Delta t_G = \frac{T_G(t) + \alpha(t)\ell(t)}{\rho U_\infty^2 c} = \pi \, \frac{A_0^2(t)}{4} \tag{72}$$

and of

$$\begin{aligned} \Delta t_{VI} &= \frac{T_{VI}(t) + \alpha(t)\ell(t))}{\rho U_\infty^2 c} = \\ &= \frac{1}{\rho U_\infty^2 c} \times \left[\rho U_\infty^2 c\pi \left(\frac{C}{2}\right)^2 - \rho \int_0^c \gamma_a w'_a \, dx - \rho \int_c^{c+U_\infty t} \gamma_w w'_w \, dx \right] = \\ &= \frac{1}{\rho U_\infty^2 c} \times \left[\rho U_\infty^2 c\pi \left(\frac{C}{2}\right)^2 - \frac{\rho}{2\pi} \int_0^{c+U_\infty t} \gamma_{a,w}(x,t) dx \int_0^{c+U_\infty t} \frac{\gamma_{a,w}(x_0,t)}{x_0 - x} \, dx_0 \right] , \end{aligned} \tag{73}$$

where $\gamma_{a,w}(x,t) = \partial \Gamma_{a,w}/\partial x$ is calculated using second-order finite differences.

Notice also that, for the case of Wagner problem in the main text, the value of $A_0(\tau)$ in (72) has been calculated by means of the analytical expression given in equation (68), see also the numerical code below, with $\phi(\tau)$ calculated using the classical approximation by R.T. Jones [1], given in equation (85).

For the case of plunging airfoils the suction force at the leading edge given in equation (72) is calculated numerically once the value of $A_0(\tau)$ is determined through equation (2.17) in the main text, which expresses the perturbed potential in the neighborhood of the leading edge is:

$$2\phi' = \Gamma = 2\, U_\infty c A_0(t) \, (r/c)^{1/2} \, \cos{(\beta/2)} \ . \tag{74}$$

Indeed, using the result in equation (74),

$$\frac{\Gamma(x/c = 3h/4)}{U_\infty c} = A_0(\tau)\sqrt{3h} \, , \tag{75}$$

with h indicating the width of the panels in the numerical method and $\Gamma(x/c = 3h/4)$ denoting the numerical value of the circulation at the point located the closest to the leading edge.

We also provide below the numerical codes used to calculate the mean thrust for the case of airfoils oscillating periodically using: i) the vortex lattice method -in this code we have made use of equation (2.22) in the main text for the case of Garrick's mean thrust and of equations (4.1) and (4.3) in the main text for the mean thrust calculated using the vortex impulse theory- and ii) the theoretical values given by the original Garrick's theory or our prediction in equation (4-20). This latter code also includes the calculation of the mean thrust coefficient deduced by Fernández-Feria as well as the exact and approximate values of Theodorsen's function as a function of the reduced frequency, see sections V and VI below.

Section V below solves Theodorsen's problem and section VI solves the so-called flutter problem using the simplified Theodorsen functions deduced in section V.

```
% JM GORDILLO, UNSTEADY VORTEX-LATTICE: WAGNER PROBLEM, PLUNGING AIRFOIL, H=
   ALPHA*COS(OMEGA*T) OR H=ALPHA*SIN(OMEGA*T)
%
clear all; close all; clc;
%
N=200;
h=1/N;

x(1:N)=h/4+((1:N)-1)*h;
```

```
x0(1:N)=3*h/4+((1:N)-1)*h;
alpha=5*pi/180;

for j=1:N
    for i=1:N
        R(j,i)=-1/(x0(j)-x(i))+1/(x0(j)-(x(i)+h));
    end
end

R=R/(2*pi);
Rinv=inv(R);

dtau=2*h;

Nsteps=20/dtau;

Clunsteadym1=0;
Cmunsteadym1=0;
sumphinant=0;
Gamma1start=0;

COS=1;
WAGNER=0;
XE=0;
DOTALPHA=0; %W'=-DOTH-ALPHA-DOTALPHA(X-XE)

for I=0:Nsteps
    for j=1:N
        if I>0
            tau=I*dtau;
            tv(I)=tau;
            if(WAGNER==1)
                Westela=-alpha; %WAGNER PROBLEM: AIRFOIL SUDDENLY CHANGES THE
                    ANGLE OF ATTACK
            else
                if(COS==1)
                    Westela=-alpha*cos(2*tv(I)); %OSCILLATING PLUNGING AIRFOIL
                        SUDDENLY SET INTO MOTION
                else
                    Westela=-alpha*sin(2*tv(I));  %OSCILLATING PLUNGING AIRFOIL
                         SMOOTHLY SET INTO MOTION
                end
            end

           b(j)=Westela;
            for i=1:I
                b(j)=b(j)+(1/(2*pi))*Gammaestela(i)*(1/(1+0.25*h+0.5*dtau*(I-i
                    +1)-x0(j))-1/(1+0.25*h+0.5*dtau*(I-i)-x0(j)));
            end
        else
            if(WAGNER==1 || COS==1)
               b(j)=-alpha;
            else
               b=zeros();
            end
        end
    end
```

```
gammai=Rinv*b';

Gammaestela(I+1)=gammai(N);

Clunsteady=0;
for j=1:N-1
    Clunsteady=Clunsteady+h*gammai(j);
end
Clunsteady=Clunsteady+gammai(N)*3*h/4;

Cmunsteady=gammai(1)*h*h*9/40;
for j=2:N-1
    Cmunsteady=Cmunsteady+h*gammai(j)*x0(j);
end
Cmunsteady=Cmunsteady+3*h/4*x0(N)*gammai(N);

Cl=2*(Clunsteady-Clunsteadym1)/dtau+gammai(N);
Cm=2*(Cmunsteady-Cmunsteadym1)/dtau+gammai(N)-Clunsteady;

Clunsteadym1=Clunsteady;
Cmunsteadym1=Cmunsteady;

if I>0
    Clv(I)=Cl/(pi*alpha);
    A0(I)=gammai(1)/sqrt(3*h);
    Clteor1(I)=1-0.165*exp(-0.0455*tau)-0.335*exp(-0.3*tau);
    A0th(I)=2*alpha*Clteor1(I);

    % CONTRIBUTION TO THE THRUST BY THE SUCTION AT THE LEADING EDGE:
    % CASE CORRESPONDING TO THE DIRECT INTEGRATION OF PRESSURE
    % DISTRIBUTION:

    if(WAGNER==1)
        destelateor(I)=pi*A0th(I)*A0th(I)/4;
    else
        destelateor(I)=pi*A0(I)*A0(I)/4;
    end
end

% VORTEX IMPULSE THEORY:
% NEXT, WE CALCULATE THE VERTICAL VELOCITY AT THE AIRFOIL AND THE
% WAKE AND ALSO THE CONTRIBUTION TO THE THRUST FORCE OF THE VORTICES
% EXTENDING ALONG THE AIRFOIL AND THE WAKE.

if I>0
    xT(1:N+I)=h/4+((1:N+I)-1)*h;
    x0T(1:N+I)=3*h/4+((1:N+I)-1)*h;
    GammaT(1:N+I)=zeros();

     wT(1:N+I)=zeros();
     for i=1:N+I
         if i<N+1
             GammaT(i)=gammai(i);
         else
             GammaT(i)=Gammaestela(1+I+N-i);
         end
     end
```

```
        for j=1:I+N
            for i=1:N+I
                wT(j)=wT(j)+(1/(2*pi))*GammaT(i)*(1/(xT(i)-x0T(j))-1/(xT(i)+h-
                    x0T(j)));
            end
        end

        % IN THE CASE W'WAKE=0, THIS WOULD BE THE ONLY TERM IN THE VORTEX
        % IMPULSE THEORY
        sumphin=Westela*GammaT(N)+DOTALPHA*Clunsteady;

        % HOWEVER, DUE TO THE FACT THAT THE VERTICAL VELOCITIES ALONG
        % THE WAKE ARE DIFFERENT FROM ZERO, IT IS NECESSARY TO ADD THE
        % CONTRIBUTION OF THE FORCE EXRTED BY THE VORTICIES IN THE WAKE

        sumphin=sumphin+0.5*wT(N)*(GammaT(N+1)-GammaT(N-1))*0.5;
        for i=N+1:N+I-1
            sumphin=sumphin+wT(i)*(GammaT(i+1)-GammaT(i-1))*0.5;
        end
        sumphin=sumphin+wT(I+N)*(3*GammaT(I+N)-4*GammaT(I+N-1)+GammaT(I+N-2))
            *0.5;


        % FOR THE CASES IN WHICH THE AIRFOIL IS SUDDENLY SET INTO MOTION
        % WE NEED TO ADD THE CONTRIBUTION OF THE STARTING VORTEX

        sumphinOK=sumphin-pi*alpha^2/4;

        % FINALLY, THE CONTRIBUTION TO THE THRUST FORCE OF THE VORTICES
        % ALONG THE AIRFOIL AND THE WAKE
        % CALCULATED USING THE VORTEX IMPULSE THEORY IS:

        destelaNOOK(I)=-sumphin;
        destelaOK(I)=-sumphinOK;
    end
end


figure
plot(x0T,GammaT,'-','linewidth',2,'Color','b');
figure
plot(x0T,wT,'-','linewidth',2,'Color','r');


figure
plot(tv,destelateor,'-','linewidth',2,'Color','b');
hold on;
if(COS==1 || WAGNER==1)
    plot(tv,destelaOK,'-','linewidth',2,'Color','black');
    hold on
end
plot(tv,destelaNOOK,'-','linewidth',2,'Color','r');


% WAGNER FUNCTION

if (WAGNER==1)
```

```
    figure
    plot(tv,Clv);
    hold on;
    plot(tv,Clteor1);
end



% JM GORDILLO, UNSTEADY VORTEX-LATTICE: MEAN THRUST CORRESPONDING TO GARRICK'S
   PREDICTION IN %EQUATION (4.21) AND TO THE PREDICTION IN EQUATION (4.20),
   DEDUCED FROM (4.1). FOR THE CASE OF %EQUATION (4.20), DEDUCED USING THE
   VORTEX-IMPULSE THEORY, WE DO NOT INCLUDE THE CONTRIBUTION OF %THE STARTING
   VORTEX AND, HENCE, THIS CONTRIBUTION IS NOT INCLUDED IN THE PRESENT NUMERICAL
    CODE.
% THIS CONTRIBUTION CAN BE EASILY ADDED USING EQUATION (2.36) IN THE MAIN TEXT
   WITH THE VALUE OF C % DEDUCED IN APPENDIX A OF THE MAIN TEXT.

clear all; close all; clc;
%
N=200;
h=1/N;

x(1:N)=h/4+((1:N)-1)*h;

x0(1:N)=3*h/4+((1:N)-1)*h;
alpha0=6*pi/180;
h0=6*pi/180;
phi=pi/2;
a=-1/2;

for j=1:N
    for i=1:N
        R(j,i)=-1/(x0(j)-x(i))+1/(x0(j)-(x(i)+h));
    end
end

R=R/(2*pi);
Rinv=inv(R);

dtau=2*h;

CTmean=zeros(1:12);
CTmeanG=zeros(1:12);
vecK=zeros(1:12);

Clunsteadym1=0;
Cmunsteadym1=0;
sumphinant=0;
Gamma1start=0;

COS=0;
WAGNER=0;
XE=0.5*(1+a); %XE=(1+a)c/2
DOTALPHA=0; %W'=-DOTH-ALPHA-DOTALPHA(X-XE)

for contK=1:12

    K=0.5*contK;
```

```
vecK(contK)=K;

Nsteps=12*pi/(K*dtau);
DeltaT=10*pi/K;
Tmean=0;
TmeanG=0;

for I=0:Nsteps

    for j=1:N
    if I>0
        tau=I*dtau;
        tv(I)=tau;
        if(WAGNER==1)
            Westela=-alpha0; %WAGNER PROBLEM: AIRFOIL SUDDENLY CHANGES THE
                ANGLE OF ATTACK
        else
            if(COS==1)
                Westela=-alpha0*cos(2*tv(I)); %OSCILLATING PLUNGING AIRFOIL
                    SUDDENLY SET INTO MOTION
            else
                Westela=-alpha0*sin(2*tv(I));   %OSCILLATING PLUNGING
                    AIRFOIL SMOOTHLY SET INTO MOTION
            end
        end


        dhdt(I)=-2*h0*K*sin(K*tv(I));
        alpha(I)=alpha0*cos(K*tv(I)+phi);
        dalpha(I)=-2*alpha0*K*sin(K*tv(I)+phi);

        b(j)=-dhdt(I)-dalpha(I)*(x0(j)-XE)-alpha(I);


        for i=1:I
            b(j)=b(j)+(1/(2*pi))*Gammaestela(i)*(1/(1+0.25*h+0.5*dtau*(I-i
                +1)-x0(j))-1/(1+0.25*h+0.5*dtau*(I-i)-x0(j)));
        end
    else
        if(WAGNER==1 || COS==1)
           b(j)=-alpha;
        else
           b=zeros();
        end
    end
end

gammai=Rinv*b';

Gammaestela(I+1)=gammai(N);

Clunsteady=0;
for j=1:N-1
    Clunsteady=Clunsteady+h*gammai(j);
end
Clunsteady=Clunsteady+gammai(N)*3*h/4;

Cmunsteady=gammai(1)*h*h*9/40;
```

```
    for j=2:N-1
        Cmunsteady=Cmunsteady+h*gammai(j)*x0(j);
    end
    Cmunsteady=Cmunsteady+3*h/4*x0(N)*gammai(N);

    Cl=2*(Clunsteady-Clunsteadym1)/dtau+gammai(N);
    Cm=2*(Cmunsteady-Cmunsteadym1)/dtau+gammai(N)-Clunsteady;

    Clunsteadym1=Clunsteady;
    Cmunsteadym1=Cmunsteady;

    if I>0
        Clv(I)=Cl/(pi*alpha0);
        A0(I)=gammai(1)/sqrt(3*h);
        Clteor1(I)=1-0.165*exp(-0.0455*tau)-0.335*exp(-0.3*tau);
        A0th(I)=2*alpha0*Clteor1(I);

        % CONTRIBUTION TO THE THRUST BY THE SUCTION AT THE LEADING EDGE:
        % CASE CORRESPONDING TO THE DIRECT INTEGRATION OF PRESSURE
        % DISTRIBUTION:

        if(WAGNER==1)
            destelateor(I)=pi*A0th(I)*A0th(I)/4-alpha(I)*Cl;
        else
            destelateor(I)=pi*A0(I)*A0(I)/2-2*alpha(I)*Cl;
        end
    end

   % MEAN THRUST CALCULATED BY MEANS OF GARRICK'S THEORY OR
   % USING THE VORTEX IMPULSE THEORY FOR THE CASE IN WHICH VORTICES IN THE WAKE
   % ARE CONVECTED WITH THE FREE STREAM VELOCITY

    if I>0
         Westela=-dhdt(I)-dalpha(I)*(1-XE)-alpha(I);
         sumphin=-2*(alpha(I)*Cl+Westela*gammai(N)+dalpha(I)*Clunsteady);
         destelaOK(I)=sumphin;
         if tv(I)>2*pi/K
             Tmean=Tmean+dtau*sumphin/DeltaT;
             TmeanG=TmeanG+dtau*destelateor(I)/DeltaT;
         end
    end
    end
    CTmean(contK)=Tmean;
    CTmeanG(contK)=TmeanG;
    CTmean(contK)
    vecK(contK)

end

plot(vecK(1:12),CTmean(1:12),'-.g','linewidth',4);
hold on;
plot(vecK(1:12),CTmeanG(1:12),'-.y','linewidth',4);


% JM Gordillo: Theodorsen functions F and G, approximate Theodorsen
% functions Fw and Gw, new functions Fg and Gg, our thrust coefficient CT
% calculated by means of the vortex impulse theory using equation (4.20),
% FF's thrust coefficient CTFF calculated using equation (4.22) and
```

```
% thrust coefficient using Garrick's theory, CTG, using equation (4.21)
%
%

clear all;
close all;
clc;


i=sqrt(-1);
Nsteps=600;
Deltak=0.01;
a0=6*pi/180;
h0c=6*pi/180;
phi=pi/2;
a=-1/2;
%h0c=0;
%a0=0;


for s=1:Nsteps
    K(s)=s*Deltak;
    F(s)=real(besselk(1,i*K(s))/(besselk(0,i*K(s))+besselk(1,i*K(s))));
    G(s)=imag(besselk(1,i*K(s))/(besselk(0,i*K(s))+besselk(1,i*K(s))));
    Gg(s)=real((exp(-i*K(s))/(i*K(s)))/(besselk(0,i*K(s))+besselk(1,i*K(s))));
    Fg(s)=-imag((exp(-i*K(s))/(i*K(s)))/(besselk(0,i*K(s))+besselk(1,i*K(s))));
    Fw(s)=real(1-0.165*i*K(s)/(i*K(s)+0.0455)-0.335*i*K(s)/(0.3+i*K(s)));
    Gw(s)=imag(1-0.165*i*K(s)/(i*K(s)+0.0455)-0.335*i*K(s)/(0.3+i*K(s)));
    Termino2(s)=(Gg(s)*sin(phi)-K(s)*Fg(s)*sin(phi)/2-Fg(s)*cos(phi));
    CT(s)=4*pi*K(s)*K(s)*Gg(s)*(h0c^2+cos(phi)*a0*h0c*(3/4-a)+a0^2*(1-a)*(1/2-a)
        /4)+2*pi*K(s)*a0*h0c*Termino2(s)-pi*Fg(s)*K(s)*a0*a0*(1-a);
    TerminoFF(s)=(F(s)-Gg(s)-K(s)*Fg(s)/2);
    Diferencia(s)=-2*pi*TerminoFF(s)*(a0^2+2*a0*h0c*K(s)*sin(phi));
    CTG(s)=4*pi*K(s)*K(s)*h0c^2*(F(s)*F(s)+G(s)*G(s))+pi*a0*a0*((F(s)*F(s)+G(s)*
        G(s))*(1+K(s)*K(s)*(1/2-a)^2)+(a-1/2)*(F(s)-1/2)*K(s)*K(s)-(a+1/2)*K(s)*G
        (s)-F(s));
    CTG(s)=CTG(s)+pi*a0*h0c*(4*K(s)*(F(s)*F(s)+G(s)*G(s))*sin(phi)+4*K(s)*K(s)
        *(1/2-a)*(F(s)*F(s)+G(s)*G(s))*cos(phi)-2*K(s)*K(s)*(G(s)*sin(phi)+F(s)*
        cos(phi))+2*K(s)*(G(s)*cos(phi)-F(s)*sin(phi))+K(s)*K(s)*cos(phi));
end

figure
semilogy(K(1:Nsteps),CT(1:Nsteps),'-r','linewidth',3)
hold on
plot(K(1:Nsteps),CT(1:Nsteps)+Diferencia(1:Nsteps),'-.k','linewidth',3)
hold on
plot(K(1:Nsteps),CTG(1:Nsteps),'-b','linewidth',3)

grid on;
```

V. THEODORSEN'S FUNCTION

In order to calculate the aerodynamic force and torque experienced by an airfoil oscillating periodically, we seek for the real parts of the time-dependent, complex functions $h(\tau)$, $\alpha(\tau)$ and also of the time derivative of the circulation around the airfoil,

$$h(\tau) = H\, e^{i\omega\tau}\,, \quad \alpha(\tau) = \bar{\alpha}\, e^{i\omega\tau}\,, \quad \dot{A}_0 + \dot{A}_1/2 = G\, e^{i\omega\tau} \tag{76}$$

with $\omega \equiv k$ the dimensionless frequency, also named reduced frequency, k, in the main text, which is related with the dimensional frequency ω^* through the equation

$$\omega\tau = \omega^* t \Rightarrow \omega\, t \frac{2\,U_\infty}{c} = \omega^*\, t \Rightarrow \omega \equiv k = \frac{\omega^* c}{2\,U_\infty}\ . \tag{77}$$

In this case, the substitution of $\dot{\Gamma}_e$ in Eq. (76) into the integral equation (28) provides with the following equation for G:

$$\begin{aligned}
&I_1 + \frac{I_2}{2} = G \int_0^\tau \frac{e^{i\omega\tau_0}\,(1 + 1/2\,(\tau-\tau_0))}{\sqrt{(\tau-\tau_0)^2 + 2\,(\tau-\tau_0)}}\, d\tau_0 = \frac{-\bar{w'}_{3/4}}{U_\infty}\, e^{i\omega\tau} \Rightarrow \\
&G\, e^{i\omega\tau} \int_0^\tau \frac{(1 + 1/2\,(\tau-\tau_0))}{\sqrt{(\tau-\tau_0)^2 + 2\,(\tau-\tau_0)}}\, e^{-i\omega(\tau-\tau_0)}\, d\tau_0 = \frac{-\bar{w'}_{3/4}}{U_\infty}\, e^{i\omega\tau} \Rightarrow \\
&G = \frac{-\bar{w'}_{3/4}}{U_\infty} \left(\int_0^\tau \frac{(1 + 1/2\,(\tau-\tau_0))}{\sqrt{(\tau-\tau_0)^2 + 2\,(\tau-\tau_0)}}\, e^{-i\omega(\tau-\tau_0)}\, d\tau_0 \right)^{-1},
\end{aligned} \tag{78}$$

where

$$\frac{1}{U_\infty}\left(U_\infty \alpha(t) + \frac{dh}{dt} - \left(x_e - \frac{3c}{4}\right)\frac{d\alpha}{dt}\right) = \frac{-w'_{3/4}}{U_\infty}(\tau) \quad \text{and} \quad \frac{-w'_{3/4}}{U_\infty}(\tau) = e^{i\omega\tau}\frac{-\bar{w'}_{3/4}}{U_\infty}\,. \tag{79}$$

We are looking for solutions corresponding to dimensionless values of τ large enough so that the solution is not affected by initial conditions and, hence, since τ is such that $1+\tau \to \infty$, we introduce the change of variables $\xi = \tau - \tau_0 + 1$ into Eq. (78) for G, which now reads:

$$G = \frac{-\bar{w'}_{3/4}}{U_\infty}\left(\frac{1}{2}\int_1^\infty \frac{(1+\xi)}{\sqrt{\xi^2-1}}\, e^{-i\omega(\xi-1)}\, d\xi\right)^{-1}. \tag{80}$$

Now that G is known as a function of $-\overline{w'_{3/4}}/U_\infty$ through Eq. (80), the circulatory component of the lift force in the limit $\tau + 1 \to \infty$ can be calculated in terms of $I_1 + I_2$ using Eq. (30):

$$\begin{aligned}
&(I_1 + I_2) = e^{i\omega\tau}\, G \int_0^\tau \frac{1 + \tau - \tau_0}{\sqrt{(\tau-\tau_0)^2 + 2\,(\tau-\tau_0)}}\, e^{-i\omega(\tau-\tau_0)}\, d\tau_0 = e^{i\omega\tau}\, G \int_1^\infty \frac{\xi}{\sqrt{\xi^2-1}}\, e^{-i\omega(\xi-1)}\, d\xi = \\
&= e^{i\omega\tau}\frac{-\bar{w'}_{3/4}}{U_\infty}\left(\int_1^\infty \frac{\xi}{\sqrt{\xi^2-1}}\, e^{-i\omega\,\xi}\, d\xi\right)\left(\frac{1}{2}\int_1^\infty \frac{(1+\xi)}{\sqrt{\xi^2-1}}\, e^{-i\omega\xi}\, d\xi\right)^{-1} = \\
&= \frac{-w'_{3/4}}{U_\infty}\left(\int_1^\infty \frac{\xi}{\sqrt{\xi^2-1}}\, e^{-i\omega\,\xi}\, d\xi\right)\left(\frac{1}{2}\int_1^\infty \frac{(1+\xi)}{\sqrt{\xi^2-1}}\, e^{-i\omega\xi}\, d\xi\right)^{-1},
\end{aligned} \tag{81}$$

where we have made use of the equation for G in Eq. (80) and of Eq. (79). Then, introducing Eq. (81) into Eq. (30), the lift force and the torque calculated at $x = 0$ for the case of oscillatory motion are given by the real parts of -see Eqs. (30) and (34):

$$\begin{aligned}
&\ell(t) = \frac{\rho c^2\pi}{4}\frac{d}{dt}\left(U_\infty\alpha(t) + \frac{dh}{dt} + \left(\frac{c}{2} - x_e\right)\frac{d\alpha}{dt}\right) + \rho U_\infty c\,\pi\left(-w'_{3/4}\right) C(\omega) = \ell_a(t) + \ell_c(t) \\
&m(t) = \frac{c}{2}\ell_a(t) + \frac{c}{4}\ell_c(t) + \frac{\rho c^4\pi}{128}\frac{d^2\alpha}{d\,t^2} + \frac{\rho U_\infty c^3\pi}{16}\frac{d\alpha}{d\,t},
\end{aligned} \tag{82}$$

with

$$\ell_c(t) = \rho U_\infty c\,\pi\left(-w'_{3/4}\right) C(\omega) \quad \text{and} \quad C(\omega) = \left(\int_1^\infty \frac{\xi}{\sqrt{\xi^2-1}}\, e^{-i\omega\,\xi}\, d\xi\right)\left(\int_1^\infty \frac{(1+\xi)}{\sqrt{\xi^2-1}}\, e^{-i\omega\xi}\, d\xi\right)^{-1}, \tag{83}$$

the so-called Theodorsen function, which can be expressed in terms of modified Bessel functions of the second kind. Indeed, using the integral form of the K_n-Bessel function of order n, Theodorsen's function, defined in Eq. (83), can be expressed as:

$$C(\omega) = \frac{K_1(i\omega)}{K_0(i\omega) + K_1(i\omega)} \,. \tag{84}$$

Theodorsen's function, which arises as the contribution of the wake to the force and torque, can also be expressed in terms of Wagner's function $\phi(\tau)$ using the Duhamel's integral in Eq. (45). Indeed, the numerical solution of Wagner's problem, depicted in figure 1, reveals that $\phi(\tau)$ can be well approximated by:

$$\phi(\tau) = 1 - 0.165\, e^{-0.0455\,\tau} - 0.335\, e^{-0.3\tau} \,, \tag{85}$$

and, hence, by virtue of Eq. (45), the circulatory lift $\ell_c(t)$, which can be calculated in the limit $\tau \to \infty$ taking $-w'_{3/4}/U_\infty = Ce^{i\omega\tau}$, is given by

$$\begin{aligned}
\ell_c(\tau) &= -\rho\, U_\infty c\pi\, w'_{3/4}(0)\phi(\tau) - \rho\, U_\infty^2 c\pi \int_0^\tau \frac{d\, w'_{3/4}(\tau_0)/U_\infty}{d\tau_0} \phi(\tau - \tau_0)\, d\tau_0 = \\
&= -\rho\, U_\infty c\pi\, w'_{3/4}(0)\phi(\tau) - \rho\, U_\infty c\pi\, w'_{3/4} \int_0^\tau i\omega\, e^{-i\omega(\tau-\tau_0)}\, \phi(\tau - \tau_0)\, d\tau_0 \simeq \\
&\simeq -\rho\, U_\infty c\pi\, w'_{3/4}(0)\phi(\tau) - \rho\, U_\infty c\pi\, w'_{3/4} \int_0^\tau i\omega\, e^{-i\omega(\tau-\tau_0)} \left(1 - 0.165\, e^{-0.0455\,(\tau-\tau_0)} - 0.335\, e^{-0.3(\tau-\tau_0)}\right) d\tau_0 \\
&= -\rho\, U_\infty c\pi\, w'_{3/4}(0)\phi(\tau) - \rho\, U_\infty c\pi\, w'_{3/4}(0) e^{i\omega\tau} \int_0^\tau i\omega\, e^{-i\omega\tau'} \left(1 - 0.165\, e^{-0.0455\tau'} - 0.335\, e^{-0.3\tau'}\right) d\tau' \,,
\end{aligned} \tag{86}$$

where we have made use of Eq. (85). Then, since $\phi(\tau \to \infty) \to 1$, see Eq. (85),

$$\begin{aligned}
\ell_c(\tau \to \infty) &= -\rho\, U_\infty c\pi\, w'_{3/4}(0) - \rho\, U_\infty c\pi\, w'_{3/4}(0) \left(-1 + e^{i\omega\tau} - \frac{0.165\, i\omega\, e^{i\omega\tau}}{i\omega + 0.0455} - \frac{0.335\, i\omega\, e^{i\omega\tau}}{i\omega + 0.3}\right) = \\
&= -\rho\, U_\infty c\pi\, w'_{3/4} \left(1 - \frac{0.165\, i\omega}{i\omega + 0.0455} - \frac{0.3355\, i\omega}{i\omega + 0.3}\right) .
\end{aligned} \tag{87}$$

Therefore, the comparison between Eq. (82) and Eq. (87) indicates that Theodorsen's function, given in Eq. (84) can be approximated by

$$C(\omega) = \frac{K_1(i\omega)}{K_0(i\omega) + K_1(i\omega)} = F(\omega) + iG(\omega) \simeq 1 - \frac{0.165\, i\omega}{i\omega + 0.0455} - \frac{0.335\, i\omega}{i\omega + 0.3} \,, \tag{88}$$

which is an excellent approximation to the exact result, as it is shown in figure 2, where the real and imaginary parts of Theodorsen's function, $C(\omega)$ in equation (88), are compared with the approximate expression, also given in equation (88).

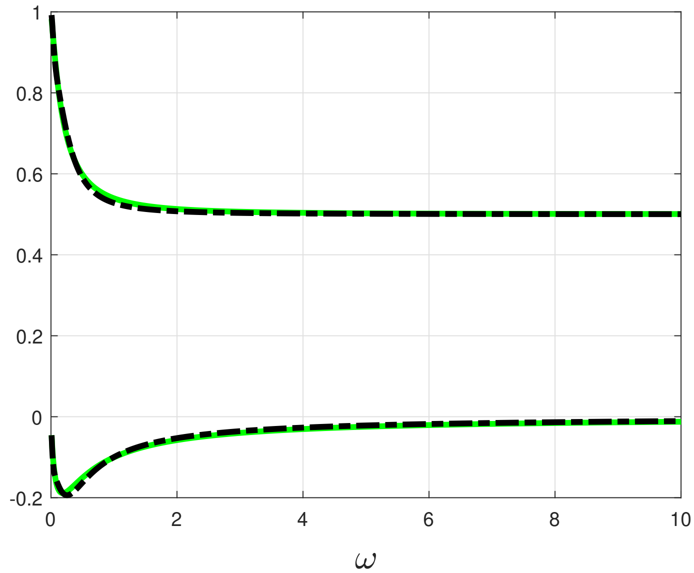

FIG. 2. The real and imaginary parts of Theodorsen's function, $C(\omega)$, plotted using green lines, are compared with the corresponding approximate expressions, plotted using black dashed lines, defined in equation (88).

VI. FLUTTER: THE RESULTS PRESENTED IN THIS SECTION DO NOT AFFECT *AT ALL* TO ANY OF THE RESULTS IN THE MAIN TEXT OR IN THE REST OF THIS SUPPLEMENTARY MATERIAL. IT IS ADDED HERE ONLY FOR THE PURPOSE OF COMPLETENESS.

This section is devoted to determine the conditions under which small perturbations on the vertical distance $h(t)$ and on the angle of attack $\alpha(t)$ of the airfoil depicted in figure 1 of the main text, whose elastic axis and center of mass are respectively located at the distances x_e and x_g from the leading edge respectively, either grow or decay in time. For this purpose, we first write the two equations characterizing the time evolution of the two degrees of freedom, which result from projecting the force and momentum balances over the unit cartesian vectors, finding that:

$$\begin{aligned} &m\left(\ddot{h}+\ddot{\alpha}\,(x_g-x_e)\right)=-\ell(t)-k_h\,h(t)\Rightarrow m\left(\ddot{h}+\ddot{\alpha}\,(x_g-x_e)\right)+k_h\,h(t)+\ell(t)=0\\ &I_e\ddot{\alpha}+m\ddot{h}\,(x_g-x_e)=-m_e(t)-k_\alpha\alpha(t)\Rightarrow I_e\ddot{\alpha}+m\ddot{h}\,(x_g-x_e)+k_\alpha\alpha(t)+m_e(t)=0\,, \end{aligned} \tag{89}$$

where m refers to the mass per unit length of the airfoil, $I_e=I_g+m(x_e-x_g)^2$ is the moment of inertia of the airfoil calculated at x_e, with I_g indicating the moment of inertia at the center of mass, k_h and k_α indicate the elastic constants corresponding to the vertical and angular deflections, $\ell(t)$ is the aerodynamic lift force and $m_e(t)$ refers to the aerodynamic torque calculated at x_e. Using the following definitions for the semi-chord, for the dimensionless distance a, and for the dimensionless distance x_α

$$b=\frac{c}{2}\,,\quad x_e=b\,(1+a)\,,\quad x_g-x_e=bx_\alpha \tag{90}$$

equations (30) and (34) read

$$m_e(t)=m(t)-x_e\ell(t)=b\ell_a(t)+\frac{b}{2}\ell_c(t)-b((1+a)\,(\ell_a(t)+\ell_c(t))+\frac{\rho b^4\pi}{8}\frac{d^2\alpha}{dt^2}+\frac{\rho U_\infty b^3\pi}{2}\frac{d\alpha}{dt}\,, \tag{91}$$

with

$$\ell(t)=\rho b^2\,\pi\frac{d}{dt}\left(U_\infty\alpha(t)+\frac{dh}{dt}+(b-b(1+a))\,\frac{d\alpha}{dt}\right)+\ell_c(t)=\rho b^2\,\pi\left(U_\infty\dot{\alpha}+\ddot{h}-ba\ddot{\alpha}\right)+\ell_c(t)=\ell_a(t)+\ell_c(t) \tag{92}$$

and, therefore, the substitution of Eq. (92) into Eq. (91) yields

$$\begin{aligned} m_e(t) &= \rho b^3 \pi a \left(-U_\infty \dot{\alpha} - \ddot{h} + b a \ddot{\alpha}\right) - b\left(\frac{1}{2}+a\right)\ell_c(t) + \frac{\rho b^4 \pi}{8}\frac{d^2\alpha}{d\,t^2} + \frac{\rho U_\infty b^3 \pi}{2}\frac{d\alpha}{d\,t} = \\ &= -\rho b^3 \pi a \ddot{h} + \rho b^4 \pi \left(\frac{1}{8}+a^2\right)\ddot{\alpha} - \rho b^3 \pi\, U_\infty \left(a - \frac{1}{2}\right)\dot{\alpha} - b\left(\frac{1}{2}+a\right)\ell_c(t) \end{aligned} \tag{93}$$

Equations (92)-(93) reveal that both the lift and the torque depend on the time-dependent variables $\alpha(t)$ and $h(t)$. Consequently, the substitution of Eqs. (92)-(93) into the two linear ordinary differential equations in Eq. (89) indicates that the resulting system for the two unknowns $h(t)$ and $\alpha(t)$, is homogeneous. Then, in order to determine whether perturbations grow in time or not, we just seek for the real parts of solutions of the type

$$\alpha(\tau) = \bar{\alpha} e^{i\omega\tau}\,,\ \ h(\tau) = b\bar{h} e^{i\omega\tau} \tag{94}$$

which implies

$$\left(\dot{h},\dot{\alpha}\right) = \frac{U_\infty}{b}\left(\frac{dh}{d\tau},\frac{d\alpha}{d\tau}\right) = i\omega\frac{U_\infty}{b}\left(b\bar{h},\bar{\alpha}\right)e^{i\omega\tau} \quad \text{and} \quad \left(\ddot{h},\ddot{\alpha}\right) = \left(\frac{U_\infty}{b}\right)^2\left(\frac{d^2h}{d\tau^2},\frac{d^2\alpha}{d\tau^2}\right) = -\omega^2\left(\frac{U_\infty}{b}\right)^2\left(b\bar{h},\bar{\alpha}\right)e^{i\omega\tau}\,. \tag{95}$$

Moreover, the type of solutions (94) imply that the circulatory lift is given by -see Eq. (83)

$$\begin{aligned} \ell_c(t) &= 2\pi\rho U_\infty b\left(-w'_{3/4}\right)C(\omega) = 2\pi\rho U_\infty b\left(\dot{h} + \alpha U_\infty + \dot{\alpha}\left(\frac{3}{2}b - b(1+a)\right)\right)C(\omega) = \\ &= 2\pi\rho U_\infty b\left(\dot{h} + \alpha U_\infty + \dot{\alpha} b\left(\frac{1}{2}-a)\right)\right)C(\omega)\,, \end{aligned} \tag{96}$$

The substitution of the type of harmonic solutions (94)-(95) into the linear system of ODEs (89) and into the equations describing both $m_e(t)$ and $\ell(t)$, given by Eqs. (92)-(93) and (96) yields:

$$\begin{aligned} & m\left(-\frac{U_\infty^2}{b}\omega^2\bar{h} - \frac{U_\infty^2}{b}x_\alpha\omega^2\bar{\alpha}\right) + k_h b\bar{h} + \rho b^2\pi\left(\frac{U_\infty^2}{b}i\omega\bar{\alpha} - \frac{U_\infty^2}{b}\omega^2\bar{h} + \frac{U_\infty^2}{b}\,a\,\omega^2\bar{\alpha}\right) + \\ & + 2\pi\rho U_\infty b\left(U_\infty i\omega\bar{h} + U_\infty\bar{\alpha} + U_\infty i\omega\left(\frac{1}{2}-a\right)\bar{\alpha}\right)\ C(\omega) = 0 \\ & - I_e\frac{U_\infty^2}{b^2}\omega^2\bar{\alpha} - mU_\infty^2 x_\alpha\omega^2\bar{h} + k_\alpha\bar{\alpha} + \rho b^2\pi a\,U_\infty^2\,\omega^2\bar{h} - \rho b^2\pi U_\infty^2\left(\frac{1}{8}+a^2\right)\omega^2\bar{\alpha} - \rho b^2\pi U_\infty^2 i\omega\left(a-\frac{1}{2}\right)\bar{\alpha} - \\ & - 2\pi\rho U_\infty b^2\left(\frac{1}{2}+a\right)\left(U_\infty i\omega\bar{h} + U_\infty\bar{\alpha} + U_\infty i\omega\left(\frac{1}{2}-a\right)\bar{\alpha}\right)\ C(\omega) = 0 \end{aligned} \tag{97}$$

Grouping terms, the system (97) for the two unknowns, $\bar{h}$ and $\bar{\alpha}$, depends on the following dimensionless parameters,

$$\bar{m} = \frac{m}{\rho\pi b^2}\,,\quad \bar{\Omega}^2 = \frac{k_h b^2}{m\,U_\infty^2}\,,\quad \bar{I} = \frac{I_e}{\rho\pi b^4}\,,\quad \bar{k} = \frac{k_\alpha}{b^2\,k_h} \tag{98}$$

and reads

$$\begin{aligned} & \left((\bar{m}+1)\,\omega^2 - 2i\omega C(\omega) - \bar{m}\bar{\Omega}^2\right)\bar{h} + \left((\bar{m}\,x_\alpha - a)\,\omega^2 - i\omega - 2\left(1 + i\omega\left(\frac{1}{2}-a\right)\right)C(\omega)\right)\bar{\alpha} = 0 \\ & \left((\bar{m}\,x_\alpha - a)\,\omega^2 + 2\left(\frac{1}{2}+a\right)i\omega\,C(\omega)\right)\bar{h} + \\ & + \left(\left(\bar{I} + \frac{1}{8} + a^2\right)\omega^2 + i\omega\left(a - \frac{1}{2}\right) - \bar{k}\bar{m}\bar{\Omega}^2 + 2\left(\frac{1}{2}+a\right)C(\omega) + 2i\omega\left(\frac{1}{4}-a^2\right)C(\omega)\right)\bar{\alpha} = 0\,. \end{aligned} \tag{99}$$

Notice that the system (99) will possess a solution different from the trivial one, $\bar{h} = \bar{\alpha} = 0$, only for certain values of $\omega(a, x_\alpha, \bar{m}, \bar{I}, \bar{\Omega}^2, \bar{k})$, with these values constituting the eigenvalues of the system. In order to simplify as much as possible the determination of such eigenvalues, we will make use here of the approximate expression of the Theodorsen function given in Eq. (88). Then, if the two algebraic equations in (99) are multiplied by $(0.0455 + i\omega)(0.3 + i\omega)$ and we define the function

$$C'(\omega) = (0.0455 + i\omega)(0.3 + i\omega)\,C(\omega) = (0.0455 + i\omega)(0.3 + i\omega) - 0.165 i\omega\,(0.3 + i\omega) - 0.335 i\omega\,(0.0455 + i\omega)\ , \tag{100}$$

the solution of the system (99) will be different from the trivial one only for those values of ω satisfying the equation

$$\begin{vmatrix} A & C \\ D & B \end{vmatrix} = 0 \tag{101}$$

namely, for values of ω satisfying the equation

$$AB - CD = 0 \tag{102}$$

where

$$\begin{aligned}
A &= (0.0455 + i\omega)(0.3 + i\omega)\left((\bar{m} + 1)\,\omega^2 - \bar{m}\bar{\Omega}^2\right) - 2i\omega C'(\omega) \\
B &= (0.0455 + i\omega)(0.3 + i\omega)\left(\left(\bar{I} + \frac{1}{8} + a^2\right)\omega^2 + i\omega\left(a - \frac{1}{2}\right) - \bar{k}\bar{m}\bar{\Omega}^2\right) + 2\left(\frac{1}{2} + a\right)\left(1 + i\omega\left(\frac{1}{2} - a\right)\right)C'(\omega) \\
C &= (0.0455 + i\omega)(0.3 + i\omega)\left((\bar{m}\,x_\alpha - a)\,\omega^2 - i\omega\right) - 2\left(1 + i\omega\left(\frac{1}{2} - a\right)\right)C'(\omega) \\
D &= (0.0455 + i\omega)(0.3 + i\omega)\left((\bar{m}\,x_\alpha - a)\,\omega^2\right) + 2\left(\frac{1}{2} + a\right)i\omega\,C'(\omega)\,.
\end{aligned} \tag{103}$$

The solution of equation (102) will provide with eight different values of ω namely, with eight different eigenvalues, for a given set of values of the dimensionless parameters $(a, x_\alpha, \bar{m}, \bar{I}, \bar{\Omega}^2, \bar{k})$. The phenomenon known as flutter will take place when any of these eight eigenvalues possesses a negative imaginary part with a real part different from zero. Those eigenvalues with a real part equal to zero and with a negative imaginary part will correspond to conditions for which the incident velocity exceeds the so-called divergence velocity.

Notice that, if the imaginary part of an eigenvalue is negative, the small perturbations on h and α will grow exponentially in time due to the fact that the value of any of such eigenvalues can be written as

$$\omega = \omega_r + i\omega_i \tag{104}$$

and, by virtue of Eq. (94)

$$\frac{h(\tau)}{b} = \Re\left(\bar{h}e^{i\omega_r\tau}\right)\,e^{-i\omega_i\tau} \quad \text{and} \quad \alpha(\tau) = \Re\left(\bar{\alpha}e^{i\omega_r\tau}\right)\,e^{-i\omega_i\tau}\,. \tag{105}$$

In order to determine the minimum value of U_∞ for which an airfoil flutters -or reaches the divergence velocity- for a given set of dimensionless parameters $(a, x_\alpha, \bar{m}, \bar{I}, \bar{k})$ we will proceed as follows: equation (102)-(103) is solved for a very large value of $\bar{\Omega}^2$, which can be viewed as the value of the dimensionless frequency corresponding to a very small value of U_∞. If all eight eigenvalues possess positive imaginary parts, then, Eq. (102)-(103) is solved for decreasing values of $\bar{\Omega}^2$ until at least one of the eigenvalues possesses an imaginary part equal to zero for a critical value of $\bar{\Omega}^2$, which we will denote here as ${\bar{\Omega}^*}^2 = {\bar{\Omega}^*}^2(a, x_\alpha, \bar{m}, \bar{I}, \bar{k})$. The critical flutter velocity is then determined as:

$$\bar{\Omega}^{*2} = \frac{k_h b^2}{m\,U_\infty^{*2}} \Rightarrow U_\infty^* = b\sqrt{\frac{k_h}{m\bar{\Omega}^{*2}}} \tag{106}$$

whereas the flutter frequency is calculated from

$$\omega^*_{flutter} = \omega^*_r \frac{U^*_\infty}{b}\ , \tag{107}$$

where ω^*_r indicates the real part of the first eigenvalue whose imaginary part is zero.

The numerical code provided below calculates the eight eigenvalues for a given set of dimensionless control parameters.

A. Eigenvalues

```
% JM GORDILLO, ALGEBRAIC EQUATION TO SEEK FOR THE EIGENVALUES IN THE FLUTTER
    PROBLEM
%
clear all; close all; clc;
%
% DEFINITION OF THE DIMENSIONAL PARAMETERS AND VARIABLES
% a xalpha madim=m/(\rho \pi b^2) Iadim=Ie/(\rho \pi b^4) kadim=kalpha/(b^2
% kh) Fh=kh b^2/(m U^2_\infty)

syms a xalpha madim Iadim kadim Fh k A B C D Ck F G Gb

kcrit=zeros(1,8);
Fhcrit=zeros(1,8);

Ck=(i*k+0.0455)*(i*k+0.3)-0.165*i*k*(0.3+i*k)-0.335*i*k*(i*k+0.0455);
A=(i*k+0.0455)*(i*k+0.3)*(madim*(k^2-Fh)+k^2)-2*i*k*Ck;
B=(i*k+0.0455)*(i*k+0.3)*(-Iadim*k^2+madim*kadim*Fh-(0.125+a^2)*k^2-(a-0.5)*i*k)
    -2*(0.5+a)*(1+i*k*(0.5-a))*Ck;
C=(i*k+0.0455)*(i*k+0.3)*(madim*k^2*xalpha-i*k-a*k^2)-2*(1+i*k*(0.5-a))*Ck;
D=(i*k+0.0455)*(i*k+0.3)*(-madim*xalpha*k^2+a*k^2)-2*(0.5+a)*i*k*Ck;

F=A*B-C*D;

% EXAMPLE: SPECIFIC VALUES OF THE %CONTROL PARAMETERS

Fnum=0.5;
b=0.5;
rho=1;
Fmin=0.5674*b^2/(1.5708);
ap=-0.47;
xalphap=0.4;
mp=1.5708/(rho*pi*b^2);
Ip=0.08/(rho*pi*b^4);
kp=1/(0.5674*b^2);

%UD=\sqrt(kalpha/(\rho b^2 2\pi %(1/2+a)): DIMENSIONLESS FREQUENCY %
    CORRESPONDING TO THE DIVERGENCE %VELOCITY


FhD=(1+2*ap)/(mp*kp);

%Ratio OF DIMENSIONLESS FREQUENCIES, R=kh Ie/(m kalpha)=Iadim/(madim
%kadim)

R=Ip/(mp*kp);

%PARAMETER INDICATING WHETHER THE %DIVERGENCE VELOCITY IS SMALLER OR %LARGER
    THAN THE FLUTTER VELOCITY
%

FhD

true=1;

while true
```

```
    Fnum=Fnum-0.005
    FhD


    G=expand(subs(F,[a xalpha madim Iadim kadim Fh],[ap xalphap mp Ip kp Fnum]))
        ;
    p=coeffs(G);
    p = fliplr(p);
    r=roots(p);
    s=double(r);

    plot(real(r),imag(r),'o')
    axis([-2 2 -0.05 0.5])
    grid on
    pause(0.01)
    hold on

    cont=0;
    for i=1:8
        if imag(s(i))<0
            cont=cont+1;
            kcrit(cont)=s(i);
            Fhcrit(cont)=Fnum;
            true=0;
        end
    end
end

for i=1:cont
    kcrit(i)
    Fhcrit(i)
end
```

[1] R. T. Jones, NACA TN 667 , 347– (1938).