Supplementary Materials for "Physical significance of artificial numerical noise in direct numerical simulation of turbulence"

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This supplemental information provides some key measures of the flow and some figures involved in the main text.

1 Key measures of the flow

For the sake of simplicity, the definitions of some statistic operators are briefly described below. The spatial average is defined by

$$\langle \ \rangle_A = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} dxdy, \tag{S1}$$

the temporal average is defined by

$$\langle \ \rangle_t = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} dt, \tag{S2}$$

and the spatiotemporal average is defined by

$$\langle \ \rangle = \frac{1}{4\pi^2(T_2 - T_1)}\int_0^{2\pi}\int_0^{2\pi}\int_{T_1}^{T_2} dxdydt, \tag{S3}$$

respectively, where $T_1 = 100$ and $T_2 = 300$ are chosen in the main text for an interval of time corresponding to a relatively stable state of the turbulent flow.

For the turbulent two-dimensional Kolmogorov flow considered in this paper, vorticity is given by the stream function

$$\omega(x,y,t) = \nabla^2\psi(x,y,t). \tag{S4}$$

We also focus on the kinetic energy

$$E(x,y,t) = \frac{1}{2}[u^2(x,y,t) + v^2(x,y,t)], \tag{S5}$$

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enstrophy

$$\Omega(x,y,t) = \frac{1}{2}\,\omega^2, \tag{S6}$$

the kinetic energy dissipation rate

$$D(x,y,t) = \frac{1}{2Re}\sum_{i,j=1,2}\left[\partial_i u_j(x,y,t) + \partial_j u_i(x,y,t)\right]^2, \tag{S7}$$

and enstrophy dissipation rate

$$D_\Omega(x,y,t) = \frac{1}{Re}|\nabla\omega|^2, \tag{S8}$$

where $u_1(x,y,t) = u(x,y,t)$, $u_2(x,y,t) = v(x,y,t)$, $\partial_1 = \partial/\partial x$, and $\partial_2 = \partial/\partial y$.

The stream function can be expanded as the Fourier series

$$\psi(x,y,t) \approx \sum_{m=-\lfloor N/3\rfloor}^{\lfloor N/3\rfloor}\ \sum_{n=-\lfloor N/3\rfloor}^{\lfloor N/3\rfloor} \Psi_{m,n}(t)\exp(\mathbf{i}\,mx)\exp(\mathbf{i}\,ny), \tag{S9}$$

where m, n are integers, $\lfloor\ \rfloor$ stands for a floor function, $\mathbf{i} = \sqrt{-1}$ denotes the imaginary unit, and for dealiasing $\Psi_{m,n} = 0$ is imposed for wavenumbers outside the above domain $\sum$. Note that for the real number ψ, $\Psi_{-m,-n} = \Psi^*_{m,n}$ must be satisfied, where $\Psi^*_{m,n}$ is the conjugate of $\Psi_{m,n}$. Therefore, the kinetic energy spectrum is defined as

$$E_k(t) = \sum_{k-1/2\le\sqrt{m^2+n^2}<k+1/2} \frac{1}{2}\,(m^2+n^2)\mid\Psi_{m,n}(t)\mid^2, \tag{S10}$$

where the wave number k is a non-negative integer. Noth that, if the stream function ψ is obtained via the difference between two velocity fields, such as $\Delta\mathbf{u} = \mathbf{u}_{\mathrm{CNS}^*} - \mathbf{u}_{\mathrm{DNS}}$, the corresponding kinetic energy spectrum is denoted by $E_\Delta(k,t)$.

Filter-Space-Technique (FST) is employed in this investigation to extract the scale-to-scale energy and enstrophy fluxes, denoted as $\Pi_E^{[l]}$ and $\Pi_Z^{[l]}$ (see definitions below), respectively. FST, initially developed for large eddy simulation in the 1970s [1], involves applying a low-pass filter to the velocity field. Mathematically, it is expressed as:

$$f^{[l]}(\mathbf{x},t) = \int G^{[l]}(\mathbf{x}-\mathbf{x}')f(\mathbf{x}',t)d\mathbf{x}', \tag{S11}$$

where f represents u or v for the two-dimensional velocity field, $\mathbf{x} = (x,y)$ denotes the coordinate vector, and $G^{[l]}$ is chosen to be a round Gaussian filter for the scale l [2–4]. For the incompressible Navier-Stokes equations, scale-to-scale energy and enstrophy fluxes can be derived analytically as:

$$\Pi_E^{[l]} = -\sum_{i,j=1,2}\left[(u_iu_j)^{[l]} - u_i^{[l]}u_j^{[l]}\right]\frac{\partial u_i^{[l]}}{\partial x_j}, \tag{S12}$$

$$\Pi_Z^{[l]} = -\sum_{i=1,2} \left[(u_i\omega)^{[l]} - u_i^{[l]}\omega^{[l]} \right] \frac{\partial \omega^{[l]}}{\partial x_i}, \tag{S13}$$

respectively. Note that the sign of $\Pi_E^{[l]}$ or $\Pi_Z^{[l]}$ reveals the direction of energy or enstrophy transfer: a positive value indicates a cascade from the larger scale ($> l$) to the smaller scale ($< l$), i.e. the direct cascade, while a negative value signifies the reverse, i.e. the inverse cascade.

2 Figures

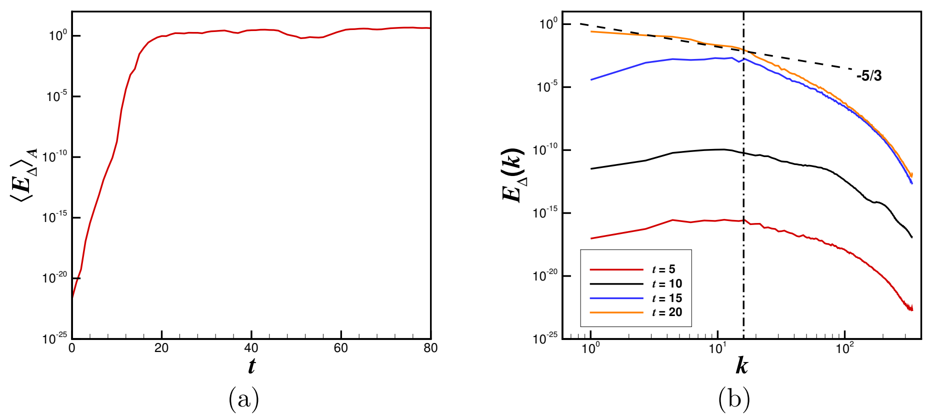
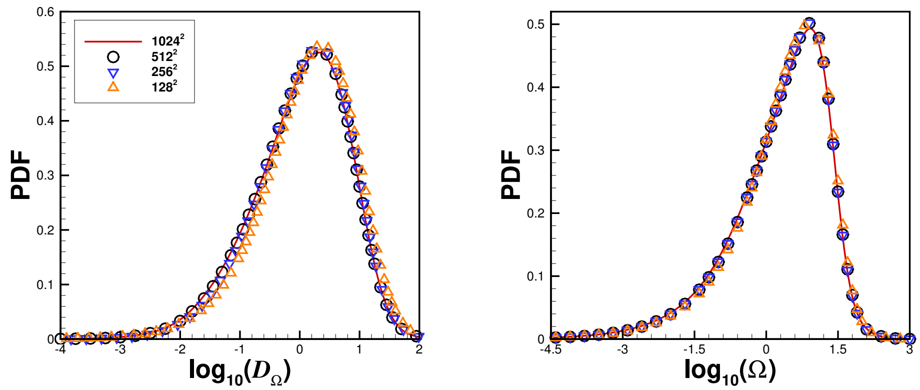


Figure S1: (a) Time history of the spatially averaged error/uncertainty energy $\langle E_\Delta \rangle_A = \langle |\Delta \mathbf{u}|^2/2 \rangle_A$. (b) Kinetic energy spectra of $\Delta \mathbf{u}$, i.e. $E_\Delta(k)$, at different times. In both (a) and (b), $\Delta \mathbf{u} = \mathbf{u}_{\mathrm{CNS}^*} - \mathbf{u}_{\mathrm{DNS}}$, where $\mathbf{u}_{\mathrm{CNS}^*}$ and $\mathbf{u}_{\mathrm{DNS}}$ correspond to the velocity fields given by CNS* and DNS, respectively.



(a) (b)

Figure S2: Probability density functions (PDFs) of (a) the enstrophy dissipation rate $D_\Omega(x, y, t)$ and (b) the enstrophy $\Omega(x, y, t)$ of the 2D turbulent Kolmogorov flow, given by DNS using the following four uniform meshes: 1024×1024 (red line), 512×512 (black circle), 256×256 (blue inverted triangle), and 128×128 (orange triangle).

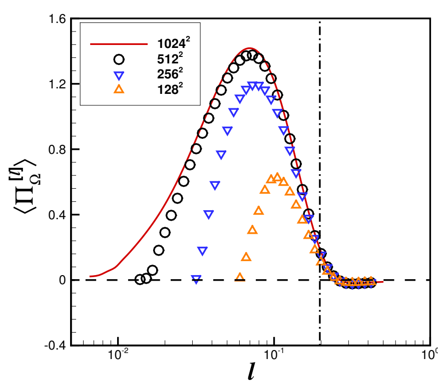


Figure S3: Spatiotemporal-averaged scale-to-scale enstrophy fluxes $\langle \Pi_\Omega^{[l]} \rangle$ of the 2D turbulent Kolmogorov flow, given by DNS using the following four uniform meshes: 1024×1024 (red line), 512×512 (black circle), 256×256 (blue inverted triangle), and 128×128 (orange triangle), where the black dashed line denotes $\langle \Pi_\Omega^{[l]} \rangle = 0$ and the black dash-dot line denotes the forcing scale $l = l_f = \pi/n_K = 0.196$.

References

[1] A. Leonard, Energy cascade in large-eddy simulations of turbulent fluid flows, Adv. Geophys. 18 (1975) 237–248.

[2] S. Chen, R. E. Ecke, G. L. Eyink, X. Wang, Z. Xiao, Physical mechanism of the two-dimensional enstrophy cascade, Phys. Rev. Lett. 91 (21) (2003) 214501.

[3] G. Boffetta, R. E. Ecke, Two-dimensional turbulence, Annu. Rev. Fluid Mech. 44 (2012) 427–51.

[4] Q. Zhou, Y. Huang, Z. Lu, Y. Liu, R. Ni, Scale-to-scale energy and enstrophy transport in two-dimensional Rayleigh-Taylor turbulence, J. Fluid Mech. 786 (2016) 294–308.