Supplementary material

1. Governing equations for first-order disturbances

In cartesian coordinates the non-divergence form of the equation for the first-order disturbances are

$$\frac{\partial \rho'}{\partial t}+\bar{u}\frac{\partial \rho'}{\partial x}+\bar{v}\frac{\partial \rho'}{\partial y}+\bar{\rho}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)+u'\frac{\partial \bar{\rho}}{\partial x}+v'\frac{\partial \bar{\rho}}{\partial y}+\rho'\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}\right)=0 \quad (1.1)$$

$$\begin{aligned}\frac{\partial u'}{\partial t}&+\bar{u}\frac{\partial u'}{\partial x}+\bar{v}\frac{\partial u'}{\partial y}+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}-\bar{\nu}\frac{\partial}{\partial x}\left[2\frac{\partial u'}{\partial x}-\frac{2}{3}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)\right]\\&-\bar{\nu}\frac{\partial}{\partial y}\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)-\bar{\nu}\frac{\partial}{\partial z}\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right)\\&-\frac{1}{\bar{\rho}}\left[2\frac{\partial u'}{\partial x}-\frac{2}{3}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)\right]\frac{\partial \bar{\mu}}{\partial x}-\frac{1}{\bar{\rho}}\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)\frac{\partial \bar{\mu}}{\partial y}\\&+\frac{1}{\bar{\rho}}\left(\rho'\bar{u}+\bar{\rho}u'\right)\frac{\partial \bar{u}}{\partial x}+\frac{1}{\bar{\rho}}\left(\rho'\bar{v}+\bar{\rho}v'\right)\frac{\partial \bar{u}}{\partial y}=0 \qquad (1.2)\end{aligned}$$

$$\begin{aligned}\frac{\partial v'}{\partial t}&+\bar{u}\frac{\partial v'}{\partial x}+\bar{v}\frac{\partial v'}{\partial y}+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y}-\bar{\nu}\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)\\&-\bar{\nu}\frac{\partial}{\partial y}\left[2\frac{\partial v'}{\partial y}-\frac{2}{3}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)\right]-\bar{\nu}\frac{\partial}{\partial z}\left(\frac{\partial v'}{\partial z}+\frac{\partial w'}{\partial y}\right)\\&-\frac{1}{\bar{\rho}}\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)\frac{\partial \bar{\mu}}{\partial x}-\frac{1}{\bar{\rho}}\left[2\frac{\partial v'}{\partial y}-\frac{2}{3}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)\right]\frac{\partial \bar{\mu}}{\partial y}\\&+\frac{1}{\bar{\rho}}\left(\rho'\bar{u}+\bar{\rho}u'\right)\frac{\partial \bar{v}}{\partial x}+\frac{1}{\bar{\rho}}\left(\rho'\bar{v}+\bar{\rho}v'\right)\frac{\partial \bar{v}}{\partial y}=0 \qquad (1.3)\end{aligned}$$

$$\begin{aligned}\frac{\partial w'}{\partial t}&+\bar{u}\frac{\partial w'}{\partial x}+\bar{v}\frac{\partial w'}{\partial y}+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z}-\bar{\nu}\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right)-\bar{\nu}\frac{\partial}{\partial y}\left(\frac{\partial v'}{\partial z}+\frac{\partial w'}{\partial y}\right)\\&-\bar{\nu}\frac{\partial}{\partial z}\left[2\frac{\partial w'}{\partial z}-\frac{2}{3}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)\right]\\&-\frac{1}{\bar{\rho}}\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right)\frac{\partial \bar{\mu}}{\partial x}-\frac{1}{\bar{\rho}}\left(\frac{\partial v'}{\partial z}+\frac{\partial w'}{\partial y}\right)\frac{\partial \bar{\mu}}{\partial y}=0 \qquad (1.4)\end{aligned}$$

$$\frac{\partial T'}{\partial t}+\bar{u}\frac{\partial T'}{\partial x}+\bar{v}\frac{\partial T'}{\partial y}+\frac{1}{\bar{\rho}c_v}\left[p'\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}\right)+\bar{p}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)\right] \tag{1.5}$$

$$\begin{aligned}
&-\frac{\gamma}{Pr}\bar{\nu}\left(\frac{\partial^2 T'}{\partial^2 x}+\frac{\partial^2 T'}{\partial^2 y}+\frac{\partial^2 T'}{\partial^2 z}\right)-\frac{\gamma}{Pr}\frac{1}{\bar{\rho}}\frac{\partial T'}{\partial x}\frac{\partial \bar{\mu}}{\partial x}-\frac{\gamma}{Pr}\frac{1}{\bar{\rho}}\frac{\partial T'}{\partial y}\frac{\partial \bar{\mu}}{\partial y}\\
&-\frac{\bar{\nu}}{c_v}\left[4\left(\frac{\partial \bar{u}}{\partial x}\frac{\partial u'}{\partial x}+\frac{\partial \bar{v}}{\partial y}\frac{\partial w'}{\partial y}\right)+2\left(\frac{\partial \bar{u}}{\partial y}+\frac{\partial \bar{v}}{\partial x}\right)\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)\right.\\
&\qquad\left.-\frac{4}{3}\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}\right)\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)\right]\\
&+\frac{1}{\bar{\rho}}\left(\rho'\bar{u}+\bar{\rho}u'\right)\frac{\partial \bar{T}}{\partial x}+\frac{1}{\bar{\rho}}\left(\rho'\bar{v}+\bar{\rho}v'\right)\frac{\partial \bar{T}}{\partial y}=0
\end{aligned} \tag{1.6}$$

$$p'=\rho' R\bar{T}+\bar{\rho}RT' \tag{1.7}$$

3 ## 2. Governing equations for second-order disturbances

The non-divergence form of the equations of the second-order disturbances are

$$\begin{aligned}
&\frac{\partial \rho''}{\partial t}+\bar{u}\frac{\partial \rho''}{\partial x}+\bar{v}\frac{\partial \rho''}{\partial y}+\bar{\rho}\left(\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)\\
&\qquad+u''\frac{\partial \bar{\rho}}{\partial x}+v''\frac{\partial \bar{\rho}}{\partial y}+\rho''\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}\right)=N_{B,\rho}
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
&\frac{\partial u''}{\partial t}+\bar{u}\frac{\partial u''}{\partial x}+\bar{v}\frac{\partial u''}{\partial y}+\frac{1}{\bar{\rho}}\frac{\partial p''}{\partial x}-\bar{\nu}\frac{\partial}{\partial x}\left[2\frac{\partial u''}{\partial x}-\frac{2}{3}\left(\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)\right]\\
&-\bar{\nu}\frac{\partial}{\partial y}\left(\frac{\partial u''}{\partial y}+\frac{\partial v''}{\partial x}\right)-\bar{\nu}\frac{\partial}{\partial z}\left(\frac{\partial u''}{\partial z}+\frac{\partial w''}{\partial x}\right)\\
&-\frac{1}{\bar{\rho}}\left[2\frac{\partial u''}{\partial x}-\frac{2}{3}\left(\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)\right]\frac{\partial \bar{\mu}}{\partial x}-\frac{1}{\bar{\rho}}\left(\frac{\partial u''}{\partial y}+\frac{\partial v''}{\partial x}\right)\frac{\partial \bar{\mu}}{\partial y}\\
&+\frac{1}{\bar{\rho}}\left(\rho''\bar{u}+\bar{\rho}u''\right)\frac{\partial \bar{u}}{\partial x}+\frac{1}{\bar{\rho}}\left(\rho''\bar{v}+\bar{\rho}v''\right)\frac{\partial \bar{u}}{\partial y}=N_{B,u}+N_{S,u}
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&\frac{\partial v''}{\partial t}+\bar{u}\frac{\partial v''}{\partial x}+\bar{v}\frac{\partial v''}{\partial y}+\frac{1}{\bar{\rho}}\frac{\partial p''}{\partial y}-\bar{\nu}\frac{\partial}{\partial x}\left(\frac{\partial u''}{\partial y}+\frac{\partial v''}{\partial x}\right)\\
&-\bar{\nu}\frac{\partial}{\partial y}\left[2\frac{\partial v''}{\partial y}-\frac{2}{3}\left(\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)\right]-\bar{\nu}\frac{\partial}{\partial z}\left(\frac{\partial v''}{\partial z}+\frac{\partial w''}{\partial y}\right)\\
&-\frac{1}{\bar{\rho}}\left(\frac{\partial u''}{\partial y}+\frac{\partial v''}{\partial x}\right)\frac{\partial \bar{\mu}}{\partial x}-\frac{1}{\bar{\rho}}\left[2\frac{\partial v''}{\partial y}-\frac{2}{3}\left(\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)\right]\frac{\partial \bar{\mu}}{\partial y}\\
&+\frac{1}{\bar{\rho}}\left(\rho''\bar{u}+\bar{\rho}u''\right)\frac{\partial \bar{v}}{\partial x}+\frac{1}{\bar{\rho}}\left(\rho''\bar{v}+\bar{\rho}v''\right)\frac{\partial \bar{v}}{\partial y}=N_{B,v}+N_{S,v}
\end{aligned} \tag{2.3}$$

$$\frac{\partial w''}{\partial t} + \bar{u}\frac{\partial w''}{\partial x} + \bar{v}\frac{\partial w''}{\partial y} + \frac{1}{\bar{\rho}}\frac{\partial p''}{\partial z} - \bar{\nu}\frac{\partial}{\partial x}\left(\frac{\partial u''}{\partial z} + \frac{\partial w''}{\partial x}\right) - \bar{\nu}\frac{\partial}{\partial y}\left(\frac{\partial v''}{\partial z} + \frac{\partial w''}{\partial y}\right)$$
$$- \bar{\nu}\frac{\partial}{\partial z}\left[2\frac{\partial w''}{\partial z} - \frac{2}{3}\left(\frac{\partial u''}{\partial x} + \frac{\partial v''}{\partial y} + \frac{\partial w''}{\partial z}\right)\right]$$
$$- \frac{1}{\bar{\rho}}\left(\frac{\partial u''}{\partial z} + \frac{\partial w''}{\partial x}\right)\frac{\partial \bar{\mu}}{\partial x} - \frac{1}{\bar{\rho}}\left(\frac{\partial v''}{\partial z} + \frac{\partial w''}{\partial y}\right)\frac{\partial \bar{\mu}}{\partial y} = N_{B,w} + N_{S,w} \tag{2.4}$$

$$\frac{\partial T''}{\partial t} + \bar{u}\frac{\partial T''}{\partial x} + \bar{v}\frac{\partial T''}{\partial y} + \frac{1}{\bar{\rho}c_v}\left[p''\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) + \bar{p}\left(\frac{\partial u''}{\partial x} + \frac{\partial v''}{\partial y} + \frac{\partial w''}{\partial z}\right)\right]$$
$$- \frac{\gamma}{Pr}\bar{\nu}\left(\frac{\partial^2 T''}{\partial^2 x} + \frac{\partial^2 T''}{\partial^2 y} + \frac{\partial^2 T''}{\partial^2 z}\right) - \frac{\gamma}{Pr}\frac{1}{\bar{\rho}}\frac{\partial T''}{\partial x}\frac{\partial \bar{\mu}}{\partial x} - \frac{\gamma}{Pr}\frac{1}{\bar{\rho}}\frac{\partial T''}{\partial y}\frac{\partial \bar{\mu}}{\partial y}$$
$$- \frac{\bar{\nu}}{c_v}\left[4\left(\frac{\partial \bar{u}}{\partial x}\frac{\partial u''}{\partial x} + \frac{\partial \bar{v}}{\partial y}\frac{\partial w''}{\partial y}\right) + 2\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)\left(\frac{\partial u''}{\partial y} + \frac{\partial v''}{\partial x}\right)\right.$$
$$\left. - \frac{4}{3}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right)\left(\frac{\partial u''}{\partial x} + \frac{\partial v''}{\partial y} + \frac{\partial w''}{\partial z}\right)\right]$$
$$+ \frac{1}{\bar{\rho}}\left(\rho''\bar{u} + \bar{\rho}u''\right)\frac{\partial \bar{T}}{\partial x} + \frac{1}{\bar{\rho}}\left(\rho''\bar{v} + \bar{\rho}v''\right)\frac{\partial \bar{T}}{\partial y} = N_{B,T} + N_{S,T} \tag{2.5}$$

$$p'' = \rho'' R\bar{T} + \bar{\rho}RT'' + \rho' RT' \tag{2.6}$$

where

$$N_{B,\rho} = -\frac{\partial \rho' u'}{\partial x} - \frac{\partial \rho' v'}{\partial y} - \frac{\partial \rho' w'}{\partial z} \tag{2.7a}$$

$$N_{B,u} = -\frac{\rho'}{\bar{\rho}}\frac{\partial u'}{\partial t} - \frac{1}{\bar{\rho}}\left(\rho'\bar{u} + \bar{\rho}u'\right)\frac{\partial u'}{\partial x} - \frac{1}{\bar{\rho}}\left(\rho'\bar{v} + \bar{\rho}v'\right)\frac{\partial u'}{\partial y} - w'\frac{\partial u'}{\partial z} - \frac{\rho'}{\bar{\rho}}u'\frac{\partial \bar{u}}{\partial x} - \frac{\rho'}{\bar{\rho}}v'\frac{\partial \bar{u}}{\partial y} \tag{2.7b}$$

$$N_{S,u} = 0 \tag{2.7c}$$

$$N_{B,v} = -\frac{\rho'}{\bar{\rho}}\frac{\partial v'}{\partial t} - \frac{1}{\bar{\rho}}\left(\rho'\bar{u} + \bar{\rho}u'\right)\frac{\partial v'}{\partial x} - \frac{1}{\bar{\rho}}\left(\rho'\bar{v} + \bar{\rho}v'\right)\frac{\partial v'}{\partial y} - w'\frac{\partial v'}{\partial z} - \frac{\rho'}{\bar{\rho}}u'\frac{\partial \bar{v}}{\partial x} - \frac{\rho'}{\bar{\rho}}v'\frac{\partial \bar{v}}{\partial y} \tag{2.7d}$$

$$N_{S,v} = 0 \tag{2.7e}$$

$$N_{B,w} = -\frac{\rho'}{\bar{\rho}}\frac{\partial w'}{\partial t} - \frac{1}{\bar{\rho}}\left(\rho'\bar{u} + \bar{\rho}u'\right)\frac{\partial w'}{\partial x} - \frac{1}{\bar{\rho}}\left(\rho'\bar{v} + \bar{\rho}v'\right)\frac{\partial w'}{\partial y} - w'\frac{\partial w'}{\partial z} \tag{2.7f}$$

$$N_{S,w} = 0 \tag{2.7g}$$

$$N_{B,T} = -\frac{\rho'}{\bar{\rho}}\frac{\partial T'}{\partial t} - \frac{1}{\bar{\rho}}\left(\rho'\bar{u} + \bar{\rho}u'\right)\frac{\partial T'}{\partial x} - \frac{1}{\bar{\rho}}\left(\rho'\bar{v} + \bar{\rho}v'\right)\frac{\partial T'}{\partial y} - w'\frac{\partial T'}{\partial z} - \frac{\rho'}{\bar{\rho}}u'\frac{\partial \bar{T}}{\partial x} - \frac{\rho'}{\bar{\rho}}v'\frac{\partial \bar{T}}{\partial y} \tag{2.7h}$$

$$N_{S,T} = -\frac{1}{\bar{\rho}c_v}p'\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)$$
$$+ \frac{\bar{\nu}}{c_v}\left[2\left(\frac{\partial u'}{\partial x}\right)^2 + 2\left(\frac{\partial v'}{\partial y}\right)^2 + 2\left(\frac{\partial w'}{\partial z}\right)^2 - \frac{2}{3}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)^2\right.$$
$$\left. + \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)^2 + \left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right)^2 + \left(\frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial z}\right)^2\right] \tag{2.7i}$$