Supplementary material:
Preferential orientation of small floaters drifting in water waves

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1 Kinematics of rotation

In this section, we detail the Euler angle convention used in our article and give the kinematic relation that links $\dot{\varphi}(t), \dot{\theta}(t), \dot{\psi}(t)$ to the components $\widetilde{\Omega}_x(t), \widetilde{\Omega}_y(t), \widetilde{\Omega}_z(t)$ of the instantaneous rotation vector in the floater frame.

The instantaneous orientation of the floater is defined with the three Euler angles $\theta(t)$ pitch, $\varphi(t)$ roll and $\psi(t)$ yaw. These angles relate to three successive rotations that bring the laboratory frame $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ to the floater frame $(\widetilde{\boldsymbol{e}}_x(t), \widetilde{\boldsymbol{e}}_y(t), \widetilde{\boldsymbol{e}}_z(t))$. Imagine looking at the wave from above, from the $\boldsymbol{e}_z$ axis. The yaw angle $\psi(t)$ is the angle over which we need to rotate the laboratory frame to align the $\boldsymbol{e}'_x$-axis of a new frame, with the front of the floater. This defines a first intermediate frame $(\boldsymbol{e}'_x, \boldsymbol{e}'_y, \boldsymbol{e}'_z)$ as

$$\begin{bmatrix} \boldsymbol{e}'_x \\ \boldsymbol{e}'_y \\ \boldsymbol{e}'_z \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_\psi} \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix}. \tag{1}$$

R_ψ is a first rotation matrix. We now change our point of view and look at the floater from the side, from the $\boldsymbol{e}'_y$ direction. The pitch angle $\theta(t)$ is the angle over which we need to rotate the $(\boldsymbol{e}'_x, \boldsymbol{e}'_y, \boldsymbol{e}'_z)$ frame to find a new $\boldsymbol{e}''_x$ axis that aligns with the front of the floater. Hence, we have a second intermediate frame $(\boldsymbol{e}''_x, \boldsymbol{e}''_y, \boldsymbol{e}''_z)$ defined as

$$\begin{bmatrix} \boldsymbol{e}''_x \\ \boldsymbol{e}''_y \\ \boldsymbol{e}''_z \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{R_\theta} \begin{bmatrix} \boldsymbol{e}'_x \\ \boldsymbol{e}'_y \\ \boldsymbol{e}'_z \end{bmatrix}. \tag{2}$$

R_θ is a second rotation matrix. In a third and final rotation, we imagine looking at the boat from the front, from the $\boldsymbol{e}''_x$ direction. The roll angle $\varphi(t)$, rotates the $(\boldsymbol{e}''_x, \boldsymbol{e}''_y, \boldsymbol{e}''_z)$ frame to the floater frame. Hence, we have

$$\begin{bmatrix} \widetilde{\boldsymbol{e}}_x \\ \widetilde{\boldsymbol{e}}_y \\ \widetilde{\boldsymbol{e}}_z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}}_{R_\varphi} \begin{bmatrix} \boldsymbol{e}''_x \\ \boldsymbol{e}''_y \\ \boldsymbol{e}''_z \end{bmatrix}, \tag{3}$$

and R_ϕ is a third rotation matrix. All combined and using the fact that the rotation matrices are orthogonal, we can explicit the passage from the laboratory frame to the floater frame as

$$\begin{bmatrix} \widetilde{\boldsymbol{e}}_x \\ \widetilde{\boldsymbol{e}}_y \\ \widetilde{\boldsymbol{e}}_z \end{bmatrix} = \underbrace{R_\varphi R_\theta R_\psi}_{R} \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = \underbrace{R_\psi^T R_\theta^T R_\varphi^T}_{R^T} \begin{bmatrix} \widetilde{\boldsymbol{e}}_x \\ \widetilde{\boldsymbol{e}}_y \\ \widetilde{\boldsymbol{e}}_z \end{bmatrix}. \tag{4}$$

Evaluating these matrix products explicitly we obtain formula (3.6a) of the article.

We now explain how to find Eq. (3.9) that links $\dot{\psi}(t), \dot{\theta}(t), \dot{\varphi}(t)$ to $\widetilde{\Omega}_x(t), \widetilde{\Omega}_y(t), \widetilde{\Omega}_z(t)$. The floater frame rotates at instantaneous speed $\boldsymbol{\Omega}(t)$ which can be expressed using the differential equation $\dot{\widetilde{\boldsymbol{e}}}_i(t) = \boldsymbol{\Omega}(t) \times \widetilde{\boldsymbol{e}}_i(t)$ for $i = x, y, z$. In other terms, we have

$$\frac{d}{dt} \begin{bmatrix} \widetilde{\boldsymbol{e}}_x \\ \widetilde{\boldsymbol{e}}_y \\ \widetilde{\boldsymbol{e}}_z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \widetilde{\Omega}_z & -\widetilde{\Omega}_y \\ -\widetilde{\Omega}_z & 0 & \widetilde{\Omega}_x \\ \widetilde{\Omega}_y & -\widetilde{\Omega}_x & 0 \end{bmatrix}}_{M} \begin{bmatrix} \widetilde{\boldsymbol{e}}_x \\ \widetilde{\boldsymbol{e}}_y \\ \widetilde{\boldsymbol{e}}_z \end{bmatrix}. \tag{5}$$

Considering the transform Eq. (3.6a), we can also compute the time-derivative of the vectors of the floater frame by deriving the rotation matrix. We then have

$$\frac{d}{dt} \begin{bmatrix} \widetilde{\boldsymbol{e}}_x \\ \widetilde{\boldsymbol{e}}_y \\ \widetilde{\boldsymbol{e}}_z \end{bmatrix} = \frac{d}{dt}\left(R_\varphi R_\theta R_\psi\right) \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = \frac{d}{dt}\left(R_\varphi R_\theta R_\psi\right)\left(R_\psi^T R_\theta^T R_\varphi^T\right) \begin{bmatrix} \widetilde{\boldsymbol{e}}_x \\ \widetilde{\boldsymbol{e}}_y \\ \widetilde{\boldsymbol{e}}_z \end{bmatrix}. \tag{6}$$

Comparing with (5) implies

$$\left(R'_\varphi R_\theta R_\psi \dot{\varphi} + R_\varphi R'_\theta R_\psi \dot{\theta} + R_\varphi R_\theta R'_\psi \dot{\psi}\right) R_\psi^T R_\theta^T R_\varphi^T = M. \tag{7}$$

We denote $R'_\varphi = \partial_\varphi R_\varphi$ and similar for the other matrices. Considering the orthogonality of the rotation matrices, this simplifies to $R'_\varphi R_\varphi^T \dot{\varphi} + R_\varphi R'_\theta R_\theta^T R_\varphi^T \dot{\theta} + R_\varphi R_\theta R'_\psi R_\psi^T R_\theta^T R_\varphi^T \dot{\psi} = M$. After evaluating all the matrix products, we then deduce that

$$\begin{bmatrix} \widetilde{\Omega}_x \\ \widetilde{\Omega}_y \\ \widetilde{\Omega}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\varphi & \cos\theta\sin\varphi \\ 0 & -\sin\varphi & \cos\theta\cos\varphi \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{8}$$

Notice that this matrix is singular, when $\theta = \pm\pi/2$. In our applications, we do not encounter such large pitch angles. Hence, we can always invert this relation to find the system of equations (3.9).

2 Extra details on the asymptotic calculation

In the asymptotic theory, we describe the idealised motion of a parallelepiped floater that is short with respect to the wavelength and flat. Hence, we suppose the ordering

$$\delta_z \ll (\delta_x, \delta_y) \ll 1 \tag{9}$$

in the non-dimensional floater sizes. Both δ_x and δ_y are kept at similar magnitude. When floaters are small with respect to the wavelength, they see the free-surface and the surrounding wave-flow as weakly varying in space. In this limit, we can construct polynomial approximations of the waveflow and the water-surface and this makes the analytical calculation of the pressure force and moment, as defined by Eqs. (4.21) and (4.22), possible. The full calculation is very long, but we provide some extra information here.

2.1 Volume integrals and polynomial approximations of the integrands

In the volume integrals of Eqs. (4.21) and (4.22), we have the $V_{sub}^{(0)}$ integral, over the equilibrium submerged volume, and $V_{sub}^{(1)}$, its perturbation caused by the first order displacement and the deformation of the interface. Both integrals are parametrised in floater frame coordinates $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ and we have

$$\int_{V_{sub}^{(0)}} \Big(\dots\Big) dV = \int_{-\delta_x/2}^{\delta_x/2}\int_{-\delta_y/2}^{\delta_y/2}\int_{-\delta_z/2}^{-\overline{z}_c} \Big(\dots\Big)\Big|_{(\overline{x}_c,\overline{z}_c,\overline{\psi})} d\widetilde{x}\, d\widetilde{y}\, d\widetilde{z}. \tag{10}$$

$$\begin{aligned}\int_{V_{sub}^{(1)}} \Big(\dots\Big) dV = & \underbrace{\int_{-\delta_x/2}^{\delta_x/2}\int_{-\delta_y/2}^{\delta_y/2}\int_{-\delta_z/2}^{-\overline{z}_c} \left(x'_c \frac{\partial}{\partial x_c} + z'_c \frac{\partial}{\partial z_c} + \psi' \frac{\partial}{\partial \psi}\right)\Big(\dots\Big)\Big|_{(\overline{x}_c,\overline{z}_c,\overline{\psi})} d\widetilde{x}\, d\widetilde{y}\, d\widetilde{z}}_{\text{deviation due to motion } x'_c,\, z'_c,\, \psi'} \\ & + \underbrace{\int_{-\delta_x/2}^{\delta_x/2}\int_{-\delta_y/2}^{\delta_y/2} \left(\widetilde{\zeta}(\widetilde{x},\widetilde{y},t) + \overline{z}_c\right)\Big(\dots\Big)\Big|_{(\overline{x}_c,\overline{z}_c,\overline{\psi})} d\widetilde{x}\, d\widetilde{y}}_{\text{deviation due to locally varying submersion}} + O(\epsilon^2). \end{aligned} \tag{11}$$

The notation $|_{(\overline{x}_c,\overline{z}_c,\overline{\psi})}$ suggests that we need to replace $x_c = \overline{x}_c, z_c = \overline{z}_c, \psi = \overline{\psi}$ in the integrand because we need to evaluate it at the equilibrium position.

The function $\widetilde{\zeta}(\widetilde{x}, \widetilde{y}, t)$ in the $V_{sub}^{(1)}$ integral defines the shape of the water surface, as seen from the floater center C. For floaters that are small with respect to the wavelength, we can use a polynomial approximation

$$\begin{aligned}\widetilde{\zeta}(\widetilde{x},\widetilde{y},t) = & \underbrace{-\overline{z}_c}_{O(\delta)} \underbrace{-z'_c + \zeta_c}_{O(\epsilon)} + \underbrace{(\theta' + c_\psi \partial_x \zeta_c)\widetilde{x} + (-\varphi' - s_\psi \partial_x \zeta_c)\widetilde{y}}_{O(\epsilon\delta)} \\ & \underbrace{+\frac{1}{2}\left(c_\psi \widetilde{x} - s_\psi \widetilde{y}\right)^2 \partial^2_{xx}\zeta_c}_{O(\epsilon\delta^2)} + \underbrace{\frac{1}{6}\left(c_\psi \widetilde{x} - s_\psi \widetilde{y}\right)^3 \partial^3_{xxx}\zeta_c}_{O(\epsilon\delta^3)} + O(\epsilon^2, \epsilon\delta^4) \end{aligned} \tag{12}$$

We group here terms that have same order of magnitude in terms of powers of δ (non-dimensional floater size) and ϵ (non-dimensional wave-amplitude).

In the integrands we find the fields $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ that are polynomial approximations of the local fluid acceleration, locally valid near C. We construct them as follows. The fluid acceleration is by definition

$$a_x = \partial_t u_x = -\epsilon e^z \cos(x-t) \quad , \quad a_z = \partial_t u_z = -\epsilon e^z \sin(x-t). \tag{13}$$

These fields depend on the lab-frame coordinates x, z but need to be expressed in terms of $\widetilde{x}, \widetilde{y}, \widetilde{z}$ variables if we want to compute the integrals analytically. Using Taylor series around the floater center, we have

$$\begin{aligned}\boldsymbol{a} &= \boldsymbol{a}_c + (x-x_c)\partial_x \boldsymbol{a}_c + (z-z_c)\partial_z \boldsymbol{a}_c \\ &\quad + \frac{1}{2}(x-x_c)^2 \partial^2_{xx}\boldsymbol{a}_c + \frac{1}{2}(z-z_c)^2\partial^2_{zz}\boldsymbol{a}_c + (x-x_c)(z-z_c)\partial^2_{xz}\boldsymbol{a}_c + O(\epsilon\delta^3).\end{aligned} \tag{14}$$

Replacing $x - x_c = c_\psi \widetilde{x} - s_\psi \widetilde{y} + (c_\psi\theta' + s_\psi\varphi')\,\widetilde{z}$ and $z - z_c = -\theta'\widetilde{x} + \varphi'\widetilde{y} + \widetilde{z}$ from the transform formula (4.12) we get

$$\boldsymbol{a} = \boldsymbol{a}^{(1)} + \boldsymbol{a}^{(2)} + O(\epsilon^3). \tag{15a}$$

There is an $O(\epsilon)$ part

$$\begin{aligned}\boldsymbol{a}^{(1)} &= \underbrace{\boldsymbol{a}_c}_{O(\epsilon)} + \underbrace{(c_\psi\widetilde{x} - s_\psi\widetilde{y})\partial_x\boldsymbol{a}_c + \widetilde{z}\partial_z\boldsymbol{a}_c}_{O(\epsilon\delta)} \\ &\quad + \underbrace{\frac{1}{2}(c_\psi\widetilde{x} - s_\psi\widetilde{y})^2\partial^2_{xx}\boldsymbol{a}_c + \frac{\widetilde{z}^2}{2}\partial^2_{zz}\boldsymbol{a}_c + (c_\psi\widetilde{x} - s_\psi\widetilde{y})\widetilde{z}\partial^2_{xz}\boldsymbol{a}_c}_{O(\epsilon\delta^2)}\end{aligned} \tag{15b}$$

and an $O(\epsilon^2)$ part

$$\begin{aligned}\boldsymbol{a}^{(2)} &= \underbrace{(c_\psi\theta' + s_\psi\varphi')\widetilde{z}\partial_x\boldsymbol{a}_c + (-\theta'\widetilde{x} + \varphi'\widetilde{y})\partial_z\boldsymbol{a}_c}_{O(\epsilon^2\delta)} \\ &\quad + \underbrace{\Big[(c_\psi\widetilde{x} - s_\psi\widetilde{y})(c_\psi\theta' + s_\psi\varphi')\widetilde{z}\partial^2_{xx}\boldsymbol{a}_c + (-\theta'\widetilde{x} + \varphi'\widetilde{y})\widetilde{z}\partial^2_{zz}\boldsymbol{a}_c + \left[(c_\psi\widetilde{x} - s_\psi\widetilde{y})(-\theta'\widetilde{x} + \varphi'\widetilde{y}) + (c_\psi\theta' + s_\psi\varphi')\widetilde{z}^2\right]\partial^2_{xz}\boldsymbol{a}_c\Big]}_{O(\epsilon^2\delta^2)}.\end{aligned} \tag{15c}$$

This second order part $\boldsymbol{a}^{(2)}$ has a non-zero average, that captures how the first order angular motion of θ' and φ' correlates to the local fluid accelaration.

2.2 First order motion

The first order motion carried by the variables $x'_c, z'_c, \theta', \varphi', \psi'$ are solutions of the evolution equations

$$m\ddot{x}'_c = F'_x \quad , \quad m\ddot{z}'_c = F'_z \quad , \quad \widetilde{I}_{xx}\ddot{\varphi}' = \widetilde{K}'_x \quad , \quad \widetilde{I}_{yy}\ddot{\theta}' = \widetilde{K}'_y \quad , \quad \widetilde{I}_{zz}\ddot{\psi}' = K'_z. \tag{16}$$

Non-dimensional mass m and moments of interia are as defined in the text. We need to compute the force and torque components

$$F'_x = \int_{V^{(0)}_{sub}} a^{(1)}_x dV \tag{17a}$$

$$F'_z = \int_{V^{(0)}_{sub}} a^{(1)}_z\, dV + \int_{V^{(1)}_{sub}} dV \tag{17b}$$

$$\widetilde{K}'_x = \int_{V^{(0)}_{sub}} \left(\widetilde{y}a^{(1)}_z + s_\psi\widetilde{z}a^{(1)}_x - \widetilde{z}\varphi'\right) dV + \int_{V^{(1)}_{sub}} \widetilde{y}dV \tag{17c}$$

$$\widetilde{K}'_y = \int_{V^{(0)}_{sub}} \left(-\widetilde{x}a^{(1)}_z + c_\psi\widetilde{z}a^{(1)}_x - \widetilde{z}\theta'\right) dV + \int_{V^{(1)}_{sub}} (-\widetilde{x})dV \tag{17d}$$

$$K'_z = -\int_{V^{(0)}_{sub}} (s_\psi\widetilde{x} + c_\psi\widetilde{y})\, a^{(1)}_x dV. \tag{17e}$$

Using the methods of the previous section, this reduces in practice to the calculation of polynomial integrals involving powers of $\widetilde{x}, \widetilde{y}, \widetilde{z}$. In the flat floater limit (9), we then ignore all $O(\delta_z)$ terms with respect to $O(\delta_x)$ and $O(\delta_y)$ terms. Injecting these force and torque components into the equations, we get

$$\ddot{x}'_c \approx -\epsilon\cos(\overline{x}_c - t) \tag{18a}$$

$$\ddot{z}'_c + \frac{1}{\beta\delta_z} z'_c \approx \epsilon\left[\frac{1}{\beta\delta_z} - 1 - \frac{1}{24\beta\delta_z}\left(\overline{c}_\psi^2\delta_x^2 + \overline{s}_\psi^2\delta_y^2\right)\right]\sin(\overline{x}_c - t) \tag{18b}$$

$$\ddot{\varphi}' + \frac{1}{\beta\delta_z}\varphi' \approx -\epsilon\overline{s}_\psi\left[\frac{1}{\beta\delta_z} - 1 - \frac{1}{\beta\delta_z}\left(\overline{s}_\psi^2\frac{\delta_y^2}{40} + \overline{c}_\psi^2\frac{\delta_x^2}{24}\right)\right]\cos(\overline{x}_c - t) \tag{18c}$$

$$\ddot{\theta}' + \frac{1}{\beta\delta_z}\theta' \approx -\epsilon\overline{c}_\psi\left[\frac{1}{\beta\delta_z} - 1 - \frac{1}{\beta\delta_z}\left(\overline{c}_\psi^2\frac{\delta_x^2}{40} + \overline{s}_\psi^2\frac{\delta_y^2}{24}\right)\right]\cos(\overline{x}_c - t) \tag{18d}$$

$$\ddot{\psi}' \approx -\epsilon\left(\frac{\delta_x^2 - \delta_y^2}{\delta_x^2 + \delta_y^2}\right)\overline{s}_\psi\overline{c}_\psi\sin(\overline{x}_c - t). \tag{18e}$$

The z'_c, φ' and θ' equations are that of forced harmonic oscillators with natural frequencies $\sqrt{1/\beta\delta_z}$. Flat floaters, with $\delta_z \ll \delta_x, \delta_y$, have nearly the same natural oscillation frequencies $\omega_z \approx \omega_\varphi \approx \omega_\theta \approx \sqrt{1/\beta\delta_z}$ [see the general Eqs. (3.14)]. Considering that $\delta_z \ll 1$, we have $\sqrt{1/\beta\delta_z} \gg 1$, implying that the incoming wave is never resonantly forcing the z'_c, φ' and θ' oscillations. Hence, the solution for the motion is the sum of free oscillations at the bobbing frequency and a harmonic response. In the theoretical calculation, we discard all free high-frequency oscillations. The harmonic solution is given by Eq. (4.26) in the text.

2.3 Second order, mean yaw motion

In the text, we have shown that according to our idealised model, the slow evolution of the yaw angle is governed by

$$\widetilde{I}_{zz}\ddot{\overline{\psi}} = \overline{K}_z. \tag{19}$$

Here $\overline{K}_z$ is vertical component of the mean yaw moment in the laboratory frame, defined by

$$\overline{K}_z = -\overline{\int_{V_{sub}^{(0)}} (s_\psi\theta' - c_\psi\varphi')\,\widetilde{z}a_x^{(1)}dV} - \overline{\int_{V_{sub}^{(0)}} (s_\psi\widetilde{x} + c_\psi\widetilde{y})\,a_x^{(2)}dV} - \overline{\int_{V_{sub}^{(1)}} (s_\psi\widetilde{x} + c_\psi\widetilde{y})\,a_x^{(1)}dV}. \tag{20}$$

In the flat floater limit $\delta_z \ll \delta_x, \delta_y$, the first term is negligible with respect to the second and third terms. This is due to the fact that $s_\psi\theta' - c_\psi\varphi' \approx O(\epsilon\delta_x^2, \epsilon\delta_y^2)$ and the presence of $\widetilde{z}$ in the integrandum. We find more precisely that

$$\overline{-\widetilde{I}_{zz}^{-1}\left(\int_{V_{sub}^{(0)}} (s_\psi\theta' - c_\psi\varphi')\,\widetilde{z}a_x^{(1)}dV\right)} \approx O(\epsilon^2\delta_z) \tag{21}$$

is indeed vanishingly small for $\delta_z \to 0$. The second term is not negligible. We find

$$\begin{aligned}\overline{-\widetilde{I}_{zz}^{-1}\left(\int_{V_{sub}^{(0)}} (s_\psi\widetilde{x} + c_\psi\widetilde{y})\,a_x^{(2)}dV\right)} &\approx \overline{-\widetilde{I}_{zz}^{-1}\left(\int_{V_{sub}^{(0)}} (s_\psi\widetilde{x} + c_\psi\widetilde{y})\,\underbrace{(-\theta'\widetilde{x} + \varphi'\widetilde{y})\partial_z a_{x,c}}_{\text{due to motion } \theta', \varphi'}\,dV\right)} \\ &\approx \frac{\epsilon^2}{2}\left(\frac{\delta_x^2 - \delta_y^2}{\delta_x^2 + \delta_y^2}\right)\overline{s}_\psi\overline{c}_\psi.\end{aligned} \tag{22}$$

In the lengthy expression of $a_x^{(2)}$, there is only one term, the one that is detailed here, that contributes at leading order. It is useful to notice how this part of the mean torque relates to the first order motion of θ' and φ'. It captures a small effective yaw moment that is due to the fact that first order rotations of the floater allows the floater to probe small vertical gradients in wave-magnitude ($\partial_z a_{x,c}$).

The third contribution to the mean yaw moment is the longest to evaluate and, considering the definition of the

$V_{sub}^{(1)}$ integral, can be split in two distinct parts

$$
\begin{aligned}
&\overline{-\widetilde{I}_{zz}^{-1}\left(\int_{V_{sub}^{(1)}}\left(s_\psi\widetilde{x}+c_\psi\widetilde{y}\right)a_x^{(1)}dV\right)} \\
&\quad\approx \underbrace{-\widetilde{I}_{zz}^{-1}\int_{-\delta_x/2}^{\delta_x/2}\int_{-\delta_y/2}^{\delta_y/2}\int_{-\delta_z/2}^{-\overline{z}_c}\overline{\left(x_c'\frac{\partial}{\partial x_c}+z_c'\frac{\partial}{\partial z_c}+\psi'\frac{\partial}{\partial\psi}\right)\left(\left(s_\psi\widetilde{x}+c_\psi\widetilde{y}\right)a_x^{(1)}\right)}\Big|_{(\overline{x}_c,\overline{z}_c,\overline{\psi})}d\widetilde{x}\,d\widetilde{y}\,d\widetilde{z}}_{\overline{T}_1\text{, due to motion }x_c',z_c',\psi'} \\
&\quad\underbrace{-\widetilde{I}_{zz}^{-1}\int_{-\delta_x/2}^{\delta_x/2}\int_{-\delta_y/2}^{\delta_y/2}\overline{\left(\widetilde{\zeta}(\widetilde{x},\widetilde{y},t)+\overline{z}_c\right)\left(\left(s_\psi\widetilde{x}+c_\psi\widetilde{y}\right)a_x^{(1)}\right)}\Big|_{(\overline{x}_c,\overline{z}_c,\overline{\psi})}d\widetilde{x}\,d\widetilde{y}}_{\overline{T}_2\text{, due to locally varying submersion}}.
\end{aligned}
\tag{23}
$$

The first part is due to the first order motion in x_c', z_c', ψ' and we find it as

$$
\widetilde{I}_{zz}^{-1}\overline{T}_1 \approx -\epsilon^2\left(\frac{\delta_x^2-\delta_y^2}{\delta_x^2+\delta_y^2}\right)\overline{s}_\psi\overline{c}_\psi\left[1+\frac{1}{2}\left(\frac{\delta_x^2-\delta_y^2}{\delta_x^2+\delta_y^2}\right)\left(\overline{c}_\psi^2-\overline{s}_\psi^2\right)\right]. \tag{24}
$$

The second part is given by

$$
\begin{aligned}
\widetilde{I}_{zz}^{-1}\overline{T}_2 \;\approx\;& \overline{\frac{\epsilon e^{\overline{z}_c}\cos(x_c-t)}{\beta\delta_z(\delta_x^2+\delta_y^2)}\Bigg[\overline{s}_\psi\left(\theta'+\epsilon\overline{c}_\psi\cos(\overline{x}_c-t)\right)\delta_x^2-\overline{c}_\psi\left(\varphi'+\epsilon\overline{s}_\psi\cos(\overline{x}_c-t)\right)\delta_y^2} \\
&+\left(\overline{c}_\psi^3\overline{s}_\psi\left(-\frac{\delta_x^4}{40}+\frac{\delta_x^2\delta_y^2}{24}\right)-\overline{s}_\psi^3\overline{c}_\psi\left(-\frac{\delta_y^4}{40}+\frac{\delta_x^2\delta_y^2}{24}\right)\right)\epsilon\cos(\overline{x}_c-t)\Bigg] \\
+\;& \overline{\frac{\epsilon e^{\overline{z}_c}\sin(\overline{x}_c-t)}{\beta\delta_z(\delta_x^2+\delta_y^2)}\Bigg[\overline{s}_\psi\overline{c}_\psi(\delta_x^2-\delta_y^2)(z_c'-\epsilon\sin(\overline{x}_c-t))} \\
&+\left(\overline{c}_\psi^3\overline{s}_\psi\left(3\frac{\delta_x^4}{40}-2\frac{\delta_x^2\delta_y^2}{24}\right)-\overline{s}_\psi^3\overline{c}_\psi\left(3\frac{\delta_y^4}{40}-2\frac{\delta_x^2\delta_y^2}{24}\right)\right)\epsilon\sin(\overline{x}_c-t)\Bigg] \\
\approx\;& \frac{\epsilon^2}{\beta\delta_z(\delta_x^2+\delta_y^2)}\left[\overline{s}_\psi\overline{c}_\psi^3\left(\frac{\delta_x^4}{60}-\frac{\delta_x^2\delta_y^2}{48}\right)-\overline{s}_\psi^3\overline{c}_\psi\left(\frac{\delta_y^4}{60}-\frac{\delta_x^2\delta_y^2}{48}\right)\right].
\end{aligned}
\tag{25}
$$

We show an intermediate formula to highlight how the first order motion z_c', θ', φ', also intervenes in this term that relates to varying submersion. When injecting the solutions of θ' and φ' as defined by Eq. (4.26c) and (4.26.d) in the article, we find that the first group of terms exactly cancels on average. From Eq. (4.26b) for z_c', we get

$$
z_c'-\epsilon\sin(\overline{x}_c-t) = -\frac{\epsilon}{24}\left(\overline{c}_\psi^2\delta_x^2+\overline{s}_\psi^2\delta_y^2\right)\sin(\overline{x}_c-t). \tag{26}
$$

The right hand side relates to the parabolic correction of the deformed interface, also discussed in Eq. (5.9) of the article. It seems tiny, but it is this weak, buoyancy-induced correction of the vertical motion that causes $\overline{T}_2$ to be non-zero. Summing up all the bits, we find the total mean yaw moment to be

$$
\begin{aligned}
\widetilde{I}_{zz}^{-1}\overline{K}_z \;=\;& \underbrace{-\frac{\epsilon^2}{2}\left(\frac{\delta_x^2-\delta_y^2}{\delta_x^2+\delta_y^2}\right)\overline{s}_\psi\overline{c}_\psi\left[1+\left(\frac{\delta_x^2-\delta_y^2}{\delta_x^2+\delta_y^2}\right)\left(\overline{c}_\psi^2-\overline{s}_\psi^2\right)\right]}_{\text{due to first order motion }x_c',z_c',\varphi',\theta',\psi'} \\
&\underbrace{+\frac{\epsilon^2}{\beta\delta_z(\delta_x^2+\delta_y^2)}\left[\overline{s}_\psi\overline{c}_\psi^3\left(\frac{\delta_x^4}{60}-\frac{\delta_x^2\delta_y^2}{48}\right)-\overline{s}_\psi^3\overline{c}_\psi\left(\frac{\delta_y^4}{60}-\frac{\delta_x^2\delta_y^2}{48}\right)\right]}_{\text{due to spatially varying submersion}}.
\end{aligned}
\tag{27}
$$

We can clearly separate two contributions. The first contribution is due to the first order motion. As the floater is moving in the wave, it can probe flow gradients and this results in a net $O(\epsilon^2)$ mean yaw moment. The second contribution of order $O(\epsilon^2\delta_x^2/\beta\delta_z)$, $O(\epsilon^2\delta_y^2/\beta\delta_z)$ is due to the spatial variation of submersion. Elongated floaters see curved waterlines and this modifies the buoyancy force significantly. In the end, we can replace $1 = \overline{c}_\psi^2 + \overline{s}_\psi^2$ in the first term and reorganise everything to find

$$\ddot{\psi} \;=\; \epsilon^2 \left[\overline{s}_\psi \overline{c}_\psi^3 \left(-\frac{(\delta_x^2 - \delta_y^2)\delta_x^2}{(\delta_x^2 + \delta_y^2)^2} + \frac{\dfrac{\delta_x^4}{60} - \dfrac{\delta_x^2 \delta_y^2}{48}}{\beta \delta_z (\delta_x^2 + \delta_y^2)} \right) - \overline{s}_\psi^3 \overline{c}_\psi \left(-\frac{(\delta_y^2 - \delta_x^2)\delta_y^2}{(\delta_x^2 + \delta_y^2)^2} + \frac{\dfrac{\delta_y^4}{60} - \dfrac{\delta_x^2 \delta_y^2}{48}}{\beta \delta_z (\delta_x^2 + \delta_y^2)} \right) \right], \tag{28}$$

which is Eq. (4.27) of our article.