9. Supplementary Information I: Mild nonuniformity

The applicability of the mild nonuniformity approximation is examined below, in the context of the ocean-seafloor system. As discussed in the main text, the acoustic speed in the water column is assumed to be a function of the vertical coordinate y, and the pressure wave and shear wave phase speeds in the seafloor are assumed to be functions of the horizontal coordinates x_1 and x_2. The speeds are assumed not to be time-dependent. Under the mild nonuniformity approximation, the linear (first-order) wave equations can be used, with the acoustic speeds appearing as functions the appropriate spatial variables (Whitham 1973). Here, some conditions are proposed under which this approximation is justified.

Letting φ represent one of the displacement potentials (ϕ, ϕ', and H_z) for the acoustic-gravity-Scholte wave system, the linear wave equation can be expressed in a self-adjoint form as,

$$\frac{\partial^2 \varphi}{\partial t^2} - \boldsymbol{\nabla} \cdot \left(c_i^2 \boldsymbol{\nabla} \varphi \right) = 0, \tag{9.1}$$

where $c_l = c_l(x_1, x_2, y)$, represents one of the speeds c_1, c_p, and c_s and is a function of the spatial coordinates. Equation (9.1) expands into,

$$\frac{\partial^2 \varphi}{\partial t^2} - \boldsymbol{\nabla} c_i^2 \cdot \boldsymbol{\nabla} \varphi - c_l^2 \nabla^2 \varphi = 0. \tag{9.2}$$

The media are thought to be mildly nonuniform when the second term in equation (9.1) is an order of magnitude smaller than the first and third terms, or,

$$\left| \boldsymbol{\nabla} c_i^2 \cdot \boldsymbol{\nabla} \varphi \right| \ll \left| \frac{\partial^2 \varphi}{\partial t^2} \right|, \left| c_i^2 \nabla^2 \varphi \right|. \tag{9.3}$$

Specifically, with the seawater acoustic speed $c_1 = c_1(y)$ in the present study, equation (9.2) becomes,

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial c_1^2}{\partial y} \frac{\partial \phi}{\partial y} - c_1^2 \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{9.4}$$

For the pressure wave and shear wave speeds $c_p(x_1, x_2)$ and $c_s(x_1, x_2)$ in the seafloor, the system becomes,

$$\begin{aligned} \frac{\partial^2 \phi'}{\partial t^2} - \frac{\partial c_p^2}{\partial x_i} \frac{\partial \phi'}{\partial x_i} - c_p^2 \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_i} \phi' \right) &= 0, \\ \frac{\partial^2 H_z}{\partial t^2} - \frac{\partial c_s^2}{\partial x_i} \frac{\partial H_z}{\partial x_i} - c_s^2 \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_i} H_z \right) &= 0. \end{aligned} \tag{9.5}$$

where z is the direction perpendicular to the propagation direction of the wave group and H_z is the only non-zero component of the vector potential $\boldsymbol{H}$ [see (Graff 1991)].

For the acoustic speed nonuniformity $c_1(y)$ in the water column, if

$$\begin{aligned} \left| \frac{\partial^2 \phi}{\partial t^2} \right|, \ c_1^2(y) \left| \frac{\partial^2 \phi}{\partial y^2} \right|, c_1(y) \left| \frac{\partial \phi}{\partial y} \right| \sim \mathcal{O}(1), \\ \text{then } \left| \frac{\partial c_1(y)}{\partial y} \right| \lesssim \mathcal{O}\left(10^{-1}\right) \end{aligned} \tag{9.6}$$

is needed. For the well known Munk profile (see Figure 5 and (Munk 1974)), c_1 changes by about 60 m/s from its nominal value of 1500 m/s over a depth of 1000 m, so the average slope is less than $\mathcal{O}\left(10^{-1}\right)$. A more exact estimation for $\partial c_1 / \partial y$ for the Munk profile can be obtained knowing that,

$$c_1(y) = c_{1o}(1+\epsilon f(y)), \text{ where } \epsilon f(y) = 7.37\times 10^{-3}\left(-\frac{2(y+1300)}{1300} - 1 + \exp(\frac{-2(y+1300)}{1300}\right). \tag{9.7}$$

Calculation of $\epsilon f'(y)$ shows that the mild nonuniformity approximation is satisfied by Munk profile when particle oscillations are of $\mathcal{O}\left(10^{-3}\right)$. If particle oscillations become larger than this but accelerations and the vertical gradients of particle oscillations do not experience equivalent increase, the present approximation may not be able to model accurately the propagation through the Munk acoustic speed profile.

Whereas the seafloor propagation speeds c_p and c_s may be difficult to quantify in some regions, it is assumed here that complex variations are amenable to expansion using polynomial basis functions. Here it is assumed further that the seafloor acoustic speeds vary linearly over a distance. Using the following variation for c_p (c_s variation being assumed to be similar),

$$c_p(\boldsymbol{x}) = c_{p0} + \frac{\Delta c_p}{|\boldsymbol{X}_c|}\boldsymbol{x}, \tag{9.8}$$

where $|\boldsymbol{X}_c|$ is the horizontal distance over which c_p changes by Δc_p. Then,

$$\boldsymbol{\nabla} c_p = \frac{\Delta c_p}{|\boldsymbol{X}_c|}\boldsymbol{\nabla}\cdot\boldsymbol{x}. \tag{9.9}$$

Now if

$$\left|\frac{\partial^2\phi'}{\partial t^2}\right|,\ c_p^2\left|\nabla^2\phi'\right| \sim \mathcal{O}(1),\ \text{then}$$

$$\left|\boldsymbol{\nabla} c_p\cdot\boldsymbol{\nabla}\cdot\phi'\right| \leqslant \frac{\Delta c_p}{|\boldsymbol{X}_{c1}|}\left|\frac{\partial\phi'}{\partial x_1}\right| + \frac{\Delta c_p}{|\boldsymbol{X}_{c2}|}\left|\frac{\partial\phi'}{\partial x_2}\right| \tag{9.10}$$

needs to be $\lesssim \mathcal{O}\left(10^{-1}\right)$ for the mild nonuniformity approximation to be valid. For example, if speed c_p changes linearly from 5200 m/s to 2800 m/s over 70 km, then $\Delta c_p/|X_c| \sim \mathcal{O}(10^{-2})$ and the present approximation is justified. Similar arguments apply to changes in c_s in the horizontal plane.

10. Supplementary Information II: Purely vertical acoustic-gravity waves

The seawater acoustic speed variation with y only appears implicitly in S_A, S_W, Π, and S_R above through the partial derivatives G_ω and $G_{\omega_k i}$. The effect of $c_1(y)$ on the vertically trapped waves can be understood more clearly by considering the case $\boldsymbol{k} = 0$, or $k_1 = 0$, and $k_2 = 0$. The function $G(\omega, \boldsymbol{k}, \alpha, \alpha', \beta', \boldsymbol{x}, y)$ then reduces to,

$$G(\omega, \alpha, \alpha', \beta', \boldsymbol{x}, y) = \left[(\omega^2 - c_1^2(y)\alpha^2)(1 + F_1(\alpha)) + (\omega^2 - c_p^2(\boldsymbol{x})\alpha'^2)F_2(\alpha, \alpha', \beta)\right] + \left[(\omega^2 - c_s^2(\boldsymbol{x})\beta'^2 F_3(\alpha, \alpha', \beta')\right]. \tag{10.1}$$

For vertical propagation with $\boldsymbol{k} = 0$, it is found more insightful to use the expansion containing dC_1 and dC_2 separately (equation (2.12)) with $k = 0$. In the analysis below, the part of the vertical wave system that goes into the seafloor but decays exponentially thereafter is neglected. The consistency conditions for downward and upward propagating waves are then, respectively,

$$\frac{\partial \alpha}{\partial t} + \overline{C}_\alpha \frac{\partial \alpha}{\partial y} = -\frac{\partial \omega}{\partial y},$$
$$\frac{\partial \alpha}{\partial t} - \overline{C}_\alpha \frac{\partial \alpha}{\partial y} = \frac{\partial \omega}{\partial y}, \tag{10.2}$$

where,

$$\overline{C}_\alpha = \frac{\partial \omega}{\partial \alpha}, \tag{10.3}$$

with the overbar indicating that $\overline{C}_\alpha$ is a vertical energy transfer rate for the downward and upward waves, which together make up the vertically trapped component of the acoustic-gravity waves.

A variational principle similar to equation (3.10) is proposed for the purely vertical system, with,

$$d\mathcal{L} = G(\omega, \alpha, \alpha', \beta', \boldsymbol{x}, y)dC_1 dC_1^* + G(\omega, \alpha, \alpha', \beta', \boldsymbol{x}, y)dC_2 dC_2^*. \tag{10.4}$$

The necessary conditions for the $k = 0$ case then are,

$$\frac{\partial(d\mathcal{L})}{\partial(dC_1)} = 0, \quad \frac{\partial(d\mathcal{L})}{\partial(dC_2)} = 0, \text{etc.},$$
$$\frac{\partial}{\partial t}\left(\frac{\partial d\mathcal{L}}{\partial \omega}\right) - \frac{\partial}{\partial y}\left(\frac{\partial d\mathcal{L}}{\partial \alpha}\right) = 0. \tag{10.5}$$

For a particular $\boldsymbol{x}$, the conditions in the first line of equation (10.5) imply that,

$$G(\omega, \alpha, \alpha', \beta', \boldsymbol{x}, y) = 0. \tag{10.6}$$

A partial differentiation with respect to α provides,

$$G_\omega \overline{C}_\alpha + G_\alpha = 0, \ \Rightarrow G_\alpha = \overline{C}_\alpha G_\omega. \tag{10.7}$$

Following a procedure similar to steps (3.19)–(3.25) independently for dC_1 and dC_2, for the downward component dC_1,

$$\frac{\partial}{\partial t}\left(dC_1 dC_1^*\right) + \frac{\partial}{\partial y}\left(\overline{C}_\alpha dC_1 dC_1^*\right) + \frac{1}{G_\omega}\left[\frac{\partial G_\omega}{\partial \alpha}\left(-\frac{\partial \omega}{\partial y}\right) + \frac{\partial G_\omega}{\partial y}\overline{C}_\alpha\right] dC_1 dC_1^* = 0. \tag{10.8}$$

For the upward component dC_2,

$$\frac{\partial}{\partial t}\left(dC_2 dC_2^*\right) - \frac{\partial}{\partial y}\left(\overline{C}_\alpha dC_2 dC_2^*\right) + \frac{1}{G_\omega}\left[\frac{\partial G_\omega}{\partial \alpha}\left(\frac{\partial \omega}{\partial y}\right) - \frac{\partial G_\omega}{\partial y}\overline{C}_\alpha\right] dC_2 dC_2^* = 0. \tag{10.9}$$

Here,

$$\frac{\partial G_y}{\partial y} = \frac{\partial G}{\partial c_1}\frac{\partial c_1}{\partial y} + \frac{\partial G}{\partial c_1}\frac{\partial c_1}{\partial y}. \tag{10.10}$$

A spectral distribution in terms of the vertical wave-number α can be expressed by dividing both equations by $d\alpha$ as,

$$\frac{\partial}{\partial t}\left(S_{C_1}(\alpha)\right) + \frac{\partial}{\partial y}\left(\overline{C}_\alpha S_{C_1}(\alpha)\right) + \frac{1}{G_\omega}\left[\frac{\partial G_\omega}{\partial \alpha}\left(-\frac{\partial \omega}{\partial y}\right) + \frac{\partial G_\omega}{\partial y}\overline{C}_\alpha\right] S_{C_1}(\alpha) = 0. \tag{10.11}$$

For the upward component dC_2,

$$\frac{\partial}{\partial t}\left(S_{C_2}(\alpha)\right) - \frac{\partial}{\partial y}\left(\overline{C}_\alpha S_{C_2}(\alpha)\right) + \frac{1}{G_\omega}\left[\frac{\partial G_\omega}{\partial \alpha}\left(\frac{\partial \omega}{\partial y}\right) - \frac{\partial G_\omega}{\partial y}\overline{C}_\alpha\right] S_{C_2}(\alpha) = 0. \tag{10.12}$$

Equations (10.11) and (10.12) just describe the distribution of upward and downward moving energy in the wave field trapped between the seafloor and the water surface, for the purely vertical free acoustic-gravity waves. The effect of acoustic speed dependence on y is represented by the third and fourth terms in each of the two equations, which capture change in ω and α for particular (ω, α) pairs and the corresponding distribution rate $\overline{C}_\alpha$ as the downward and upward components travel through the water column.