Math Note
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section 4

$$\left\langle \overline{\left(\omega_{z,0}+\omega_{z,1}+\omega_{z,2}\right)^3}\right\rangle = \Big\langle \overset{\text{small}}{\overline{\omega_{z,0}^3}} + \overset{=0}{\overline{\omega_{z,1}^3}} + \overset{=0}{\overline{\omega_{z,2}^3}} + \overset{=0}{\overline{3\omega_{z,0}^2\omega_{z,1}}} + \overset{\checkmark}{\overline{3\omega_{z,0}\omega_{z,1}^2}} + \overset{\checkmark}{\overline{3\omega_{z,1}^2\omega_{z,2}}}$$

$$+ \overset{=0}{\overline{3\omega_{z,1}\omega_{z,2}^2}} + \overset{=0}{\overline{3\omega_{z,0}^2\omega_{z,2}}} + \overset{\text{small}}{\overline{3\omega_{z,0}\omega_{z,2}^2}} + \overset{=0}{\overline{6\omega_{z,0}\omega_{z,1}\omega_{z,2}}}\Big\rangle$$

$$= 3\langle \omega_{z,1}^2\omega_{z,0}\rangle + 3\langle \omega_{z,1}^2\omega_{z,2}\rangle$$

n=1

$$\boxed{\begin{aligned}
&\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)\omega_{z,1} = \frac{1}{Ro}\frac{\partial w_1}{\partial z}\\
&\cancel{\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)}\delta_1 = -\nabla_h^2 p_1 + \frac{1}{Ro}\omega_{z,1} \quad \text{QG}\\
&\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right) w_1 = -\frac{\partial p_1}{\partial z} + T_1\\
&\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right) T_1 = w_1
\end{aligned}}$$

$\nu_e = Ra^{-1/2}$

$$\boxed{\begin{aligned}
&\widetilde{Ra}^{1/2} = Ra^{1/2}E^{2/3}\\
&\Uparrow\\
&\widetilde{Ra} = Ra\,E^{4/3}, \quad Ro = E\left(\frac{Ra}{Pr}\right)^{1/2} = E\,Ra^{1/2}Pr^{-1/2}\\
&\frac{Ro}{\widetilde{Ra}^{1/2}} = \frac{E\,Ra^{1/2}Pr^{-1/2}}{Ra^{1/2}E^{2/3}} = E^{\frac{1}{3}}Pr^{-1/2}\\
&\Rightarrow E^{\frac{1}{3}} = Ro\,\widetilde{Ra}^{-1/2}Pr^{1/2}\\
&\quad E = Ro^3\,\widetilde{Ra}^{-3/2}Pr^{3/2}
\end{aligned}}$$

n=2

$$\boxed{\begin{aligned}
&\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)\omega_{z,2} = \frac{1}{Ro}\frac{\partial w_2}{\partial z}\\
&\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right) w_2 = -\frac{\partial p_2}{\partial z} + T_2 - (\vec{u}_1\cdot\nabla)w_1\\
&\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right) T_2 = -(\vec{u}_1\cdot\nabla)T_1 + w_2\\
&\nabla_h^2 p_2 - \frac{1}{Ro}\omega_{z,2} = 0
\end{aligned}}$$

n=0

$$\boxed{\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)\omega_{z,0} = -(\vec{u}_1\cdot\nabla)\omega_{z,1} + (\vec{\omega}_{h,1}\cdot\nabla)w_1}$$

$$\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)T_2 = -(\vec{u}_1\cdot\nabla)T_1 + w_2$$

$$\Big\downarrow \leftarrow T_2 = \frac{\partial p_2}{\partial z} + \left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)w_2 + (\vec{u}_1\cdot\nabla)w_1$$

$$\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)\frac{\partial p_2}{\partial z} = -\vec{u}_1\cdot\nabla T_1 + w_2 - \left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)^2 w_2 - \left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)(\vec{u}_1\cdot\nabla)w_1$$

$$\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)\frac{\partial}{\partial z}(\nabla_h^2 p_2) = -\nabla_h^2(\vec{u}_1\cdot\nabla T_1) + \nabla_h^2 w_2 - \left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)^2\nabla_h^2 w_2 - \left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)\nabla_h^2\left[(\vec{u}_1\cdot\nabla)w_1\right]$$

$$\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)\frac{\partial}{\partial z}\left(\frac{1}{Ro}\omega_{z,2}\right) = -\nabla_h^2(\vec{u}_1\cdot\nabla T_1) + \nabla_h^2 w_2 - \left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)^2\nabla_h^2 w_2 - \left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)\nabla_h^2\left[(\vec{u}_1\cdot\nabla)w_1\right]$$

(The Omega equation for nonhydrostatic QG)

$$\frac{1}{Ro}\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)\omega_{z,2} = \frac{1}{Ro^2}\frac{\partial w_2}{\partial z} \longrightarrow \left[\underbrace{\frac{1}{Ro^2}\frac{\partial^2}{\partial z^2}}_{\text{from } p_2} \underbrace{-\nabla_h^2}_{\text{from } T\text{adv}} + \left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)^2\nabla_h^2\right]w_2 = \underbrace{-\left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)\nabla_h^2(\vec{u}_1\cdot\nabla w_1)}_{\text{momentum adv}} \underbrace{-\nabla_h^2(\vec{u}_1\cdot\nabla T_1)}_{\text{temperature adv.}}$$

The follow-up derivation in Appendix C, using the single-scale approx. if RHS=0, reduce to the linear eigenvalue problem

Along the axis, $$\left[-\frac{4\pi^2}{Ro^2} + 2k_m^2 - \sigma_2^2\,2k_m^2\right]w_2 = 2k_m^2\left[\sigma_2(\vec{u}_1\cdot\nabla)w_1 + (\vec{u}_1\cdot\nabla)T_1\right] \leftarrow \sigma_1 T_1 = w_1 \leftarrow \left(\frac{\partial}{\partial t}-\nu_e\nabla_h^2\right)T_1 = w_1$$

$$= 2k_m^2\left[\sigma_2(\vec{u}_1\cdot\nabla)w_1 + \frac{1}{\sigma_1}(\vec{u}_1\cdot\nabla)w_1\right]$$

$$\boxed{\begin{aligned} &\text{for } n=1, \quad \frac{1}{Ro^2}(-\pi^2)+K^2+(\sigma+\nu_e K^2)^2(-K^2)=0 \\ &(\sigma+\nu_e K^2)^2 K^2 = K^2-\frac{\pi^2}{R_o^2} \\ &\sigma = \left(\frac{K^2-\frac{\pi^2}{Ro^2}}{K^2}\right)^{1/2} - \nu_e k^2 \end{aligned}}$$

$$\downarrow$$

$$\sigma_1^2 = \frac{K^2-\frac{\pi^2}{Ro^2}}{K^2}$$

$$\sigma_2 w_2 = -(\vec{u}_1\cdot\nabla)w_1 \cdot \underbrace{\frac{2K_m^2\left(\sigma_2^2+\frac{\sigma_2}{\sigma_1}\right)}{\frac{4\pi^2}{Ro^2}+2K_m^2\left(\sigma_2^2-1\right)}}_{\mu}$$

$$\begin{aligned} \mu &= \frac{2K_m^2(4\sigma_1^2+2)}{\frac{4\pi^2}{Ro^2}+2K_m^2(4\sigma_1^2-1)} \\ &= \frac{2K_m^2\left(4-\frac{4\pi^2}{Ro^2K_m^2}+2\right)}{\frac{4\pi^2}{Ro^2}+2K_m^2\left(4-\frac{4\pi^2}{Ro^2K_m^2}-1\right)} \\ &= \frac{2K_m^2\left(6-\frac{4\pi^2}{Ro^2K_m^2}\right)}{-\frac{4\pi^2}{Ro^2}+6K_m^2} \\ &= \frac{12K_m^2-\frac{8\pi^2}{Ro^2}}{6K_m^2-\frac{4\pi^2}{Ro^2}} \\ &= 2 \end{aligned}$$

Section 5:

$$3\omega_z^2(\vec{u}\cdot\nabla)\omega_z = (\vec{u}\cdot\nabla)\omega_z^3 \Rightarrow \langle \overline{3\omega_z^2(\vec{u}\cdot\nabla)\omega_z}\rangle = \langle \overline{\nabla\cdot(\vec{u}\,\omega_z^3)}\rangle = 0$$

$$\begin{aligned} &3\omega_z^2\nabla^2\omega_z \\ &= 3\omega_z^2\frac{\partial^2}{\partial x^2}\omega_z + 3\omega_z^2\frac{\partial^2}{\partial y^2}\omega_z + 3\omega_z^2\frac{\partial^2}{\partial z^2}\omega_z \\ &= 3\omega_z\frac{\partial}{\partial x}\left(\omega_z\frac{\partial\omega_z}{\partial x}\right) - 3\omega_z\left(\frac{\partial\omega_z}{\partial x}\right)^2 + 3\omega_z\frac{\partial}{\partial y}\left(\omega_z\frac{\partial\omega_z}{\partial y}\right) - 3\omega_z\left(\frac{\partial\omega_z}{\partial y}\right)^2 + 3\omega_z\frac{\partial}{\partial z}\left(\omega_z\frac{\partial\omega_z}{\partial z}\right) - 3\omega_z\left(\frac{\partial\omega_z}{\partial z}\right)^2 \\ &= 3\frac{\partial}{\partial x}\left(\omega_z^2\frac{\partial\omega_z}{\partial x}\right) - 6\omega_z\left(\frac{\partial\omega_z}{\partial x}\right)^2 + 3\frac{\partial}{\partial y}\left(\omega_z^2\frac{\partial\omega_z}{\partial y}\right) - 6\omega_z\left(\frac{\partial\omega_z}{\partial y}\right)^2 + 3\frac{\partial}{\partial z}\left(\omega_z^2\frac{\partial\omega_z}{\partial z}\right) - 6\omega_z\left(\frac{\partial\omega_z}{\partial z}\right)^2 \\ &= \frac{\partial^2}{\partial x^2}\omega_z^3 + \frac{\partial^2}{\partial y^2}\omega_z^3 + \frac{\partial^2}{\partial z^2}\omega_z^3 - 6\omega_z\left[\left(\frac{\partial\omega_z}{\partial x}\right)^2+\left(\frac{\partial\omega_z}{\partial y}\right)^2+\left(\frac{\partial\omega_z}{\partial z}\right)^2\right] \end{aligned}$$

$$\begin{aligned} \langle\overline{3\omega_z^2\nabla^2\omega_z}\rangle &= \langle\overline{\nabla_h^2\omega_z^3}\rangle - 6\langle\overline{\omega_z|\nabla\omega_z|^2}\rangle \\ &= \left\langle\overline{\frac{\partial^2\omega_z^3}{\partial z^2}}\right\rangle - 6\langle\overline{\omega_z|\nabla\omega_z|^2}\rangle \\ &= \left\langle \frac{\partial}{\partial z}\omega_z^3\Big|_{z=1} - \frac{\partial}{\partial z}\omega_z^3\Big|_{z=0}\right\rangle - 6\langle\overline{\omega_z|\nabla\omega_z|^2}\rangle \\ &= \left\langle \underbrace{3\omega_z^2\frac{\partial\omega_z}{\partial z}\Big|_{z=1}}_{=0} - \underbrace{3\omega_z^2\frac{\partial\omega_z}{\partial z}\Big|_{z=0}}_{=0}\right\rangle - 6\langle\overline{\omega_z|\nabla\omega_z|^2}\rangle \\ &= -6\langle\overline{\omega_z|\nabla\omega_z|^2}\rangle \end{aligned}$$

Appendix B:

$$u_{r,1} = \frac{\partial \phi_1}{\partial r}, \quad u_{\theta,1} = \frac{\partial \psi_1}{\partial r}$$

$$w_1 = -\int \nabla_h^2 \phi_1 \, dz, \quad \omega_{z,1} = \nabla_h^2 \psi_1$$

$$\nabla_h^2 \phi_1 = -K_m^2 \phi_1, \quad \nabla_h^2 \psi_1 = -K_m^2 \psi_1$$

$$\phi_1 = \Phi_1(r)\cos(\pi z), \quad \psi_1 = -\gamma\, \Phi_1(r)\cos(\pi z)$$

$$-u_{r,1}\frac{\partial \omega_{z,1}}{\partial r} - \frac{\partial u_{\theta,1}}{\partial z}\frac{\partial w_1}{\partial r}$$

$$= -\frac{\partial \phi_1}{\partial r}\frac{\partial}{\partial r}\nabla_h^2 \psi_1 - \frac{\partial}{\partial z}\left(\frac{\partial \psi_1}{\partial r}\right)\left(-\int \nabla_h^2 \frac{\partial \phi}{\partial r} dz\right)$$

$$= -\frac{d\Phi_1}{dr}(-K_m^2)\left(-\gamma\frac{d\Phi_1}{dr}\right)\cos^2(\pi z) - \left(-\gamma\frac{d\Phi_1}{dr}(-\pi)\sin(\pi z)\right)\left(-\frac{d\Phi_1}{dr}(-K_m^2)\int \cos \pi z \, dz\right)$$

$$= -\gamma K_m^2\left(\frac{d\Phi_1}{dr}\right)^2\cos^2(\pi z) - \gamma K_m^2\left(\frac{d\Phi_1}{dr}\right)^2 \pi \sin(\pi z)\cdot\frac{1}{\pi}\sin(\pi z)$$

$$= -\gamma K_m^2\left(\frac{d\Phi_1}{dr}\right)^2\cos^2(\pi z) - \gamma K_m^2\left(\frac{d\Phi_1}{dr}\right)^2\sin^2(\pi z)$$

$$= -\gamma K_m^2\left(\frac{d\Phi_1}{dr}\right)^2$$

Appendix C

$$\sigma_1 \Omega_1 \cos(\pi z) \sim \frac{1}{R_0}\frac{d W_1 \sin(\pi z)}{dz} \rightarrow \sigma_1 \Omega_1 \sim \frac{\pi}{R_0} W_1 \rightarrow W_1 \sim \frac{R_0}{\pi}\cdot \sigma_1 \Omega_1$$

$$\sigma_2 \Omega_2 \cos(2\pi z) \sim \frac{1}{R_0}\frac{d W_2 \sin(2\pi z)}{dz} \rightarrow \sigma_2 \Omega_2 \sim \frac{2\pi}{R_0} W_2 \rightarrow W_2 \sim \frac{R_0}{2\pi}\sigma_2 \Omega_2$$

$$\sigma_2 W_2 \sin(2\pi z) \sim -\mu W_1 \sin(\pi z)\frac{d W_1 \sin(\pi z)}{dz}$$

$$\sim -\mu W_1^2 \pi \sin(\pi z)\cos(\pi z)$$

$$\sim -\frac{\mu}{2} W_1^2 \pi \sin(2\pi z)$$

$$\downarrow$$

$$\sigma_2 W_2 \sim -\frac{\mu}{2}\pi W_1^2 \longrightarrow \sigma_2 \frac{R_0}{2\pi}\sigma_2 \Omega_2 \sim -\frac{\mu}{2}\pi \frac{R_0^2}{\pi^2}\sigma_1^2 \Omega_1^2$$

$$\sigma_2^2 R_0 \Omega_2 \sim -\mu R_0^2 \sigma_1^2 \Omega_1^2$$

$$\Omega_2 \sim -\mu R_0^2 \frac{\sigma_1^2}{\sigma_2^2}\frac{1}{R_0}\Omega_1^2$$

$$\sim -\mu R_0 \frac{\sigma_1^2}{\sigma_2^2}\Omega_1^2$$

$$\sim -2 \times \left(\frac{1}{2}\right)^2 \times R_0 \Omega_1^2$$

$$\sim -\frac{Ro}{2}\Omega_1^2$$

$$\sigma_2 \Omega_0 \sim -W_1 \sin(\pi z)\frac{d}{dz}\left(\Omega_1 \cos(\pi z)\right) + \Omega_1 \cos(\pi z)\frac{d\, W_1 \sin(\pi z)}{dz}$$

$$\sim \pi W_1 \Omega_1 \sin^2(\pi z) + \pi W_1 \Omega_1 \cos^2(\pi z)$$

$$\sim \pi W_1 \Omega_1$$

$$\Omega_0 \sim \frac{\pi W_1}{\sigma_2}\Omega_1 \sim \frac{\pi}{\sigma_2}\frac{Ro}{\pi}\sigma_1 \Omega_1 \Omega_1 \sim Ro\frac{\sigma_1}{\sigma_2}\Omega_1^2 \sim \frac{Ro}{2}\Omega_1^2$$

$$S \approx 3\frac{\overline{\langle \omega_{z,1}^2 \omega_{z,2}\rangle}}{\langle \overline{\omega_{z,1}^2}\rangle^{3/2}} + 3\frac{\overline{\langle \omega_{z,1}^2 \omega_{z,0}\rangle}}{\langle \overline{\omega_{z,1}^2}\rangle^{3/2}}$$

$$\sim 3\frac{\Omega_1^2\Omega_2}{\Omega_1^3}\frac{\overline{\cos^2(\pi z)\cos(2\pi z)}}{\overline{\cos^2(\pi z)}^{3/2}} + 3\frac{\Omega_1^2\Omega_0}{\Omega_1^3}\frac{\overline{\cos^2(\pi z)}}{\overline{\cos^2(\pi z)}^{3/2}}$$

$$\sim 3\frac{\Omega_2}{\Omega_1}\cdot\frac{\frac{1}{4}}{(\frac{1}{2})^{3/2}} + 3\frac{\Omega_0}{\Omega_1}\frac{\frac{1}{2}}{(\frac{1}{2})^{3/2}} \quad \leftarrow \Omega_2 \sim -\frac{Ro}{2}\Omega_1^2,\ \Omega_0 \sim \frac{Ro}{2}\Omega_1^2$$

$$\sim -3\frac{Ro\,\Omega_1}{2}\frac{2\sqrt{2}}{4} + 3\frac{Ro\,\Omega_1}{2}\frac{2\sqrt{2}}{2}$$

$$\sim Ro\,\Omega_1\left(\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{4}\right)$$

$$\sim Ro\,\Omega_1\frac{3\sqrt{2}}{4}$$

$$\boxed{\begin{aligned} \overline{\cos^2(\pi z)} &= \frac{1}{2} \\ \overline{\cos^2(\pi z)\cos(2\pi z)} &= \overline{\frac{1+\cos(2\pi z)}{2}\cos(2\pi z)} \\ &= \frac{1}{2}\overline{\cos(2\pi z)\cos(2\pi z)} \\ &= \frac{1}{4} \end{aligned}}$$

$$Ro_g \approx \frac{\langle \overline{\omega_{z,1}^2}\rangle^{1/2}}{Ro^{-1}}$$

$$\sim Ro\,\Omega_1\,\overline{\cos^2(\pi z)}^{1/2}$$

$$\sim Ro\,\Omega_1\frac{\sqrt{2}}{2}$$

$$\Rightarrow \frac{S}{Ro_g} \sim \frac{3}{2}$$