Supplementary Material: Full derivation of asymmetric Marangoni thin film equations

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1 Introduction

The full derivation for the thin film equations given at the end of section 3 in the paper is given in this supplementary material. It should be explicitly noted that the derivation is analogous to the derivations previously given in the literature [1, 2].

Sections 2-5 in this supplementary material are identical to sections 2-3 in the paper. Section 6 gives the full detail of the next-order expansion that was skipped over in the paper (i.e., the details of section 3.4 in the paper).

2 Governing equations and boundary conditions

We consider an incompressible, two-dimensional Newtonian film (viscosity μ, density ρ) with the horizontal direction given by the x axis and the vertical direction given by the z axis. The top surface of the film is given by $z = H(x,t) + \frac{1}{2}h(x,t)$ and the bottom surface of the film is given by $z = H(x,t) - \frac{1}{2}h(x,t)$ (see Fig. 1). We assume that the fluid of the thin film is not coupled to the surrounding fluid (typically, the thin film is surrounded by air otherwise at rest).

Let $\epsilon := \frac{\mathcal{H}}{\mathcal{L}}$ be the aspect ratio between the characteristic vertical scale $\mathcal{H}$ (e.g., the film thickness) and horizontal scale $\mathcal{L}$ (e.g., a typical wavelength of a perturbation). Assume that $\epsilon \ll 1$, i.e., a thin film. The velocity field is denoted by (u, w) where u is the horizontal velocity and w is the vertical velocity. As we focus on the influence of asymmetries, we consider variable surface tension fields on the top and bottom interfaces, given by $\sigma_+(x,t)$ and $\sigma_-(x,t)$, respectively.

The two-dimensional continuity and Navier-Stokes equations apply to the fluid between $H - \frac{1}{2}h \leq z \leq H + \frac{1}{2}h$,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{2}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right). \tag{3}$$

The outward normals of the top and bottom surfaces are, respectively, denoted $\boldsymbol{n}_\pm$ and the corresponding unit tangent vectors (in the direction of increasing x) are denoted, respectively, $\boldsymbol{t}_\pm$ (see Fig. 1). For each of the two surfaces, $z = H(x,t) \pm \frac{1}{2}h(x,t)$, we have one kinematic boundary condition

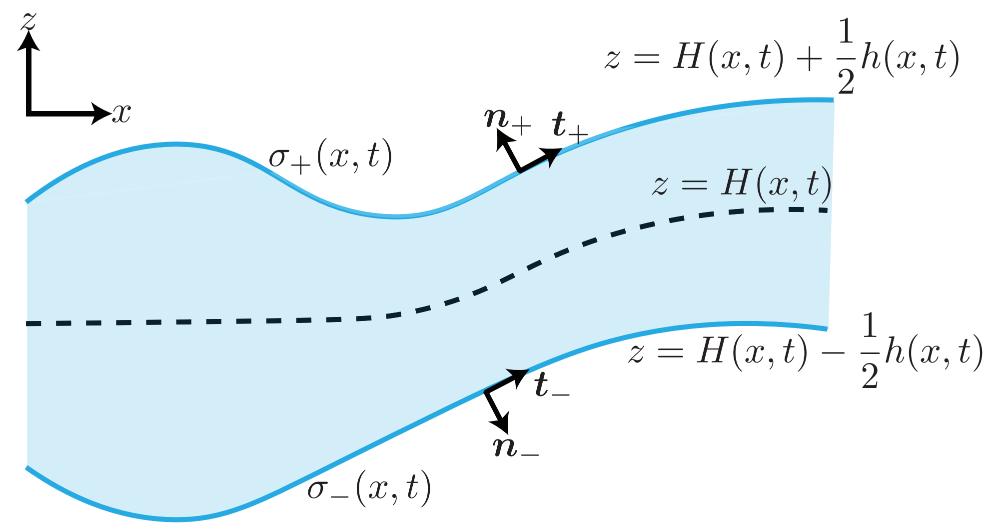

Figure 1: Schematic of a thin film with curved centreline $H(x,t)$ and thickness $h(x,t)$ (the x axis is in the horizontal direction, the z axis is in the vertical direction, and t is time). The top surface of the film is given by $z = H(x,t) + \frac{1}{2}h(x,t)$ and the bottom surface of the film is given by $z = H(x,t) - \frac{1}{2}h(x,t)$. The outward normal of the top/bottom surface is denoted $\boldsymbol{n}_\pm$ and the tangential vector (in the direction of increasing x) on the top/bottom surface is denoted $\boldsymbol{t}_\pm$. Surface tension at the top/bottom of the sheet is given by $\sigma_\pm(x,t)$.

and two dynamic boundary conditions. For notational convenience, let $z^\pm := H(x,t) \pm \frac{1}{2}h(x,t)$.The kinematic boundary conditions at the top and bottom surfaces, $z = z^\pm$, are given by

$$w|_{z=z^\pm} = \left(\frac{\partial H}{\partial t} \pm \frac{1}{2}\frac{\partial h}{\partial t}\right) + u|_{z=z^\pm}\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right). \tag{4}$$

Consider the $\boldsymbol{n}_\pm$ and $\boldsymbol{t}_\pm$ directions. Denote the curvatures of the surfaces by $\kappa_\pm$. The normal and tangential stress boundary conditions at the top and bottom surfaces, $z = z^\pm$ are given, respectively, by the two equations

$$\begin{aligned} \sigma_\pm \kappa_\pm &= \Bigg(p - \mu \frac{2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}{1 + \left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}\frac{\partial u}{\partial x} \\ &\quad + \mu \frac{2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)}{1 + \left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \mu \frac{2}{1 + \left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}\frac{\partial w}{\partial z}\Bigg)\Bigg|_{z=z^\pm}, \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{\partial \sigma_\pm}{\partial x} &= \Bigg(\pm\mu \frac{2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)}{\sqrt{1 + \left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}}\left(\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}\right) \\ &\quad \pm\mu \frac{1 - \left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}{\sqrt{1 + \left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\Bigg)\Bigg|_{z=z^\pm}, \end{aligned} \tag{6}$$

where the curvatures $\kappa_\pm$ of the top and bottom surfaces, $z = z^\pm$, are given by

$$\kappa_\pm = \frac{\mp\frac{\partial^2 H}{\partial x^2} - \frac{1}{2}\frac{\partial^2 h}{\partial x^2}}{\left(1 + \left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2\right)^{\frac{3}{2}}}. \tag{7}$$

3 Non-dimensionalisation

We first non-dimensionalise the equations according to

$$x = \mathcal{L}\tilde{x}, u = \mathcal{U}\tilde{u}, H = \epsilon\mathcal{L}\tilde{H}, t = \frac{\mathcal{L}}{\mathcal{U}}\tilde{t}, y = \epsilon\mathcal{L}\tilde{y}, w = \epsilon\mathcal{U}\tilde{w}, h = \epsilon\mathcal{L}\tilde{h}, p = \frac{\mu\mathcal{U}}{\mathcal{L}}\tilde{p}, \sigma = \Sigma + \Delta\Sigma\tilde{\sigma} \tag{8}$$

where $\mathcal{U}$ is some constant characteristic velocity, Σ is some constant characteristic surface tension and $\Delta\Sigma$ is some characteristic surface tension variation of interest in the problem. There are three dimensionless parameters: Reynolds number $Re = \frac{\rho\mathcal{U}\mathcal{L}}{\mu}$, Marangoni number $M = \frac{\Delta\Sigma}{\epsilon\mu\mathcal{U}}$, capillary number $C = \frac{\epsilon\mu\mathcal{U}}{\Sigma}$. The parameter ϵ is included in the definition of M and C in a way such that the later discussed thresholds for the extensional flow conditions are $O(1)$ (see (17) and (18)). Note that in the nondimensionalisation (8), we are considering timescales $\frac{\mathcal{L}}{\mathcal{U}}$. Henceforth, the tilde $(\tilde{\ })$ will be omitted.

3.1 Non-dimensional equations

The bulk equations (1, 2, 3) are

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{9}$$

$$Re\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 u}{\partial z^2}\right), \tag{10}$$

$$Re\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) = -\frac{1}{\epsilon^2}\frac{\partial p}{\partial z} + \left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 w}{\partial z^2}\right). \tag{11}$$

The kinematic boundary conditions (4) are given by

$$w|_{z=z^\pm} = \left(\frac{\partial H}{\partial t} \pm \frac{1}{2}\frac{\partial h}{\partial t}\right) + u|_{z=z^\pm}\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right). \tag{12}$$

The normal and tangential stress boundary conditions (5, 6) are given by

$$\frac{\epsilon^2}{C}(1 + MC\sigma_\pm)\kappa_\pm = \Bigg(p - \frac{2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}{1 + \epsilon^2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}\epsilon^2\frac{\partial u}{\partial x} + \frac{2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)}{1 + \epsilon^2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}\left(\frac{\partial u}{\partial z} + \epsilon^2\frac{\partial w}{\partial x}\right) - \frac{2}{1 + \epsilon^2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}\frac{\partial w}{\partial z}\Bigg)\Bigg|_{z=z^\pm} \tag{13}$$

$$\epsilon^2 M\frac{\partial \sigma_\pm}{\partial x} = \Bigg(\pm\frac{2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)\epsilon^2}{\sqrt{1 + \epsilon^2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}}\left(\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}\right) \pm \frac{1 - \epsilon^2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}{\sqrt{1 + \epsilon^2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2}}\left(\frac{\partial u}{\partial z} + \epsilon^2\frac{\partial w}{\partial x}\right)\Bigg)\Bigg|_{z=z^\pm}, \tag{14}$$

where the curvatures $\kappa_\pm$ (7) are given by

$$\kappa_\pm = \frac{\mp\frac{\partial^2 H}{\partial x^2} - \frac{1}{2}\frac{\partial^2 h}{\partial x^2}}{\left(1 + \epsilon^2\left(\frac{\partial H}{\partial x} \pm \frac{1}{2}\frac{\partial h}{\partial x}\right)^2\right)^{\frac{3}{2}}}. \tag{15}$$

4 Conditions for extensional flow

In order to have extensional flow, i.e., $u(x,z,t) \approx u(x,t)$ and $p(x,z,t) \approx p(x,t)$ to leading order in ϵ, we take three conditions. The first condition is that inertia does not play a dominant role:

$$Re\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = O(1) \text{ or less.} \tag{16}$$

This condition yields $\frac{\partial^2 u}{\partial z^2} = 0$ to leading order - see (20) below. The second condition is that the tangential stress is not too large:

$$M\frac{\partial \sigma_\pm}{\partial x} = O(1) \text{ or less.} \tag{17}$$

This condition yields the horizontal velocity field $u(x,z,t) \approx u(x,t)$ - see (21) below. The third condition is that the capillary pressure due to the surface tension is not too large:

$$\frac{1}{C}\left(1 + MC\sigma_\pm\right)\kappa_\pm = O(1) \text{ or less.} \tag{18}$$

This condition yields the pressure field $p(x,z,t) \approx p(x,t)$ - see (26) below.

5 Leading order asymptotic expansion

5.1 Derivation of leading order equations

We expand the horizontal velocity as

$$u(x,z,t) = u_0(x,z,t) + \epsilon^2 u_1(x,z,t) + \ldots \tag{19}$$

and consider analogous expansions for w, H, h, and p. We consider the leading-order expansion first where we ignore relative errors of order $O(\epsilon^2)$. The horizontal momentum equation (10) using the first condition (16) gives

$$\frac{\partial^2 u_0}{\partial z^2} = 0. \tag{20}$$

The tangential boundary conditions (14) using the second condition (17) give

$$\left.\frac{\partial u_0}{\partial z}\right|_{z=z^\pm} = 0, \tag{21}$$

which show that the leading-order horizontal velocity is independent of z:

$$u_0 = u_0(x,t). \tag{22}$$

Then, continuity (9) gives

$$w_0 = \overline{w}(x,t) - (z - H_0)\frac{\partial u_0}{\partial x} \tag{23}$$

where $\overline{w}$ denotes the average of w in the vertical direction $\left(\overline{f} := \frac{1}{h}\int_{z^-}^{z^+} f(x,z,t)dz\right)$. Next, turning to the vertical momentum equation (11), using (16) gives

$$\frac{\partial p_0}{\partial z} = \frac{\partial^2 w_0}{\partial z^2}, \tag{24}$$

which upon substitution of (23) yields

$$p_0 = p_0(x,t). \tag{25}$$

Now, we consider the normal stress boundary conditions (13) where continuity (9) gives:

$$0 = p_0 + 2\frac{\partial u_0}{\partial x}. \tag{26}$$

Finally, the kinematic boundary conditions (12), using (23) for w_0, give

$$\overline{w} \mp \frac{1}{2}h_0 u_0 = \left(\frac{\partial H_0}{\partial t} \pm \frac{1}{2}\frac{\partial h_0}{\partial t}\right) + u_0\left(\frac{\partial H_0}{\partial x} \pm \frac{1}{2}\frac{\partial h_0}{\partial x}\right), \tag{27}$$

where (27) yields

$$\frac{\partial H_0}{\partial t} + u_0\frac{\partial H_0}{\partial x} = \overline{w} \tag{28}$$

and

$$\frac{\partial h_0}{\partial t} + \frac{\partial}{\partial x}(u_0 h_0) = 0. \tag{29}$$

5.2 Complete leading-order equations

The leading-order equations are given in this subsection for completeness. They are

$$\frac{\partial h_0}{\partial t} + \frac{\partial}{\partial x}(u_0 h_0) = 0, \tag{30}$$

$$\frac{\partial H_0}{\partial t} + u_0\frac{\partial H_0}{\partial x} = \overline{w}, \tag{31}$$

$$p_0 = -2\frac{\partial u_0}{\partial x}, \tag{32}$$

$$u_0 = u_0(x,t), \tag{33}$$

$$w_0 = \overline{w}(x,t) - (z - H_0)\frac{\partial u_0}{\partial x}. \tag{34}$$

6 Next-order asymptotic expansion

In this section, we consider the next-order terms (i.e. $O(\epsilon^2)$). Since the algebra becomes long, a symbolic manipulation package is used (Mathematica). The goal is to give a closed set of PDEs for $h_0(x,t)$, $H_0(x,t)$, $u_0(x,t)$ and $\overline{w}(x,t)$.

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6.1 Derivation of next order equations

6.1.1 Horizontal momentum equation and tangential stress

From considering the $O(\epsilon^2)$ terms of the horizontal momentum equation (10), we deduce using (32,33) that

$$\frac{\partial^2 u_1}{\partial z^2} = Re\left(\frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x}\right) - 3\frac{\partial^2 u_0}{\partial x^2}. \tag{35}$$

Integrating (35) once, we have

$$\frac{\partial u_1}{\partial z} = G(x,t) + (z - H_0)\left(Re\left(\frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x}\right) - 3\frac{\partial^2 u_0}{\partial x^2}\right) \tag{36}$$

for some $G(x,t)$.

The tangential stress boundary condition at the top (14) give

$$M\frac{\partial \sigma_\pm}{\partial x} = \pm 2\left(\frac{\partial H_0}{\partial x} \pm \frac{1}{2}\frac{\partial h_0}{\partial x}\right)\left(\frac{\partial w_0}{\partial z}|_{z=z^\pm} - \frac{\partial u_0}{\partial x}|_{z=z^\pm}\right) \pm \frac{\partial u_1}{\partial z}|_{z=z^\pm} \pm \frac{\partial w_0}{\partial x}|_{z=z^\pm} \tag{37}$$

which upon rearranging gives

$$\frac{\partial u_1}{\partial z}|_{z=z^\pm} = \pm M\frac{\partial \sigma_\pm}{\partial x} - 2\left(\frac{\partial H_0}{\partial x} \pm \frac{1}{2}\frac{\partial h_0}{\partial x}\right)\left(\frac{\partial w_0}{\partial z}|_{z=z^\pm} - \frac{\partial u_0}{\partial x}|_{z=z^\pm}\right) - \frac{\partial w_0}{\partial x}|_{z=z^\pm}. \tag{38}$$

Upon substitution of (33, 34, 36) into (38) we find an evolution equation for u_0 and an expression for G:

$$Re h_0\left(\frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x}\right) = M\left(\frac{\partial \sigma_+}{\partial x} + \frac{\partial \sigma_-}{\partial x}\right) + 4\frac{\partial}{\partial x}\left(h_0\frac{\partial u_0}{\partial x}\right), \tag{39}$$

$$G(x,t) = \frac{M}{2}\left(\frac{\partial \sigma_+}{\partial x} - \frac{\partial \sigma_-}{\partial x}\right) - \frac{\partial \overline{w}}{\partial x} + 3\frac{\partial H_0}{\partial x}\frac{\partial u_0}{\partial x}. \tag{40}$$

6.1.2 Vertical momentum equation and normal stress

The $O(\epsilon^2)$ terms of (11) gives

$$Re\left(\frac{\partial w_0}{\partial t} + u_0\frac{\partial w_0}{\partial x} + w_0\frac{\partial w_0}{\partial z}\right) = -\frac{\partial p_1}{\partial z} + \frac{\partial^2 w_1}{\partial z^2} + \frac{\partial^2 w_0}{\partial x^2}. \tag{41}$$

Before taking next steps, we rearrange the left-hand side of (41) as:

$$\begin{aligned} Re\left(\frac{\partial w_0}{\partial t} + u_0\frac{\partial w_0}{\partial x} + w_0\frac{\partial w_0}{\partial z}\right) &= Re\left(\frac{\partial \overline{w}}{\partial t} + u_0\frac{\partial \overline{w}}{\partial x}\right) \\ &+ Re\left(-\overline{w}\frac{\partial u_0}{\partial x} + \frac{\partial H_0}{\partial t}\frac{\partial u_0}{\partial x} + u_0\frac{\partial H_0}{\partial x}\frac{\partial u_0}{\partial x} + (z-H_0)\left(\left(\frac{\partial u_0}{\partial x}\right)^2 - \frac{\partial^2 u_0}{\partial x \partial t} - u_0\frac{\partial^2 u_0}{\partial x^2}\right)\right). \end{aligned} \tag{42}$$

The use of (31) can be used to eliminate $\frac{\partial H_0}{\partial t}$:

$$Re\left(\frac{\partial w_0}{\partial t} + u_0\frac{\partial w_0}{\partial x} + w_0\frac{\partial w_0}{\partial z}\right) = Re\left(\frac{\partial \overline{w}}{\partial t} + u_0\frac{\partial \overline{w}}{\partial x}\right) + Re(z-H_0)\left(\left(\frac{\partial u_0}{\partial x}\right)^2 - \frac{\partial^2 u_0}{\partial x \partial t} - u_0\frac{\partial^2 u_0}{\partial x^2}\right). \tag{43}$$

Also, (34) gives

$$\frac{\partial^2 w_0}{\partial x^2} = \frac{\partial u_0}{\partial x}\frac{\partial^2 H_0}{\partial x^2} + \frac{\partial^2 \overline{w}}{\partial x^2} + 2\frac{\partial H_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2} - (z-H_0)\frac{\partial^3 u_0}{\partial x^3}. \tag{44}$$

We can then integrate (41) to get

$$\begin{aligned} p_1 = F(x,t) + \frac{\partial w_1}{\partial z} - Re(z-H_0)\left(\frac{\partial \overline{w}}{\partial t} + u_0\frac{\partial \overline{w}}{\partial x}\right) - Re\frac{(z-H_0)^2}{2}\left(\left(\frac{\partial u_0}{\partial x}\right)^2 - \frac{\partial^2 u_0}{\partial x \partial t} - u_0\frac{\partial^2 u_0}{\partial x^2}\right) \\ + (z-H_0)\left(\frac{\partial u_0}{\partial x}\frac{\partial^2 H_0}{\partial x^2} + \frac{\partial^2 \overline{w}}{\partial x^2} + 2\frac{\partial H_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}\right) - \frac{(z-H_0)^2}{2}\frac{\partial^3 u_0}{\partial x^3} \end{aligned} \tag{45}$$

for some $F(x,t)$. The normal stress conditions (13) give

$$\begin{aligned} \frac{1}{C}\left(1 + MC\sigma_\pm\right)\kappa_\pm = p_1|_{z=z^\pm} - 2\left(\frac{\partial H_0}{\partial x} \pm \frac{1}{2}\frac{\partial h_0}{\partial x}\right)^2\frac{\partial u_0}{\partial x} + 2\left(\frac{\partial H_0}{\partial x} \pm \frac{1}{2}\frac{\partial h_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial z}|_{z=z^\pm} + \frac{\partial w_0}{\partial x}|_{z=z^\pm}\right) \\ - 2\left(\frac{\partial H_0}{\partial x} \pm \frac{1}{2}\frac{\partial h_0}{\partial x}\right)^2\frac{\partial u_0}{\partial x} - 2\frac{\partial w_1}{\partial z}|_{z=z^\pm}, \end{aligned} \tag{46}$$

which using (45) gives

$$\frac{1}{C}\left(1+MC\sigma_{+}\right)\kappa_{+}-\frac{1}{C}\left(1+MC\sigma_{-}\right)\kappa_{-}+Reh_0\left(\frac{\partial\overline{w}}{\partial t}+u_0\frac{\partial\overline{w}}{\partial x}\right)+\frac{\partial w_1}{\partial z}|_{z=z^+}-\frac{\partial w_1}{\partial z}|_{z=z^-}$$
$$-h_0\left(\frac{\partial u_0}{\partial x}\frac{\partial^2 H_0}{\partial x^2}+\frac{\partial^2\overline{w}}{\partial x^2}+2\frac{\partial H_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}\right)=\frac{\partial h_0}{\partial x}M\left(\frac{\partial\sigma_{+}}{\partial x}-\frac{\partial\sigma_{-}}{\partial x}\right)$$
$$+2\frac{\partial H_0}{\partial x}M\left(\frac{\partial\sigma_{+}}{\partial x}+\frac{\partial\sigma_{-}}{\partial x}\right)+8\frac{\partial h_0}{\partial x}\frac{\partial H_0}{\partial x}\frac{\partial u_0}{\partial x}. \quad (47)$$

Now, we only need to work out $Q(x,t) := \frac{\partial w_1}{\partial z}|_{z=z^+}-\frac{\partial w_1}{\partial z}|_{z=z^-}$ in order to simplify (47). From continuity (9), we have:

$$Q(x,t)=\frac{\partial u_1}{\partial x}|_{z=z^-}-\frac{\partial u_1}{\partial x}|_{z=z^+}, \quad (48)$$

which using (36) gives

$$Q(x,t)=-h_0\frac{\partial G}{\partial x}+\frac{\partial H_0}{\partial x}h_0\left(Re\left(\frac{\partial u_0}{\partial t}+u_0\frac{\partial u_0}{\partial x}\right)-3\frac{\partial^2 u_0}{\partial x^2}\right). \quad (49)$$

Then, substitute (39,40), which yields:

$$Q(x,t)=\frac{\partial H_0}{\partial x}M\left(\frac{\partial\sigma_{+}}{\partial x}+\frac{\partial\sigma_{-}}{\partial x}\right)+\frac{Mh_0}{2}\left(-\frac{\partial^2\sigma_{+}}{\partial x^2}+\frac{\partial^2\sigma_{-}}{\partial x^2}\right)+4\frac{\partial h_0}{\partial x}\frac{\partial H_0}{\partial x}\frac{\partial u_0}{\partial x}$$
$$-3h_0\frac{\partial u_0}{\partial x}\frac{\partial^2 H_0}{\partial x^2}+h_0\frac{\partial^2\overline{w}}{\partial x^2}-2h_0\frac{\partial H_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}. \quad (50)$$

Substituting (50) into (47) yields

$$\frac{1}{C}\left(1+MC\sigma_{+}\right)\kappa_{+}-\frac{1}{C}\left(1+MC\sigma_{-}\right)\kappa_{-}+Reh_0\left(\frac{\partial\overline{w}}{\partial t}+u_0\frac{\partial\overline{w}}{\partial x}\right)=\frac{\partial H_0}{\partial x}M\left(\frac{\partial\sigma_{+}}{\partial x}+\frac{\partial\sigma_{-}}{\partial x}\right)$$
$$+\frac{\partial h_0}{\partial x}M\left(\frac{\partial\sigma_{+}}{\partial x}-\frac{\partial\sigma_{-}}{\partial x}\right)+\frac{Mh_0}{2}\left(\frac{\partial^2\sigma_{+}}{\partial x^2}-\frac{\partial^2\sigma_{-}}{\partial x^2}\right)$$
$$+4\frac{\partial h_0}{\partial x}\frac{\partial H_0}{\partial x}\frac{\partial u_0}{\partial x}+3h_0\frac{\partial u_0}{\partial x}\frac{\partial^2 H_0}{\partial x^2}-h_0\frac{\partial^2\overline{w}}{\partial x^2}+2h_0\frac{\partial H_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}+h_0\left(\frac{\partial u_0}{\partial x}\frac{\partial^2 H_0}{\partial x^2}+\frac{\partial^2\overline{w}}{\partial x^2}+2\frac{\partial H_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}\right). \quad (51)$$

Rearranging (51),

$$Reh_0\left(\frac{\partial\overline{w}}{\partial t}+u_0\frac{\partial\overline{w}}{\partial x}\right)=-\frac{1}{C}\left(1+MC\sigma_{+}\right)\kappa_{+}+\frac{1}{C}\left(1+MC\sigma_{-}\right)\kappa_{-}+\frac{\partial H_0}{\partial x}M\left(\frac{\partial\sigma_{+}}{\partial x}+\frac{\partial\sigma_{-}}{\partial x}\right)$$
$$+\frac{\partial h_0}{\partial x}M\left(\frac{\partial\sigma_{+}}{\partial x}-\frac{\partial\sigma_{-}}{\partial x}\right)+\frac{Mh_0}{2}\left(\frac{\partial^2\sigma_{+}}{\partial x^2}-\frac{\partial^2\sigma_{-}}{\partial x^2}\right)+4\frac{\partial}{\partial x}\left(h_0\frac{\partial H_0}{\partial x}\frac{\partial u_0}{\partial x}\right). \quad (52)$$

Next, substitute (15) into (52) to deduce:

$$Reh_0\left(\frac{\partial\overline{w}}{\partial t}+u_0\frac{\partial\overline{w}}{\partial x}\right)=\frac{1}{2}M\frac{\partial^2}{\partial x^2}\left(h_0(\sigma_{+}-\sigma_{-})\right)+\frac{\partial}{\partial x}\left(\frac{\partial H_0}{\partial x}\left(\frac{2}{C}+M\left(\sigma_{+}+\sigma_{-}\right)\right)\right)+4\frac{\partial}{\partial x}\left(h_0\frac{\partial H_0}{\partial x}\frac{\partial u_0}{\partial x}\right). \quad (53)$$

6.2 Complete next-order equations

In this subsection, we summarize the equations of interest from this section:

$$Reh_0\left(\frac{\partial u_0}{\partial t}+u_0\frac{\partial u_0}{\partial x}\right)=M\left(\frac{\partial \sigma_+}{\partial x}+\frac{\partial \sigma_-}{\partial x}\right)+4\frac{\partial}{\partial x}\left(h_0\frac{\partial u_0}{\partial x}\right), \tag{54}$$

$$Reh_0\left(\frac{\partial \overline{w}}{\partial t}+u_0\frac{\partial \overline{w}}{\partial x}\right)=\frac{1}{2}M\frac{\partial^2}{\partial x^2}\left(h_0(\sigma_+-\sigma_-)\right)+\frac{\partial}{\partial x}\left(\frac{\partial H_0}{\partial x}\left(\frac{2}{C}+M\left(\sigma_++\sigma_-\right)\right)\right)+4\frac{\partial}{\partial x}\left(h_0\frac{\partial H_0}{\partial x}\frac{\partial u_0}{\partial x}\right). \tag{55}$$

7 Total equations

We then have leading-order evolution equations for $h_0, H_0, \overline{w}, u_0$ (errors are relative $O(\epsilon^2)$):

$$\frac{\partial h_0}{\partial t}+\frac{\partial}{\partial x}\left(u_0h_0\right)=0, \tag{56}$$

$$\frac{\partial H_0}{\partial t}+u_0\frac{\partial H_0}{\partial x}=\overline{w}, \tag{57}$$

$$Reh_0\left(\frac{\partial u_0}{\partial t}+u_0\frac{\partial u_0}{\partial x}\right)=M\left(\frac{\partial \sigma_+}{\partial x}+\frac{\partial \sigma_-}{\partial x}\right)+4\frac{\partial}{\partial x}\left(h_0\frac{\partial u_0}{\partial x}\right), \tag{58}$$

$$Reh_0\left(\frac{\partial \overline{w}}{\partial t}+u_0\frac{\partial \overline{w}}{\partial x}\right)=\frac{1}{2}M\frac{\partial^2}{\partial x^2}\left(h_0(\sigma_+-\sigma_-)\right)+\frac{\partial}{\partial x}\left(\frac{\partial H_0}{\partial x}\left(\frac{2}{C}+M\left(\sigma_++\sigma_-\right)\right)\right)+4\frac{\partial}{\partial x}\left(h_0\frac{\partial H_0}{\partial x}\frac{\partial u_0}{\partial x}\right). \tag{59}$$

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