Supplementary materials of: Two-phase volume-averaged predictive theory of dilute ferrofluid spin-up flow in a rotating magnetic field

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1 Two-phase average equation

Consider a two-phase system composed of a liquid, denoted by the subscript l, and a dispersed phase (particles), denoted by the subscript p. Fig. 1 is a schematic representation of the global system REV, over which the two phase transport equations will be averaged.

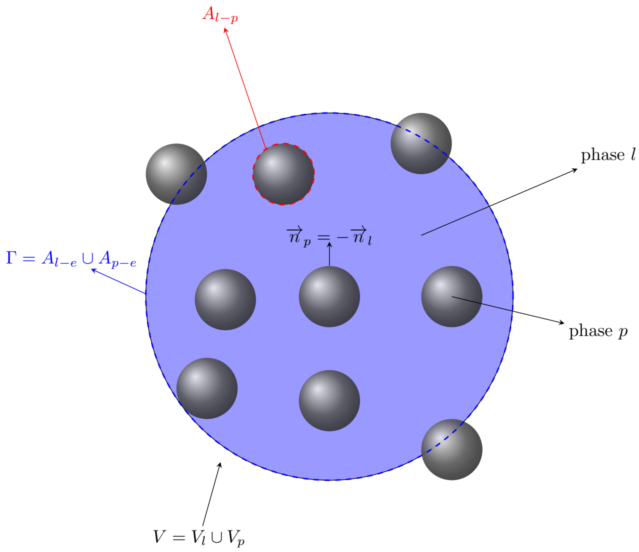

Figure 1: Schematic description of an Elementary Representative Volume (REV) of a suspension composed of a liquid and rigid spherical particles Larachi and Desvigne (2006).

The two phases (l and p) respectively occupy the volumes V_l, V_p in the global volume V bounded by the surface $\Gamma = A_{p-e} \cup A_{l-e}$. The interface surface separating the two phases is denoted by A_{l-p}. This interface is defined with respect to the unit vector normal to it $\vec{n}_l = -\vec{n}_p$. Let $\vec{x}$ be the macroscopic position vector characterizing the position of the elementary volume with respect to the macroscopic system, and $\vec{y}$ be the microscopic position vector at the scale of the REV, defined with respect to its centroid. For example, consider a scalar field Φ_l transported in phase l, defined with respect to the position $\vec{x}+\vec{y}$. Using AVT, the average field $\langle\Phi_l\rangle$ with respect to REV is given by Gray (1975), Whitaker (1999):

$$\langle\Phi_l\rangle|_{\vec{x}} = \frac{1}{V}\int_V \chi_l \Phi_l|_{\vec{x}+\vec{y}}\, dV \tag{1}$$

where χ_l denotes the phase indicator describing the spatial distribution of phase l in the REV. The latter is defined by :

$$\begin{cases} \chi_l = 1 & \vec{x}+\vec{y} \in V_l \cup A_{l-p} \\ 0 & \text{elsewhere} \end{cases} \tag{2}$$

The general form of a convective-diffusive transport equation at the microscopic scale of the scalar field Φ_l can be written as follows:

$$\frac{\partial \Phi_l}{\partial t} + \vec{\nabla}.\left(\chi_l \vec{v}_l \Phi_l\right) = \vec{\nabla}.\vec{J}_l + \sigma_l \tag{3}$$

where $\vec{v}_l$ is the velocity of the liquid. $\vec{J}_l$ is a vector representing the transport of Φ_l by diffusion, and σ_l is a volume source term.

A second averaging expression of Φ_l, called the intrinsic average and characterizing the mean of Φ_l within V_l, is defined by the following relationship:

$$\langle \Phi_l \rangle^l = \frac{1}{V_l} \int_{V_l} \chi_l \Phi_l dV \tag{4}$$

In the liquid phase, for example, the intrinsic average is calculated with respect to the volume of this phase. However, this average may have a non-zero value at points within the dispersed phase's volume. If we denote the volume fraction occupied by the l phase as $\epsilon_l = V_l/V$, the intrinsic average of Φ_l can be related to the average in the REV by the following relation:

$$\langle \Phi_l \rangle = \epsilon_l \langle \Phi_l \rangle^l \tag{5}$$

In the case where $\langle \Phi_l \rangle$ and $\langle \Phi_l \rangle^l$ are considered as Well-Behaved functions, the following relations also holds Gray (1975):

$$\begin{cases} \langle \langle \Psi_l \rangle^l \rangle = \langle \langle \Psi_l \rangle^l \rangle^l = \langle \Psi_l \rangle^l \\ \langle \langle \Psi_l \rangle \rangle = \langle \langle \Psi_l \rangle \rangle^l = \langle \Psi_l \rangle \end{cases} \tag{6}$$

The AVT applied to the Φ_l gradient can be related to the surface quantities at the liquid-particle interface as follows Gray (1975), Whitaker (1999) :

$$\langle \vec{\nabla} \chi_l \Phi_l \rangle = \vec{\nabla} \langle \Phi_l \rangle + \frac{1}{V} \int_{A_{l-p}} \chi_l \Phi_l \vec{n}_l dA \tag{7}$$

where it is important to note that the gradient of $\chi_l \Phi_l$, on the LHS of eq. (7), is performed on with respect to the microscopic position $\vec{y}$ before averaging. Whereas, on the RHS the gradient is with respect to the macroscopic position $\vec{x}$. Other expressions similar to eq. (7) relating to the averages of the gradients and divergences are expressed with respect to their surface quantities by the following relations Larachi and Desvigne (2006) :

$$\langle \vec{\vec{\nabla}} \vec{\Phi}_l \rangle = \vec{\vec{\nabla}} \langle \vec{\Phi}_l \rangle + \frac{1}{V} \int_{A_{l-p}} \vec{\Phi}_l \otimes \vec{n}_l dA \tag{8}$$

$$\langle \vec{\nabla}.\vec{\Phi}_l \rangle = \vec{\nabla}.\langle \vec{\Phi}_l \rangle + \frac{1}{V} \int_{A_{l-p}} \vec{\Phi}_l.\vec{n}_l dA \tag{9}$$

$$\langle \vec{\nabla} \times \vec{\Phi}_l \rangle = \vec{\nabla} \times \langle \vec{\Phi}_l \rangle + \frac{1}{V} \int_{A_{l-p}} \vec{n}_l \times \vec{\Phi}_l dA \tag{10}$$

$$\langle \vec{\nabla}.\vec{\vec{\Phi}}_l \rangle = \vec{\nabla}.\langle \vec{\vec{\Phi}}_l \rangle + \frac{1}{V} \int_{A_{l-p}} \vec{\vec{\Phi}}_l.\vec{n}_l dA \tag{11}$$

The average of the time derivative of a scalar field Φ is expressed as:

$$\left\langle \frac{\partial \Phi_l}{\partial t} \right\rangle = \frac{\partial \langle \Phi_l \rangle}{\partial t} - \frac{1}{V} \int_{A_{l-p}} \Phi_l \vec{v}_{I,l} \vec{n}_l dA \tag{12}$$

where $\vec{v}_{I,l}$ denotes the rate of change (deformation, phase change ... etc.) of the $l-p$ interface surface. In the case where the p phase is assumed to be rigid, the identity (12) can be written in the following form:

$$\left\langle \frac{\partial \Phi_l}{\partial t} \right\rangle = \frac{\partial \langle \Phi_l \rangle}{\partial t} \tag{13}$$

By applying the AVT to the microscopic transport equation (3), the average macroscopic transport equation over the REV is obtained, as follows:

$$\frac{\partial \langle \Phi_l \rangle}{\partial t} + \vec{\nabla}.\left\langle \chi_l \vec{v}_l \Phi_l \right\rangle = \vec{\nabla}.\left\langle \vec{J}_l \right\rangle + \frac{1}{V}\int_{A_{l-p}} \left(\vec{J}_l - \vec{v}_l \Phi_l\right).\vec{n}_l dA + \langle \sigma_l \rangle \tag{14}$$

where it is important to note that the superficial quantities, on the RHS of equation (14), are derived from the identities eqs. (7,9).

One can see through eq. (14) that the average of the advective term $\langle \vec{v}_l \Phi_l \rangle$ cannot be written as the product of the averages, $\langle \vec{v}_l \rangle$ and $\langle \Phi_l \rangle$, due to its nonlinear nature. A statistical treatment of the $\vec{v}_l$ and Φ_l fields is therefore required to overcome this problem. According to Gray (1975), the local mean field can be decomposed by a linear superposition, called Gray's decomposition, of an average field over l and a perturbed field by the presence of the p phase, such that:

$$\begin{cases} \vec{v}_l = \langle \vec{v}_l \rangle^l + \hat{v}_l \text{ and } \Phi_l = \langle \Phi_l \rangle^l + \hat{\Phi}_l & \text{where } \vec{x} \in V_l \\ \vec{v}_l = \hat{\vec{v}}_l = 0 \quad \text{ and } \Phi_l = \hat{\Phi}_l = 0 & \text{where } \vec{x} \in V_p \end{cases} \tag{15}$$

Using equation (5), the average of the advective term can be expressed in terms of the intrinsic average as follows:

$$\langle \chi_l \vec{v}_l \Psi_l \rangle = \epsilon_l \left\langle \vec{v}_l \Psi_l \right\rangle^l \tag{16}$$

The decomposition proposed by Gray (1975), eq. (15), allows to write the advective term equation eq. (16) in the following form:

$$\langle \chi_l \vec{v}_l \Phi_l \rangle = \epsilon_l \langle \vec{v}_l \rangle^l \langle \Phi_l \rangle^l + \epsilon_l \left\langle \hat{\vec{v}}_l \hat{\Phi}_l \right\rangle^l \tag{17}$$

The final form of the macroscopic transport eq. (14) is obtained by replacing the advective term with the one obtained after using Gray's decomposition eq. (17), as follows:

$$\begin{aligned} \frac{\partial \langle \Phi_l \rangle}{\partial t} + \vec{\nabla}.\left(\epsilon_l \langle \vec{v}_l \rangle^l \langle \Phi_l \rangle^l\right) + \vec{\nabla}.\left(\epsilon_l \left\langle \hat{\vec{v}}_l \hat{\Phi}_l \right\rangle^l\right) = \\ \vec{\nabla}.\left\langle \vec{J}_l \right\rangle + \frac{1}{V}\int_{A_{l-p}} \left(\vec{J}_l - \vec{v}_l \Phi_l\right).\vec{n}_l dA + \langle \sigma_l \rangle \end{aligned} \tag{18}$$

2 Two-phase transport equation within the ferrofluid

The average transport equations on the REV within the ferrofluid can be obtained using the same procedure as in Sec. (1).

2.1 Average mass conservation equations

The volume average theorem applied to the incompressible liquid mass conservation equation at the microscopic scale,

$$\vec{\nabla}.\left(\chi_l \vec{v}_l\right) = 0 \tag{19}$$

yields:

$$\vec{\nabla} \cdot \langle \vec{v}_l \rangle^l + \frac{1}{V}\int_{A_{l-p}} \vec{v}_l . n_l dA = 0 \tag{20}$$

The average particle mass on the VER can be obtained adding the mass of each particle i,

$$m_p^i = \int_{V_p} \rho_p dV = \rho_p V_p \tag{21}$$

by assigning a weighting ponderation function to each of them.

In this case, the phase indicator χ_p^i of the particle i can be replaced by a Heaviside function located at the center of each particle $\vec{y}_i$, such as:

$$\chi_p^i = \mathcal{H}\left(\vec{y} - \vec{y}_i\right) = \begin{cases} 1 & 0 < |\vec{y} - \vec{y}_i| < R_p \\ 0 & \text{ailleur} \end{cases} \tag{22}$$

where R_p denotes the radius of the particle.

Using the eq. (1) the average of the total mass of the particles can be expressed as follows:

$$\langle \chi_p m_p \rangle = \int_V \chi_p m_p dV \tag{23}$$

m_p refers to the total mass of the particles.

In the case of a suspension consisting of monodisperse rigid spheres, the integral of eq. (23) can be replaced by a sum, as follows:

$$\langle \chi_p m_p \rangle = \sum_i m_p^i \chi_p^i \equiv \rho_p V_p \sum_i \mathcal{H}\left(\vec{y} - \vec{y}_i\right) \tag{24}$$

The intrinsic average of the total mass of the particles in the volume V can be expressed as follows:

$$\langle \chi_p m_p \rangle = \epsilon_p \rho_p V_p \tag{25}$$

The total mass of the particles must remain invariant. Therefore, the derivative with respect to time of the total mass is zero in the considered REV, where the rigidity assumption allows neglecting the variation over time of $\epsilon_p \rho_p V_p$.

$$\frac{d}{dt} \sum_i \chi_p^i = 0 \tag{26}$$

from which it is straightforward to demonstrate that:

$$\sum_i \vec{\nabla}.\left(\vec{v}_p^i \chi_p^i\right) = 0 \tag{27}$$

where $\vec{v}_p^i$ denotes the velocity of the particle i.

Similarly with the total mass of the particles, the term in the left hand side of eq. (27) can be written in the following form:

$$\sum_i \vec{\nabla}.\left(\vec{v}_p^i \chi_p^i\right) = \langle \vec{\nabla}.\chi_p \vec{v}_p \rangle \tag{28}$$

Using eq. (9), eq. (29) can be written as follows:

$$\langle \vec{\nabla}.\chi_p \vec{v}_p \rangle = \vec{\nabla}.\left(\epsilon_p \langle \vec{v}_p \rangle^p\right) + \frac{1}{V} \int_{A_{l-p}} \vec{v}_p \vec{n}_p dA \tag{29}$$

Substituting eq. (29) into eq. (27) gives the final form of the local particle mass conservation equation.

$$\vec{\nabla}.\left(\epsilon_p \langle \vec{v}_p \rangle^p\right) + \frac{1}{V} \int_{A_{l-p}} \vec{v}_p n_p dA = 0 \tag{30}$$

2.2 Average momentum conservation equations

2.2.1 Average linear momentum conservation equation for the liquid phase

The volume average theorem applied to the liquid linear momentum conservation equation,

$$\rho_l \left(\frac{\partial \chi_l \vec{v}_l}{\partial t} + \vec{\nabla}.(\chi_l \vec{v}_l \otimes \vec{v}_l) \right) = -\vec{\nabla}.\vec{\vec{\mathcal{T}}}_l + \vec{F}_l^e \tag{31}$$

taking into account the Grays' decomposition Gray (1975), yields:

$$\rho_l \left(\frac{\partial \langle \vec{v}_l \rangle}{\partial t} + \vec{\nabla}.\left(\epsilon_l \langle \vec{v}_l \rangle^l \otimes \langle \vec{v}_l \rangle^l\right) + \vec{\nabla}.\left[\epsilon_l \left\langle \vec{\hat{v}}_l \otimes \vec{\hat{v}}_l \right\rangle^l \right] \right) = \left\langle \vec{\nabla}.\vec{\vec{\mathcal{T}}}_l \right\rangle + \left\langle \vec{F}_l^e \right\rangle \tag{32}$$

For a Newtonian and non-polar liquid, the stress tensor $\vec{\vec{\mathcal{T}}}_l$ is given by the following relation Rosensweig (2013):

$$\vec{\vec{\mathcal{T}}}_l = -\chi_l P \vec{\vec{I}} + \mu \left(\vec{\vec{\nabla}} \chi_l \vec{v}_l + \vec{\vec{\nabla}} \chi_l \vec{v}_l^t \right) \tag{33}$$

where μ refers to the viscosity of the liquid.

Substituting the stress tensor expression (eq. 33) into the momentum conservation eq. (32) gives:

$$\rho_l\left(\frac{\partial\langle\vec{v}_l\rangle}{\partial t}+\vec{\nabla}.\left(\epsilon_l\langle\vec{v}_l\rangle^l\otimes\langle\vec{v}_l\rangle^l\right)+\vec{\nabla}.\left[\epsilon_l\left\langle\hat{\vec{v}}_l\otimes\hat{\vec{v}}_l\right\rangle^l\right]\right)=\left\langle-\vec{\nabla}\chi_l P+\mu\vec{\nabla}.\vec{\vec{\nabla}}\chi_l\vec{v}_l\right\rangle+\left\langle\vec{F}_l^e\right\rangle \tag{34}$$

Using the volume average theorem and the decomposition of Gray (1975) (eq. 15), the average pressure gradient can be expressed as follows:

$$\left\langle\vec{\nabla}\chi_l P\right\rangle=\vec{\nabla}\left(\langle P\rangle\right)+\frac{1}{V}\int_{A_{l-p}}\left(\langle P\rangle^l+\chi_l\hat{P}\right).\vec{n}_l dA \tag{35}$$

Using eq. (7), the expression for the average pressure gradient reduces to:

$$\left\langle\vec{\nabla}\chi_l P\right\rangle=\epsilon_l\vec{\nabla}\left\langle P\right\rangle^l+\frac{1}{V}\int_{A_{l-p}}\chi_l\hat{P}.\vec{n}_l dA \tag{36}$$

The average volume theorem applied on the term describing the viscous shear gives:

$$\left\langle\vec{\nabla}.\vec{\vec{\nabla}}\vec{v}_l\right\rangle=\vec{\nabla}.\left\langle\vec{\vec{\nabla}}\vec{v}_l\right\rangle+\frac{1}{V}\int_{A_{l-p}}\vec{\vec{\nabla}}\vec{v}_l.\vec{n}_l dA \tag{37}$$

Similarly with the pressure gradient, the use of the decomposition of Gray (1975) (eq. 15) as well as eq. (7) allows us to write the average of the viscous shear term (eq. 37) as follows:

$$\left\langle\vec{\nabla}.\vec{\vec{\nabla}}\chi_l\vec{v}_l\right\rangle=\epsilon_l\nabla^2\left\langle\vec{v}_l\right\rangle^l+\vec{\vec{\nabla}}\left\langle\vec{v}_l\right\rangle^l.\vec{\nabla}\epsilon_l+\nabla^2\epsilon_l\left\langle\vec{v}_l\right\rangle^l+\frac{1}{V}\int_{A_{l-p}}\vec{\vec{\nabla}}\chi_l\hat{\vec{v}}_l.\vec{n}_l dA \tag{38}$$

The substitution of eqs. (36,38) into eq. (34) gives the final form of the macroscopic momentum equation for the liquid phase.

$$\begin{aligned}\rho_l\left(\frac{\partial\langle\vec{v}_l\rangle}{\partial t}+\vec{\nabla}.\left(\epsilon_l\langle\vec{v}_l\rangle^l\otimes\langle\vec{v}_l\rangle^l\right)+\vec{\nabla}.\left[\epsilon_l\left\langle\hat{\vec{v}}_l\otimes\hat{\vec{v}}_l\right\rangle^l\right]\right)&=-\epsilon_l\vec{\nabla}\left\langle P\right\rangle^l\\+\mu\left(\epsilon_l\nabla^2\left\langle\vec{v}_l\right\rangle^l+\vec{\vec{\nabla}}\left\langle\vec{v}_l\right\rangle^l.\vec{\nabla}\epsilon_l+\left\langle\vec{v}_l\right\rangle^l\nabla^2\epsilon_l\right)+\left\langle\vec{f}_l^e\right\rangle+\frac{1}{V}\int_{A_{l-p}}&\left(-\hat{P}I+\mu\vec{\vec{\nabla}}\chi_l\hat{\vec{v}}_l\right).\vec{n}_l dA\end{aligned} \tag{39}$$

2.2.2 Average linear momentum equation for particles

The conservation equation for the linear momentum of a Brownian particle i, also known as the Langevin equation, is written as follows:

$$\rho_p\left.\frac{d\vec{v}_p^i}{dt}\right|_{\vec{y}=\vec{y}_i}=\vec{F}_i^{lp}+\vec{F}_i^{pp}+\vec{F}_i^e \tag{40}$$

where $\vec{v}_p^i(y_i)$ is the velocity of the particle i evaluated at its center of mass $\vec{y}=\vec{y}_i$. $\vec{F}_i$ denotes a force per unit volume of the particle i. The subscripts lp,pp denote the liquid-particle and particle-particle interaction forces, respectively.

Using the Volume Average Theorem (eq. 1), the average particle velocity field is written:

$$\langle\chi_p\vec{v}_p\rangle=\frac{1}{V}\int_V\chi_p\vec{v}_p dV \tag{41}$$

The integral in eq. (41) can be transformed to a sum using the discrete form of the phase indicator (eq. 22).

$$\langle\vec{v}_p\rangle=\sum_i\chi_p^i\vec{v}_p^i \tag{42}$$

The macroscopic conservation of momentum equation for the particles in the VER can be obtained in the same manner as the mean particle velocity field (eq. 42).

$$\rho_p\sum_i\left(\chi_p^i\left.\frac{d\vec{v}_p^i}{dt}\right|_{\vec{y}=\vec{y}_i}\right)=\sum_i\left(\chi_p^i\vec{F}_i^{lp}\right)+\sum_i\left(\chi_p^i\vec{F}_i^{pp}\right)+\sum_i\left(\chi_p^i\vec{F}_i^e\right) \tag{43}$$

The averages of the forces exerted on the particles can be represented as the average velocity field of the particles, as follows:

$$\rho_p \sum_i \left(\chi_p^i \frac{d\vec{v}_p^i}{dt} \bigg|_{\vec{y}=\vec{y}_i} \right) = \langle \vec{F}^{lp} \rangle + \langle \vec{F}^{pp} \rangle + \langle \vec{F}_p^e \rangle \tag{44}$$

The time derivative of the average particle velocity field (eq. 42) gives:

$$\frac{\partial \langle \vec{v}_p \rangle}{\partial t} = \sum_i \left(\chi_p^i \frac{d\vec{v}_p^i}{dt} \bigg|_{y=y_i} \right) - \vec{\nabla}. \left(\sum_i \chi_p^i \vec{v}_p^i \otimes \vec{v}_p^i \right) \tag{45}$$

hence,

$$\sum_i \left(\chi_p^i \frac{d\vec{v}_p^i}{dt} \bigg|_{\vec{y}=\vec{y}_i} \right) = \frac{\partial \epsilon_p \langle \vec{v}_p \rangle^p}{\partial t} + \vec{\nabla}. \langle \chi_p \vec{v}_p \otimes \vec{v}_p \rangle \tag{46}$$

The substitution of eq. (46) into the particle momentum conservation eq. (44) gives:

$$\rho_p \left(\frac{\partial \epsilon_p \langle \vec{v}_p \rangle^p}{\partial t} + \vec{\nabla}. \langle \chi_p \vec{v}_p \otimes \vec{v}_p \rangle \right) = \langle \vec{F}^{lp} \rangle + \langle \vec{F}^{pp} \rangle + \langle \vec{F}_p^e \rangle \tag{47}$$

Using the decomposition of Gray (1975) (eq. 15) over the advective term of eq. (47), the average linear momentum equation for particles is written:

$$\rho_p \left(\frac{\partial \langle \vec{v}_p \rangle}{\partial t} + \vec{\nabla}. \left(\epsilon_p \langle \vec{v}_p \rangle^p \otimes \langle \vec{v}_p \rangle^p \right) + \vec{\nabla}. \left(\epsilon_p \left\langle \tilde{\vec{v}}_p \otimes \tilde{\vec{v}}_p \right\rangle^p \right) \right) = \langle \vec{F}^{lp} \rangle + \langle \vec{F}^{pp} \rangle + \langle \vec{F}_p^e \rangle \tag{48}$$

2.2.3 Average angular momentum equation for particles

The Langevin equation describing the angular momentum conservation for a single particle i is written :

$$I_p \frac{d\vec{w}_p^i}{dt} \bigg|_{\vec{y}=\vec{y}_i} = \vec{\Gamma}_i^{lp} + \vec{\Gamma}_i^{pp} + \vec{\Gamma}_i^e \tag{49}$$

where $\vec{w}_p^i$ is the angular velocity of the particle i evaluated at $\vec{y} = \vec{y}_i$. $I_p = \frac{2}{5}\rho_p R_p^2$ is the inertia moment of the particle i per unit volume. $\vec{\Gamma}_i^{lp}$ is the moment exerted by the liquid on the particle i. $\vec{\Gamma}_i^{pp}$ is the moment resulting from the interactions between the particle i and the surrounding particles. $\vec{\Gamma}_i^e$ denotes the external moment exerted on the particle i.

Using eq. (1), the average angular velocity field of the particles is written:

$$\langle \vec{w}_p \rangle = \frac{1}{V} \int_V \chi_p \vec{w}_p dV \tag{50}$$

The integral in the expression of the average particle angular velocity field can be transformed to a sum using the discrete representation of the phase function (eq. 22), such that:

$$\langle \vec{w}_p \rangle = \sum_i \chi_p^i \vec{w}_p^i \tag{51}$$

The angular momentum conservation equation for the particles can be obtained using the same procedure with the average particle velocity field (50).

$$I_p \sum_i \left(\chi_p^i \frac{d\vec{w}_p^i}{dt} \bigg|_{\vec{y}=\vec{y}_i} \right) = \sum_i \left(\chi_p^i \vec{\Gamma}_i^{lp} \right) + \sum_i \left(\chi_p^i \vec{\Gamma}_i^{pp} \right) + \sum_i \left(\chi_p^i \vec{\Gamma}_i^e \right) \tag{52}$$

The averages of the moments acting on the particles can be represented in the same manner as the average field of the particles' angular velocity (eq. 51).

$$I_p \sum_i \left(\chi_p^i \frac{d\vec{w}_p^i}{dt} \bigg|_{\vec{y}=\vec{y}_i} \right) = \langle \vec{\Gamma}^{lp} \rangle + \langle \vec{\Gamma}^{pp} \rangle + \langle \vec{\Gamma}^e \rangle \tag{53}$$

The time derivation of the eq.(51) gives:

$$\frac{\partial \langle \vec{w}_p \rangle}{\partial t} = \sum_i \left(\chi_p^i \frac{d\vec{w}_p^i}{dt}\bigg|_{\vec{y}=\vec{y}_i} \right) - \vec{\nabla}. \left(\sum_i \chi_p^i \vec{v}_p^i \otimes \vec{w}_p^i \right) \tag{54}$$

thus,

$$\sum_i \left(\chi_p^i \frac{d\vec{w}_p^i}{dt}\bigg|_{\vec{y}=\vec{y}_i} \right) = \frac{\partial \langle \vec{w}_p \rangle}{\partial t} + \vec{\nabla}. \langle \chi_p \vec{v}_p \otimes \vec{w}_p \rangle \tag{55}$$

Substituting eq. (55) into eq. (53) gives the average angular momentum conservation equation for particles.

$$I_p \left(\frac{\partial \langle \vec{w}_p \rangle}{\partial t} + \vec{\nabla}. \langle \chi_p \vec{v}_p \otimes \vec{w}_p \rangle \right) = \langle \vec{\Gamma}^{lp} \rangle + \langle \vec{\Gamma}^{pp} \rangle + \langle \vec{\Gamma}^{e} \rangle \tag{56}$$

Applying the decomposition of Gray (1975) (eq. 15) on the advective term of the eq. (56) gives the final form of the conservation equation of the angular momentum.

$$I_p \left(\frac{\partial \langle \vec{w}_p \rangle}{\partial t} + \vec{\nabla}. \left(\epsilon_p \langle \vec{v}_p \rangle^p \otimes \langle \vec{w}_p \rangle^p \right) + \vec{\nabla}. \left(\epsilon_p \left\langle \hat{\vec{v}}_p \otimes \hat{\vec{w}}_p \right\rangle^p \right) \right) = \langle \vec{\Gamma}^{lp} \rangle + \langle \vec{\Gamma}^{pp} \rangle + \langle \vec{\Gamma}^{e} \rangle \tag{57}$$

2.3 Average Maxwell equations

The Maxwell-Ampere equations at the microscopic scale in liquid and particles are written:

$$\vec{\nabla} \times \left(\chi_l \vec{H}_l \right) = 0 \quad \in V_l \tag{58}$$

$$\vec{\nabla} \times \left(\chi_p \vec{H}_p \right) = 0 \quad \in V_p \tag{59}$$

$\vec{H}_l$ refer to the magnetic fields in the liquid and the particles, respectively.

Using the average volume theorem, the average Maxwell-Ampere equations for the both phases are written:

$$\vec{\nabla} \times \left(\epsilon_l \left\langle \vec{H}_l \right\rangle^l \right) + \frac{1}{V} \int_{A_{lp}} \chi_l \vec{H}_l \times \vec{n}_{l,p} \, dA = 0 \tag{60}$$

$$\vec{\nabla} \times \left(\epsilon_p \left\langle \vec{H}_p \right\rangle^p \right) + \frac{1}{V} \int_{A_{l-p}} \chi_p \vec{H}_p \times \vec{n}_{p,l} \, dA = 0 \tag{61}$$

Combining eqs. (60,61) gives:

$$\vec{\nabla} \times \left\langle \vec{H} \right\rangle + \frac{1}{V} \int_{A_{l-p}} \left(\chi_l \vec{H}_l - \chi_p \vec{H}_p \right) \times \vec{n}_{l,p} \, dA = 0 \tag{62}$$

where $\left\langle \vec{H} \right\rangle$ is the field averaged over the suspension which is written:

$$\left\langle \vec{H} \right\rangle = \epsilon_l \left\langle \vec{H}_l \right\rangle^l + \epsilon_p \left\langle \vec{H}_l \right\rangle^p \tag{63}$$

The Maxwell-flux equations at the microscopic scale for liquid and particles are given by:

$$\vec{\nabla}. \left(\chi_l \vec{H}_l \right) = 0 \quad \in V_l \tag{64}$$

$$\vec{\nabla}. \chi_p \left(\vec{H}_p + M_d \vec{m} \right) = 0 \quad \in V_p \tag{65}$$

where M_d denotes the domain magnetization of the particles. $\vec{m}$ refers to the vector of the particles' magnetic moment orientations.

Using the average volume theorem, the average Maxwell-flux equations for the both phases are written:

$$\vec{\nabla}. \left(\epsilon_l \left\langle \vec{H}_l \right\rangle^l \right) + \frac{1}{V} \int_{A_{lp}} \chi_l \vec{B}_l . \vec{n}_{l,p} = 0 \tag{66}$$

$$\vec{\nabla}. \left(\epsilon_l \left\langle \vec{H}_l \right\rangle^p + M_d \langle \vec{m} \rangle \right) + \frac{1}{V} \int_{A_{lp}} \chi_p \vec{B}_p . \vec{n}_{p,l} \, dA = 0 \tag{67}$$

$\vec{B}_l$ and $\vec{B}_p$ are the magnetic induction vectors of the liquid and the particles, respectively.

Combining the eqs. (66,67) gives:

$$\vec{\nabla}.\left(\left\langle \vec{H} \right\rangle + M_d \left\langle \vec{m} \right\rangle\right) + \frac{1}{V}\int_{A_{lp}} \left(\chi_p \vec{B}_p - \chi_l \vec{B}_l\right).\vec{n}_{p,l}\, dA = 0 \tag{68}$$

3 Angular operators

A sphere $\varpi_R \in \mathbb{R}^3$ with radius $R > 0$, can be defined as follows:

$$\varpi_R = \left\{\vec{x} \in \mathbb{R}^3, |x| = R\right\} \tag{69}$$

where the unit sphere is denoted ϖ, such that $\varpi = \varpi_1$. The total area of the unit sphere corresponds exactly to the solid angle 4π, such that $|\varpi| = \int_\varpi d\varpi = 4\pi$. For each $\vec{x} \in \mathbb{R}^3$ we can write $\vec{x} = R\vec{u}$, where $R = |x|$ and $\vec{u} = (u_x, u_y, u_z)^T \in \varpi$. In this case $\vec{x}$ in Cartesian coordinates can be written as follows:

$$\vec{x} = R \begin{pmatrix} \sqrt{1-\zeta^2}\cos(\phi) \\ \sqrt{1-\zeta^2}\sin(\phi) \\ \zeta \end{pmatrix} \tag{70}$$

where $\zeta = \cos(\theta)$. For a unit sphere, the local angular coordinates can be obtained using a Jacobian coordinate transformation. This transformation allows to construct a moving coordinate base, $(\vec{e}_r, \vec{e}_\phi, \vec{e}_\phi)$, on the surface of the sphere ϖ. This basis is written:

$$\vec{e}_r = \begin{pmatrix} \sqrt{1-\zeta^2}\cos(\phi) \\ \sqrt{1-\zeta^2}\cos(\phi) \\ \zeta \end{pmatrix}, \quad \vec{e}_\phi = \begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \\ 0 \end{pmatrix}, \quad \vec{e}_\zeta = \begin{pmatrix} -\zeta\cos(\phi) \\ -\zeta\sin(\phi) \\ \sqrt{1-\zeta^2} \end{pmatrix} \tag{71}$$

where $\vec{e}_r \times \vec{e}_\phi = \vec{e}_t$. Note that the unit vector e_r represents the orientation $\vec{u}$ of the unit sphere.

The canonical basis of the vector of $\vec{x} \in \mathbb{R}^3$ in a unit sphere can be expressed as a function of $(\vec{e}_r, \vec{e}_\phi, \vec{e}_\phi)$ as follows:

$$\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} = \begin{bmatrix} \sqrt{1-\zeta^2}\cos(\phi) & -\sin(\phi) & -\zeta\cos(\phi) \\ \sqrt{1-\zeta^2}\sin(\phi) & -\cos(\phi) & -\zeta\sin(\phi) \\ \zeta & 0 & \sqrt{1-\zeta^2} \end{bmatrix} \begin{pmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_\zeta \end{pmatrix} \tag{72}$$

In Cartesian coordinate the gradient operator is written:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T \tag{73}$$

and the Laplacian operator ∇^2, which can be written formally $\nabla^2 = \vec{\nabla}.\vec{\nabla}$, is written:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{74}$$

Using angular (or spherical) coordinates, the gradient and Laplacian operators can be decomposed to two parts. The first is the purely radial part, and the second represents the angular part. In this case, the gradient operator, eq. (73), is written:

$$\vec{\nabla} = \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\vec{\nabla}_u \tag{75}$$

where $\vec{\nabla}_u$ refers the angular gradient on the surface ϖ. In spherical coordinates, the angular gradient is written:

$$\vec{\nabla}_u = \frac{1}{\sin(\theta)}\frac{\partial}{\partial \phi}\vec{e}_\phi + \frac{\partial}{\partial \theta}\vec{e}_\theta \tag{76}$$

The decomposition of the operator into radial and angular parts gives:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\nabla_u^2 \tag{77}$$

where ∇_u^2 is the Beltrami operator which represents the angular part of the Laplace operator. In spherical coordinates the Beltrami operator is written:

$$\nabla_u^2 = \frac{1}{\sin(\theta)}\frac{\partial}{\partial \theta}\sin(\theta)\frac{\partial}{\partial \theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial \phi^2} \tag{78}$$

The curl surface gradient, $\vec{\mathcal{L}}_u$, is an angular rotational operator defined on the surface ϖ, as follows:

$$\vec{\mathcal{L}}_u = \vec{u} \times \vec{\nabla}_u \tag{79}$$

In spherical coordinates the operator $\vec{\mathcal{L}}_u$ is written:

$$\vec{\mathcal{L}}_u = -\frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\vec{e}_\theta + \frac{\partial}{\partial\theta}\vec{e}_\phi \tag{80}$$

Consider $\vec{m} \in \varpi$ as a vectorial quantity independent of $\vec{u}$. The application of the operators, $\vec{\nabla}_u$, ∇_u^2, $\vec{\mathcal{L}}_u$, on the scalar product $\vec{m}.\vec{u}$ gives Freeden and Schreiner (2008):

$$\vec{\nabla}_u(\vec{u}.\vec{m}) = \vec{m} - (\vec{u}.\vec{m})\vec{u} \tag{81}$$

$$\nabla_u^2(\vec{u}.\vec{m}) = -2(\vec{u}.\vec{m}) \tag{82}$$

$$\vec{\mathcal{L}}_u(\vec{u}.\vec{m}) = \vec{u} \times \vec{m} \tag{83}$$

More generally, lets consider the function $F \in \mathcal{C}^{(2)}[-1,1]$ as well as the vectors, $\vec{m}$ and $\vec{u}$, defined in the unit sphere ϖ. The application of the three angular operators defined before on $F(\vec{u}.\vec{m})$ gives Freeden and Schreiner (2008):

$$\vec{\nabla}F(u.m) = F'(u.m)\,(m - (u.m)u) \tag{84}$$

$$\nabla_u^2 F(u.m) = -2F'(u.m)(u.m) + \left(1 - (u.m)^2\right) F''(u.m) \tag{85}$$

$$\vec{\mathcal{L}}_u F(u.m) = F'(u.m)(u \times m) \tag{86}$$

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