Supplementary Materials

S.1. *Fluidisation model for an ideal gas*

For an ideal gas, an affine relationship between pressure and density will be imposed:

$$p_f = Q(\rho_f) \quad \text{with} \quad Q(\rho_f) = p_{\text{atm}}\Big(\frac{\rho_f}{\rho_f^0} - 1\Big), \tag{S.1.1}$$

where $p_{\text{atm}} = 1.013 \times 10^5$ Pa is the atmospheric pressure, and $\rho_f^0 = 1\,\text{kg.m}^{-3}$ corresponds to the density of air at atmospheric pressure. Note that this formulation is equivalent to the one given in Goren *et al.* (2010):

$$\rho_f = \rho_f^0\Big(1 + \frac{p_f}{p_{\text{atm}}}\Big).$$

Recall that it is possible to choose any regular function H in equation (3.3). If we first choose $H(\rho_f) = Q(\rho_f)$, then we deduce an equation describing the evolution of the pressure p_f, namely

$$\partial_t((1-\phi)p_f) + \text{div}((1-\phi)p_f\,\mathbf{u}) + p_{\text{atm}}\text{div}\,\mathbf{u} = \text{div}(\kappa(\phi)(p_{\text{atm}} + p_f)\nabla p_f).$$

Considering that the pore gas pressure in the granular medium is negligible compared to the atmospheric pressure, that is $p_f \ll p_{\text{atm}}$, it is reasonable to approximate $p_f + p_{\text{atm}}$ by p_{atm} in the right-hand side of the last equation. As a consequence, we obtain the following pressure "diffusion" equation (3.6):

$$\partial_t((1-\phi)p_f) + \text{div}((1-\phi)p_f\,\mathbf{u}) + p_{\text{atm}}\text{div}\,\mathbf{u} = p_{\text{atm}}\text{div}(\kappa(\phi)\nabla p_f). \tag{S.1.2}$$

As in the case of a general state law, it is also possible to estimate the term A in the energy equation (2.13). To do this, we now choose the function H in (3.3) so that $\rho_f H'(\rho_f) - H(\rho_f) = Q(\rho_f)$ (that is the function H given by using (3.5)), namely

$$H(\rho_f) = p_{\text{atm}}\frac{\rho_f}{\rho_f^0}\Big[\ln\Big(\frac{\rho_f}{\rho_f^0}\Big) - 1\Big] = (p_{\text{atm}} + p_f)\Big[\ln\Big(1 + \frac{p_f}{p_{\text{atm}}}\Big) - 1\Big].$$

We get

$$-A = \frac{d}{dt}\Big(\int (1-\phi)(p_{\text{atm}} + p_f)\Big[\ln\Big(1 + \frac{p_f}{p_{\text{atm}}}\Big) - 1\Big]\Big) + \int \kappa(\phi)|\nabla p_f|^2.$$

Remark 1. *This convection-diffusion equation* (S.1.2) *for the pore gas pressure is frequently used, sometimes in slightly different forms. Thus, in Goren* et al. *(2010, equation (7)) or in McNamara* et al. *(2000, equation (7)) corrected in Anghel* et al. *(2006), the authors use its non-conservative form (using the approximation $p_{\text{atm}} + p_f \approx p_{\text{atm}}$), namely*

$$(1-\phi)(\partial_t p_f + \mathbf{u}\cdot\nabla p_f) + p_{\text{atm}}\text{div}\,\mathbf{u} = p_{\text{atm}}\text{div}(\kappa(\phi)\nabla p_f).$$

A similar equation is often used in the incompressible case, i.e. when ϕ is constant and $\text{div}\,\mathbf{u} = 0$. *It then reduces to a "classical" convection/diffusion equation, and even to a diffusion equation if the transport term is not taken into account. This is for example the case in Montserrat* et al. *(2012) and Roche (2012).*

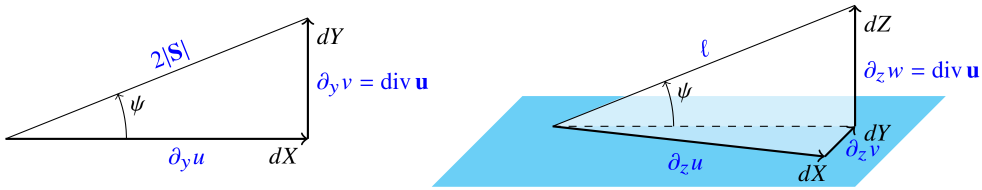

Figure 3: Schematic configuration of a two-dimensional (left) or three-dimensional (right) displacement.

S.2. *Dilatancy law and dilatancy angle*

As for the friction angle, some authors use the tangent function instead of the sine function. Specifically, the dilatancy angle is the angle of motion relative to the horizontal arising from displacement, with $dY = \tan(\psi)dX$ where dY and dX are the vertical and horizontal displacements. This definition of ψ is specific to planar shear but can be generalised by the relation

$$\operatorname{div}\mathbf{u} = 2|\mathbf{S}|\sin(\psi). \tag{S.2.1}$$

On the left-hand side of the Figure 3, the dilatancy angle is represented for a two-dimensional flow whose velocity field depends only on the vertical variable y. In that case and assuming that the velocity field is written $\mathbf{u} = (u(y), v(y))$, we have

$$\operatorname{div}\mathbf{u} = \partial_y v\ , \quad \mathbf{S} = \frac{1}{2}\begin{pmatrix} -\partial_y v & \partial_y u \\ \partial_y u & \partial_y v \end{pmatrix} \quad \text{and} \quad 2|\mathbf{S}| = \sqrt{(\partial_y u)^2 + (\partial_y v)^2}.$$

In this case, the definition of ψ *via* $\tan(\psi) = \frac{dY}{dX}$ exactly coincides with (S.2.1).

Although this article is devoted to two-dimensional models, it may be interesting to say a few words about extending the definition of the dilatancy angle ψ to the three-dimensional case. In Andreotti *et al.* (2012, p.150-151), the authors define ψ by the formula (S.2.1) by replacing the constant 2 by 3, while explaining that this definition no longer coincides with that given in the case of simple shear. Nevertheless, it is possible to do the same reasoning as in dimension 2. Assuming that $\mathbf{u} = (u(z), v(z), w(z))$, we have

$$\operatorname{div}\mathbf{u} = \partial_z w, \quad \mathbf{S} = \begin{pmatrix} -\frac{1}{3}\partial_z w & 0 & \frac{1}{2}\partial_z u \\ 0 & -\frac{1}{3}\partial_z w & \frac{1}{2}\partial_z v \\ \frac{1}{2}\partial_z u & \frac{1}{2}\partial_z v & \frac{2}{3}\partial_z w \end{pmatrix}, \quad 2|\mathbf{S}| = \sqrt{(\partial_z u)^2 + (\partial_z v)^2 + \frac{4}{3}(\partial_z w)^2},$$

so that the length ℓ (see the right-hand side of the Figure 3) writes $\ell^2 = 4|\mathbf{S}|^2 - \frac{1}{3}(\operatorname{div}\mathbf{u})^2$. We obtain

$$\sin(\psi) = \frac{\operatorname{div}\mathbf{u}}{\ell} = \frac{\operatorname{div}\mathbf{u}}{\sqrt{4|\mathbf{S}|^2 - \frac{1}{3}(\operatorname{div}\mathbf{u})^2}}.$$

It is then possible to express the divergence of the velocity field as follows

$$\operatorname{div}\mathbf{u} = \frac{2|\mathbf{S}|\sin(\psi)}{\sqrt{1 + \frac{1}{3}\sin^2(\psi)}}.$$

When the dilatancy angle is small, the above relation is a close approximation to (S.2.1). The coefficient 2 remains valid in the three-dimensional case. There would therefore be no reason to propose 3 as coefficient in (S.2.1), even for a three-dimensional model

S.3. *Proof of the stability result: Theorem 3*

We recall the stability result stated in the Theorem 3 (on page 12) and detail its proof.

THEOREM. *Under the conditions* (5.7)*,* (5.8) *and* (5.9)*, the model* (4.4)–(4.7) *is linearly stable.*

Proof. To prove this result, we adopt the ideas of the proof made in Barker *et al.* (2017, Section 3) or in Barker *et al.* (2023, section 3). We must estimate the effects of the additional equation describing the evolution of the pore gas pressure, namely (4.5), and its coupling with the momentum conservation equation through the gradient term $-\nabla p_f$ in the right-hand side of (4.6).

Let us consider a solution $\mathbf{V}^0 = (\phi^0, \mathbf{u}^0, p^0, p_f^0)$ of the system of equations (4.4)–(4.7). The first step is to linearize this system around $\mathbf{V}^0$ by looking for a perturbed solution in the form $\mathbf{V} = \mathbf{V}^0 + \widetilde{\mathbf{V}}$. As in Barker *et al.* (2017), we retain only the terms that are linear in the perturbation $\widetilde{\mathbf{V}}$ and neglect most of the terms that are not of maximal order. As an example, the linearized version of equation (4.5) describing the evolution of p_f writes

$$\partial_t \widetilde{p_f} = c\Delta \widetilde{p_f} \qquad \text{where } \ c = p_{\text{atm}} \frac{\kappa(\phi^0)}{1-\phi^0}. \tag{S.3.1}$$

In the next step, the coefficients in the resulting linear system are frozen and we look for exponential solutions $\widetilde{\mathbf{V}}(t,x) = \mathrm{e}^{\mathrm{i}\xi\cdot x + \lambda t}\widehat{\mathbf{V}}$, in order to obtain an eigenvalue problem that can be written as $\lambda\widehat{\mathbf{V}} = \mathcal{M}\widehat{\mathbf{V}}$.

By specifying the unknown p_f, *i.e.* by decomposing $\widehat{\mathbf{V}} = \left(\widehat{\mathbf{U}}, \widehat{p_f}\right)$, the matrix $\mathcal{M}$ takes the following form

$$\mathcal{M} = \left(\begin{array}{ccc|c} & & & 0 \\ & -\mathcal{N} & & \mathrm{i}\xi \\ & & & 0 \\ \hline 0 & 0 & 0 & -c|\xi|^2 \end{array}\right),$$

where $\mathcal{N}$ exactly corresponds to the matrix obtained when the fluidisation by the pore gas pressure is not modeled. The term $\mathrm{i}\xi$ comes from the pressure gradient ∇p_f involved in the momentum equation (4.6) whereas the last line of the matrix $\mathcal{M}$ corresponds to the equation (S.3.1).

In Barker *et al.* (2017), it is proved that the conditions (5.7), (5.8) and (5.9) imply that all eigenvalues of the matrix $\mathcal{N}$ are positive. Since $c|\xi|^2 \geqslant 0$, these same conditions ensure the negativity of all the eigenvalues of the matrix $\mathcal{M}$, i.e. the stability of the solution $(\phi^0, \mathbf{u}^0, p^0, p_f^0)$. □

REMARK 2. *The second condition* (5.8) *is not exactly the same as the one given in Barker* et al. *(2017), which is stronger since it requires a strict inequality $\partial_I Z > 0$. Looking in details at the proof in Barker* et al. *(2017), it is clear that the result remains valid if $I\partial_I Z + Z \geqslant 0$. Indeed, this condition is used to show that the trace of a matrix, namely*

$$\mathrm{Tr}(\mathrm{M}+\mathrm{N}) = \left(\frac{I\partial_I Y}{2\|D^\star\|} + \frac{Y}{2\|D^\star\|}\right)|\xi|^2 + \frac{1}{\Gamma}\left((1+B)^2\xi_1^2 + (1-B)^2\xi_2^2\right),$$

is strictly positive. Note the above equation is written with the notations used in Barker et al. *(2017, Proof of lemma 4.1). Since $\Gamma > 0$, according to* (5.9)*, the second term in the right-hand* *side of the above equality is always strictly positive. Therefore, the condition* $I\partial_I Y + Y \geqslant 0$ *is sufficient to ensure the positiveness of the trace. This condition* (5.8) *does not alter the rest of the proof proposed in Barker* et al. *(2017). In particular, we still have* $\mathrm{Det}(\mathrm{M}+\mathrm{N}) > 0$. *Note that the conditions for the linear stability of the model established in Barker* et al. *(2017) are only sufficient conditions. There is no evidence that they are optimal.*