Online Appendix
“Consistent Backtesting Systemic Risk Measures”

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Summary and notations

The mathematical proof sketched here is inspired by Hong (1996) and Banulescu-Radu, Hurlin, Leymarie and Scaillet (2021). The main added technicality with respect to Hong (1996) is treating the impacts of estimation risk in our backtests, whereas that with respect to Banulescu-Radu et al. (2021) is showing that both the new backtests are fully consistent. Throughout the appendices, the following notations are used. The Euclidean norm is denoted using $||\cdot||$. The notations O_p and o_p are the usual order in probability notations. The scalar c denotes a positive finite generic constant that may differ from place to place. Let $(\rho_n(j), \gamma_n(j))$ be defined as $(\hat{\rho}_n(j), \hat{\gamma}_n(j))$ with $\hat{\boldsymbol{\theta}}_T$ replaced by $\boldsymbol{\theta}_0$.

Appendix A Assumptions

Assumption 1. The stochastic process $\boldsymbol{Y}_t = (Y_{1,t}, Y_{2,t})'$ is strictly stationary and ergodic, with unknown twice continuously differentiable conditional distribution function $F_Y(\boldsymbol{y}|\Omega_{t-1}) = \mathbb{P}(Y_{1,t} \leq y_1, Y_{2,t} \leq y_2|\Omega_{t-1})$, where $\boldsymbol{y} = (y_1, y_2) \in \mathbb{R}^2$.

Assumption 2. The estimator $\hat{\boldsymbol{\theta}}_T$ is consistent for $\boldsymbol{\theta}_0$, that is $\sqrt{T}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) \xrightarrow{d} \mathrm{N}(0, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ is a positive definite bounded matrix. Besides, there exists $\hat{\boldsymbol{\Sigma}}$ such that $\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}} = o_p(1)$.

Assumption 3. The quantity $\boldsymbol{R}(j)$ is summable with bounded smoothing estimator $\boldsymbol{R}_n(j, \boldsymbol{\theta})$ that is consistent at the optimal rate $n^{-2/5}$, and $\boldsymbol{R}_n(j, \hat{\boldsymbol{\theta}}_T) = \hat{\boldsymbol{R}}(j)$. That is, (i) $\sum_{j=0}^{\infty} \|\boldsymbol{R}(j)\| < \infty$, (ii) $\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} \mathbb{E}\, \|\nabla_{\boldsymbol{\theta}}[\boldsymbol{R}_n(j, \boldsymbol{\theta})]\| = O_p(1),\ 0 \leq j \leq n-1$, and (iii) $\boldsymbol{R}_n(j, \boldsymbol{\theta}_0) - \boldsymbol{R}(j) = O_p(n^{-2/5})$.

Assumption 4. The kernel function $k : \mathbb{R} \to [-1, 1]$ is a symmetric function that is continuous at 0 and all points except a finite number of points on $\mathbb{R}$, with $k(0) = 1$ and $\int_{-\infty}^{\infty} k^2(z)\mathrm{d}z < \infty$.

Assumption 5. The autocorrelation statistics have bounded derivatives. That is, for $0 \leq j \leq n-1$, $\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} \mathbb{E}\|\nabla_{\boldsymbol{\theta}}[\rho(j)]\|^2 = O(1)$, and $\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} \mathbb{E}\|\nabla_{\boldsymbol{\theta}}[\rho_n(j)] - \nabla_{\boldsymbol{\theta}}[\rho(j)]\|^2 = O(1)$.

Assumption 6. $\{H_t(\alpha, \boldsymbol{\theta}_0)\}$ is a fourth order stationary process with $\sum_{j=1}^{\infty} \gamma^2(j) < \infty$ and $\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |\kappa_1(j,k,l)| < \infty$, where $\kappa_1(j,k,l)$ is the fourth order cumulant of the joint distribution of $\{H_t(\alpha, \boldsymbol{\theta}_0), H_{t+j}(\alpha, \boldsymbol{\theta}_0), H_{t+k}(\alpha, \boldsymbol{\theta}_0), H_{t+l}(\alpha, \boldsymbol{\theta}_0)\}$.

Appendix B Proof of Theorem 1

To begin we recall the Q formula

$$Q = \frac{n \sum_{j=1}^{n-1} k^2(j/m) \hat{\rho}_n^2(j) - \hat{M}_{1n}(k)}{\hat{V}_{1n}(k)^{1/2}},$$

where

$$\hat{M}_{1n}(k) = \sum_{j=1}^{n-1} k^2(j/m)[1 + \omega \hat{\boldsymbol{R}}(j)' \hat{\boldsymbol{\Sigma}} \hat{\boldsymbol{R}}(j)],$$
$$\hat{V}_{1n}(k) = 2 \sum_{j=1}^{n-1} k^4(j/m)[1 + \omega \hat{\boldsymbol{R}}(j)' \hat{\boldsymbol{\Sigma}} \hat{\boldsymbol{R}}(j)]^2 + 2 \sum_{j=1}^{n-1} \sum_{\substack{i=1 \\ i \neq j}}^{n-1} k^2(i/m) k^2(j/m) [\omega \hat{\boldsymbol{R}}(i)' \hat{\boldsymbol{\Sigma}} \hat{\boldsymbol{R}}(j)]^2.$$

Proof of Theorem 1. Recall the theorem

Theorem 1. Suppose Assumptions 1–4 in the online appendix hold, and let $T \to \infty$, $n \to \infty$, $n/T \to \omega < \infty$, $m \to \infty$, $m/n \to 0$. Then $Q \xrightarrow{d} \mathrm{N}(0,1)$ under $\mathbb{H}_0$.

The theorem can be given by the following lemmas

Lemma B.1. Suppose the conditions of Theorem 1 hold. Then

$$\frac{n \sum_{j=1}^{n-1} k^2(j/m) \left\{ \rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)' \nabla_{\boldsymbol{\theta}} [\rho_n(j)] \right\}^2 - M_{1n}(k)}{V_{1n}(k)^{1/2}} \xrightarrow{d} \mathrm{N}(0,1),$$

where

$$M_{1n}(k) = \sum_{j=1}^{n-1} k^2(j/m)[1 + \omega \boldsymbol{R}(j)' \boldsymbol{\Sigma} \boldsymbol{R}(j)],$$
$$V_{1n}(k) = 2 \sum_{j=1}^{n-1} k^4(j/m)[1 + \omega \boldsymbol{R}(j)' \boldsymbol{\Sigma} \boldsymbol{R}(j)]^2 + 2 \sum_{j=1}^{n-1} \sum_{\substack{i=1 \\ i \neq j}}^{n-1} k^2(i/m) k^2(j/m) [\omega \boldsymbol{R}(i)' \boldsymbol{\Sigma} \boldsymbol{R}(j)]^2.$$

Lemma B.2. Suppose the conditions of Theorem 1 hold. Then

$$\frac{n\sum_{j=1}^{n-1} k^2(j/m)\left\{\hat{\rho}_n^2(j) - \left\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\right\}^2\right\}}{V_{1n}(k)^{1/2}} = o_p(1).$$

Lemma B.3. Suppose the conditions of Theorem 1 hold. Then

$$\frac{\sum_{j=1}^{n-1} k^2(j/m)[1+\omega\boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)] - \sum_{j=1}^{n-1} k^2(j/m)[1+\omega\hat{\boldsymbol{R}}(j)'\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{R}}(j)]}{V_{1n}(k)^{1/2}} = o_p(1).$$

Lemma B.4. Suppose the conditions of Theorem 1 hold. Then

$$\frac{V_{1n}(k)^{1/2}}{\hat{V}_{1n}(k)^{1/2}} \xrightarrow{p} 1.$$

□

Proof of Lemma B.1. It is useful to first calculate the conditional distribution of $u_{2,t}(\boldsymbol{\theta}_0)$ and $u_{12,t}(\boldsymbol{\theta}_0)$. We recall the expression of $H_t(\alpha, \boldsymbol{\theta}_0)$, and to reduce notational burden dependence on the fixed quantity $\boldsymbol{\theta}_0$ is dropped

$$\begin{aligned} H_t(\alpha, \boldsymbol{\theta}_0) = H_t(\alpha) &= \mathbb{1}[F_{Y_2}(Y_{2,t}|\Omega_{t-1}) \leq \alpha] \times [1 - F_{Y_1|Y_2 \leq VaR_2(\alpha)}(Y_{1,t}|\Omega_{t-1})] \\ &\equiv \mathbb{1}(u_{2,t} \leq \alpha) \times (1 - u_{12,t}). \end{aligned}$$

In the following we calculate the conditional distribution of $u_{2,t}$ and $u_{12,t}$. In particular, we shall show using probability integral transform that

$$u_{2,t}|\Omega_{t-1} \sim \mathrm{U}[0,1], \tag{B.1}$$

$$u_{12,t}|\{\Omega_{t-1}, u_{2,t} \leq \alpha\} \sim \mathrm{U}[0,1]. \tag{B.2}$$

We begin with $u_{2,t}$, which by definition $u_{2,t} = F_{Y_2}(Y_{2,t}|\Omega_{t-1})$. We let $F_{u_2}(\cdot|\Omega_{t-1})$ denote the conditional distribution of $u_{2,t}$ given Ω_{t-1}. We have for any $u \in \mathbb{R}$

$$\begin{aligned} F_{u_2}(u|\Omega_{t-1}) &= \mathbb{P}(u_{2,t} \leq u|\Omega_{t-1}) \\ &= \mathbb{P}(F_{Y_2}(Y_{2,t}|\Omega_{t-1}) \leq u|\Omega_{t-1}) \\ &= \mathbb{P}(Y_{2,t} \leq F_{Y_2}^{-1}(u|\Omega_{t-1})|\Omega_{t-1}) \\ &= F_{Y_2}(F_{Y_2}^{-1}(u|\Omega_{t-1})|\Omega_{t-1}) \\ &= u. \end{aligned}$$

This implies $u_{2,t}|\Omega_{t-1} \sim \mathrm{U}[0,1]$. We now focus on $u_{12,t} = F_{Y_1|Y_2 \leq VaR_2(\alpha)}(Y_{1,t}|\Omega_{t-1})$, which can be written explicitly as $u_{12,t} = F_{Y_1|Y_2\leq VaR_2(\alpha)}(Y_{1,t}|\{Y_{2,t} \leq VaR_{2,t}(\alpha), \Omega_{t-1}\}) = F_{Y_1|Y_2\leq VaR_2(\alpha)}(Y_{1,t}|\{u_{2,t} \leq \alpha, \Omega_{t-1}\})$. By similar reasoning, we let $F_{u_{12}|u_2\leq\alpha}(\cdot|\{u_{2,t} \leq \alpha, \Omega_{t-1}\})$ denote the conditional distribution of $u_{12,t}$ given $u_{2,t} \leq \alpha$ and Ω_{t-1}. We have for any $u \in \mathbb{R}$

$$\begin{aligned}
F_{u_{12}}(u|\{u_{2,t} \leq \alpha, \Omega_{t-1}\}) &= \mathbb{P}(u_{12,t} < u|\{u_{2,t} \leq \alpha, \Omega_{t-1}\}) \\
&= \mathbb{P}(F_{Y_1|Y_2\leq VaR_2(\alpha)}(Y_{1,t}|\{u_{2,t} \leq \alpha, \Omega_{t-1}\}) < u|\{u_{2,t} \leq \alpha, \Omega_{t-1}\}) \\
&= \mathbb{P}(Y_{1,t} < F^{-1}_{Y_1|Y_2\leq VaR_2(\alpha)}(u|\{u_{2,t} \leq \alpha, \Omega_{t-1}\})|\{u_{2,t} \leq \alpha, \Omega_{t-1}\}) \\
&= F_{Y_1|Y_2\leq VaR_2(\alpha)}\left(F^{-1}_{Y_1|Y_2\leq VaR_2(\alpha)}(u|\{u_{2,t} \leq \alpha, \Omega_{t-1}\})\Big|\{u_{2,t} \leq \alpha, \Omega_{t-1}\}\right) \\
&= u.
\end{aligned}$$

This implies $u_{12,t}|\{\Omega_{t-1}, u_{2,t} \leq \alpha\} \sim \mathrm{U}[0, 1]$, and verifies (B.1)–(B.2).

Put $\{\rho_n(j)\}_{j=1}^m$ in the vector $\boldsymbol{\rho}_n^{(m)}$. Under $\mathbb{H}_0$, we have for any (possibly countably infinite) collection of $\boldsymbol{\rho}_n^{(m)}$ that

$$\boldsymbol{K}_m\sqrt{n}\boldsymbol{\rho}_n^{(m)} \xrightarrow{d} \mathrm{N}(\mathbf{0}, \boldsymbol{K}_m\boldsymbol{K}_m), \tag{B.3}$$

where $\boldsymbol{K}_m$ is a diagonal matrix collecting the kernel sequence $\{k(j/m)\}_{j=1}^m$. The result follows directly from Proposition 2.9 of Hayashi (2000) that $\sqrt{n}\boldsymbol{\rho}_n^{(m)} \xrightarrow{d} \mathrm{N}(\mathbf{0}, \boldsymbol{I}_m)$ under $\mathbb{H}_0$, where $\boldsymbol{I}_m$ is the identity matrix of order m. The result holds provided (i) $H_t(\alpha)$ can be decomposed as $\varepsilon_t + \mu$, where μ is a constant and ε_t is a martingale difference sequence (mds); (ii) ε_t has constant and finite conditional second moment, that is, $\mathbb{E}(\varepsilon_t^2|\Omega_{t-1}) = \sigma^2 < \infty$. To verify the conditions we let $\mu = \alpha/2$, and we let $\varepsilon_t = (H_t(\alpha) - \alpha/2)$. The proof of (i) completes by noting the mds property of $\{H_t(\alpha) - \alpha/2\}$, that is we have for all t, $\mathbb{E}(\varepsilon_t|\Omega_{t-1}) = \mathbb{E}(H_t(\alpha) - \alpha/2|\Omega_{t-1}) = 0$. The result also implies $\mathbb{E}((H_t(\alpha) - \alpha/2)^2|\Omega_{t-1}) = \mathbb{E}(H_t(\alpha)^2|\Omega_{t-1}) - \alpha^2/4$. Thus, to verify (ii) it suffices to show that $\mathbb{E}(H_t(\alpha)^2|\Omega_{t-1}) = c < \infty$. We write

$$\begin{aligned}
\mathbb{E}(H_t(\alpha)^2|\Omega_{t-1}) &= \mathbb{E}\left\{\mathbb{E}\left[\mathbb{1}(u_{2,t} \leq \alpha) \times (1 - u_{12,t})^2|\{\Omega_{t-1}, u_{2,t} \leq \alpha\}\right]|\Omega_{t-1}\right\} \\
&= \mathbb{E}\left\{\mathbb{1}(u_{2,t} \leq \alpha)\mathbb{E}\left[(1 - u_{12,t})^2|\{\Omega_{t-1}, u_{2,t} \leq \alpha\}\right]|\Omega_{t-1}\right\} \\
&= \alpha/3 < \infty.
\end{aligned} \tag{B.4}$$

The first equality follows by the tower property that $\mathbb{E}(X|F_1) = \mathbb{E}(\mathbb{E}(X|F_2)|F_1)$ for $F_1 \subseteq F_2$. The last equality makes use of the results in (B.1)–(B.2) that $u_{2,t}|\Omega_{t-1} \sim \mathrm{U}[0, 1]$ and $u_{12,t}|\{\Omega_{t-1}, u_{2,t} \leq \alpha\} \sim \mathrm{U}[0, 1]$. This implies, for instance, that $\mathbb{1}(u_{2,t} \leq \alpha)|\Omega_{t-1} \sim \mathrm{Bernoulli}(\alpha)$ and that $\mathbb{E}[u_{12,t}^2|\{\Omega_{t-1}, u_{2,t} \leq \alpha\}] = 1/3$.

We now evaluate the limiting distribution of $\sqrt{n}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]$. To this we first evaluate the probability limit of $\nabla_{\boldsymbol{\theta}}[\rho_n(j)]$. By the quotient rule, we have

$$\nabla_{\boldsymbol{\theta}}[\rho_n(j)] = \gamma_n(0)^{-1}\nabla_{\boldsymbol{\theta}}[\gamma_n(j)] - \gamma_n(0)^{-1}\rho_n(j)\nabla_{\boldsymbol{\theta}}[\gamma_n(0)]. \tag{B.5}$$

The second term in (B.5) is $o_p(1)$ by noting $\rho_n(j) \xrightarrow{p} 0$, which is implied by $\gamma_n(j) \xrightarrow{p} 0$, which in turns follows by Markov's inequality and $\mathbb{E}(\gamma_n(j)) = 0$ under $\mathbb{H}_0$. For the first term, we have the quantity $\gamma_n(0)^{-1} \xrightarrow{p} [\alpha(1/3-\alpha/4)]^{-1}$, which follows by continuous mapping theorem, Markov's inequality and

$$\begin{aligned}
\mathbb{E}[\gamma_n(0)] &= \mathbb{E}[\mathbb{E}[\gamma_n(0)|\Omega_{t-1}]] \\
&= \mathbb{E}\left[\frac{1}{n}\sum_{t=T+1}^{T+n}\mathbb{E}[(H_t(\alpha)-\alpha/2)^2|\Omega_{t-1}]\right] \\
&= \mathbb{E}\left[\frac{1}{n}\sum_{t=T+1}^{T+n}\mathbb{E}[H_t(\alpha)^2-\alpha^2/4|\Omega_{t-1}]\right] \\
&= \alpha(1/3-\alpha/4),
\end{aligned} \tag{B.6}$$

where we make use of the results in (B.4). We now focus on the quantity $\nabla_{\boldsymbol{\theta}}[\gamma_n(j)]$. By the law of iterated expectation, we have $\mathbb{E}[\nabla_{\boldsymbol{\theta}}[\gamma_n(j)]] = \mathbb{E}[\mathbb{E}[\nabla_{\boldsymbol{\theta}}[\gamma_n(j)]|\Omega_{t-1}]]$. Thus we evaluate

$$\begin{aligned}
&\mathbb{E}[\nabla_{\boldsymbol{\theta}}[\gamma_n(j)]|\Omega_{t-1}] \\
&= \frac{1}{n-j}\sum_{t=T+1+j}^{T+n}\mathbb{E}\{\nabla_{\boldsymbol{\theta}}[(H_t(\alpha)-\alpha/2)\times(H_{t-j}(\alpha)-\alpha/2)]|\Omega_{t-1}\} \\
&= \frac{1}{n-j}\sum_{t=T+1+j}^{T+n}\mathbb{E}\{(H_{t-j}(\alpha)-\alpha/2)\nabla_{\boldsymbol{\theta}}[H_t(\alpha)]|\Omega_{t-1}\} + \nabla_{\boldsymbol{\theta}}[H_{t-j}(\alpha)]\mathbb{E}\{(H_t(\alpha)-\alpha/2)|\Omega_{t-1}\} \\
&= \frac{1}{n-j}\sum_{t=T+1+j}^{T+n}\mathbb{E}\{(H_{t-j}(\alpha)-\alpha/2)\nabla_{\boldsymbol{\theta}}[H_t(\alpha)]|\Omega_{t-1}\},
\end{aligned}$$

where the second equality makes use of the product rule, and the last equality follows by $\mathbb{E}\{(H_t(\alpha)-\alpha/2)|\Omega_{t-1}\} = 0$. Therefore, we have $\nabla_{\boldsymbol{\theta}}[\gamma_n(j)] \xrightarrow{p} \frac{1}{n-j}\sum_{t=T+1+j}^{T+n}\mathbb{E}\{(H_{t-j}(\alpha)-\alpha/2)\nabla_{\boldsymbol{\theta}}[H_t(\alpha)]\}$, which follows by Markov's inequality and

$$\mathbb{E}[\nabla_{\boldsymbol{\theta}}[\gamma_n(j)]] = \mathbb{E}[\mathbb{E}[\nabla_{\boldsymbol{\theta}}[\gamma_n(j)]|\Omega_{t-1}]] = \mathbb{E}\{(H_{t-j}(\alpha)-\alpha/2)\nabla_{\boldsymbol{\theta}}[H_t(\alpha)]\}.$$

Therefore we have

$$\nabla_{\boldsymbol{\theta}}[\rho_n(j)] \xrightarrow{p} \frac{1}{\alpha(1/3-\alpha/4)}\mathbb{E}\{(H_{t-j}(\alpha)-\alpha/2)\nabla_{\boldsymbol{\theta}}[H_t(\alpha)]\} = \boldsymbol{R}(j).$$

Given $\nabla_{\boldsymbol{\theta}}[\rho_n(j)] \xrightarrow{p} \boldsymbol{R}(j)$, $\sqrt{n} \to \sqrt{\omega}\sqrt{T}$, and Assumption 2, we have by Slutsky's theorem

$$k(j/m)\sqrt{n}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)] \xrightarrow{d} \mathrm{N}(0, k^2(j/m)\omega\boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)), \tag{B.7}$$

under the null. Since $\nabla_{\boldsymbol{\theta}}[\rho_n(j)]$ converges to the constant $\boldsymbol{R}(j)$, Slutsky's theorem also implies that the statistical randomness of the term $k(j/m)\sqrt{n}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]$ is induced by the $\hat{\boldsymbol{\theta}}_T$ estimate which depends on the in-sample. By contrast, the randomness of $k(j/m)\sqrt{n}\rho_n(j)$ is induced by the out-sample iid generalized errors (see, e.g., Rosenblatt, 1952). Given the statistical independence, we have

$$k(j/m)\sqrt{n}\left\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\right\} \xrightarrow{d} \mathrm{N}(0, k^2(j/m)[1 + \omega\boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]). \tag{B.8}$$

Next, we study the dependence structure across time. Although $\{k(j/m)\sqrt{n}\rho_n(j)\}_{j=1}^m$ converges to a sequence of independent random variables as shown in (B.3), that is generally not the case for $\{k(j/m)\sqrt{n}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\}_{j=1}^m$. However, we have the sum of squares

$$n\sum_{j=1}^{m} k^2(j/m)\left\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\right\}^2,$$

which we denote in short by $n\sum_{j=1}^m X_n^2(j) \equiv n\boldsymbol{X}_n^{(m)'}\boldsymbol{X}_n^{(m)}$, converging to a sum of independent sequence. This follows by first observing that we have from (B.7)–(B.8)

$$\sqrt{n}\boldsymbol{X}_n^{(m)} \xrightarrow{d} \mathrm{N}(\boldsymbol{0}, \boldsymbol{\Delta}_k), \tag{B.9}$$

where the (i,j)-th element of $\boldsymbol{\Delta}_k$ is given by

$$\boldsymbol{\Delta}_k(i,j) = k(i/m)k(j/m)[\delta(i,j) + \omega\boldsymbol{R}(i)'\boldsymbol{\Sigma}\boldsymbol{R}(j)], \tag{B.10}$$

with $\delta(i,j)$ being the Kronecker delta. Secondly, because the real symmetric matrix $\boldsymbol{\Delta}_k$ can be decomposed into $\boldsymbol{\Delta}_k = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}'$, where $\boldsymbol{V}$ is an orthogonal matrix and $\boldsymbol{D}$ is a diagonal matrix containing the eigenvalues of $\boldsymbol{\Delta}_k$, we have

$$\boldsymbol{V}'\sqrt{n}\boldsymbol{X}_n^{(m)} \xrightarrow{d} \mathrm{N}(\boldsymbol{0}, \boldsymbol{D}). \tag{B.11}$$

The diagonal matrix $\boldsymbol{D}$ shows that $\boldsymbol{V}'\sqrt{n}\boldsymbol{X}_n^{(m)}$ converges to a random vector of independent variables. This implies $n\boldsymbol{X}_n^{(m)'}\boldsymbol{X}_n^{(m)} = n\sum_{j=1}^m k^2(j/m)\left\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\right\}^2$, which is a sum of the squared variables in $\boldsymbol{V}'\sqrt{n}\boldsymbol{X}_n^{(m)}$, converges to a sum of variables

$\sum_{j=1}^{m} Z^2(j)$, where $\{Z(j)\}_{j=1}^{m}$ is a sequence of independent Gaussian variables. Let $D(j)$ denote the j-th element of $\boldsymbol{D}$. Now, noting that $\sum_{j=1}^{m} D(j)$ is equal to the trace of $\boldsymbol{\Delta}_k$ and that $\sum_{j=1}^{m} D^2(j)$ is equal to the trace of $\boldsymbol{\Delta}_k \times \boldsymbol{\Delta}_k$, we calculate using the Gaussian moment generating function

$$\begin{aligned}
\mathbb{E}\left[\sum_{j=1}^{m} Z^2(j)\right] &= \sum_{j=1}^{m} k^2(j/m)[1+\omega \boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)], \\
\mathbb{V}\left[\sum_{j=1}^{m} Z^2(j)\right] &= 2\sum_{j=1}^{m} k^4(j/m)[1+\omega \boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2 + 2\sum_{j=1}^{m}\sum_{\substack{i=1\\ i\neq j}}^{m} k^2(i/m)k^2(j/m)[\omega \boldsymbol{R}(i)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2.
\end{aligned}$$

Finally, as $m \to \infty$, which implies $n \to \infty$, we have by the Lindeberg central limit theorem that

$$\frac{n\sum_{j=1}^{n-1} k^2(j/m)\left\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\right\}^2 - M_{1n}(k)}{V_{1n}(k)^{1/2}} \xrightarrow{d} \mathrm{N}(0,1),$$

where

$$\begin{aligned}
M_{1n}(k) &= \sum_{j=1}^{n-1} k^2(j/m)[1+\omega \boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)], \\
V_{1n}(k) &= 2\sum_{j=1}^{n-1} k^4(j/m)[1+\omega \boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2 + 2\sum_{j=1}^{n-1}\sum_{\substack{i=1\\ i\neq j}}^{n-1} k^2(i/m)k^2(j/m)[\omega \boldsymbol{R}(i)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2.
\end{aligned}$$

This completes the proof of Lemma B.1.

□

Proof of Lemma B.2. Recall Lemma B.2

$$\frac{n\sum_{j=1}^{n-1} k^2(j/m)\left\{\hat{\rho}_n^2(j) - \left\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\right\}^2\right\}}{V_{1n}(k)^{1/2}} = o_p(1).$$

First, we show that the denominator $V_{1n}(k)^{1/2} = O(m^{1/2})$. It suffices to show $V_{1n}(k)/2 = O(m)$. We write

$$\begin{aligned}
V_{1n}(k)/2 &= \sum_{j=1}^{n-1} k^4(j/m)[1+\omega \boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2 \\
&\quad + \sum_{j=1}^{n-1}\sum_{\substack{i=1\\ i\neq j}}^{n-1} k^2(i/m)k^2(j/m)[\omega \boldsymbol{R}(i)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2. \qquad \text{(B.12)}
\end{aligned}$$

We begin by showing that the first term in (B.12) is $O(m)$. We write

$$\begin{aligned}
&\sum_{j=1}^{n-1} k^4(j/m)[1+\omega \boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2 \\
&= \sum_{j=1}^{n-1} k^4(j/m) + 2\omega \sum_{j=1}^{n-1} k^4(j/m)\boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j) + \omega^2 \sum_{j=1}^{n-1} k^4(j/m)[\boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2. \quad \text{(B.13)}
\end{aligned}$$

By Assumption 4 and as $m \to \infty$, $n \to \infty$, $m/n \to 0$, we have $m^{-1}\sum_{j=1}^{n-1} k^4(j/m) \to \int_0^\infty k^4(z)\mathrm{d}z < \infty$. Therefore, the first term in (B.13) is $O(m)$. In the following we show that the second term is $O(1)$

$$\begin{aligned}
&2\omega \sum_{j=1}^{n-1} k^4(j/m)\boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j) \\
&\le 2\omega\|\boldsymbol{\Sigma}\| \sum_{j=1}^{n-1} k^4(j/m)\|\boldsymbol{R}(j)\|^2 \\
&= 2\omega\|\boldsymbol{\Sigma}\| \left\{ \sum_{j=1}^{n-1} \|\boldsymbol{R}(j)\|^2 + \sum_{j=1}^{n-1} [k^4(j/m)-1]\|\boldsymbol{R}(j)\|^2 \right\} \\
&= O(1), \qquad \text{(B.14)}
\end{aligned}$$

where we make use of Assumption 2, Assumption 3(i), dominated convergence theorem, $\lim_{m\to\infty} k^4(j/m) - 1 \to 0$ and $|k^4(j/m)-1| \le 1$. It remains to show that the third term in (B.13) is $O(1)$. By Cauchy-Schwarz inequality we write

$$\begin{aligned}
&\omega^2 \sum_{j=1}^{n-1} k^4(j/m)[\boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2 \\
&\le \omega^2\|\boldsymbol{\Sigma}\|^2 \left\{ \sum_{j=1}^{n-1} k^2(j/m)\|\boldsymbol{R}(j)\|^2 \right\} \left\{ \sum_{j=1}^{n-1} k^2(j/m)\|\boldsymbol{R}(j)\|^2 \right\} \\
&= O(1) \left\{ \sum_{j=1}^{n-1} k^2(j/m)\|\boldsymbol{R}(j)\|^2 \right\}^2 \\
&= O(1) \left\{ \left[\sum_{j=1}^{n-1} \|\boldsymbol{R}(j)\|^2 + \sum_{j=1}^{n-1} [k^2(j/m)-1]\|\boldsymbol{R}(j)\|^2 \right] \right\}^2 \\
&= O(1), \qquad \text{(B.15)}
\end{aligned}$$

where we make use of Assumption 2, Assumption 3(i), dominated convergence theorem, $\lim_{m\to\infty} k^2(j/m) - 1 \to 0$ and $|k^2(j/m)-1| \le 1$. We shall now show that the second term in (B.12) is $O(m)$. By Cauchy-Schwarz inequality, we write

$$\begin{aligned}
&\sum_{j=1}^{n-1}\sum_{\substack{i=1\\i\neq j}}^{n-1} k^2(i/m)k^2(j/m)[\omega \boldsymbol{R}(i)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2\\
&\leq \omega^2 \sum_{j=1}^{n-1}\sum_{i=1}^{n-1} k^2(i/m)k^2(j/m)[\boldsymbol{R}(i)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2\\
&\leq \omega^2\|\boldsymbol{\Sigma}\| \sum_{j=1}^{n-1}\sum_{i=1}^{n-1} k^2(i/m)\|\boldsymbol{R}(i)\|^2 k^2(j/m)\|\boldsymbol{R}(j)\|^2\\
&= \omega^2\|\boldsymbol{\Sigma}\| \left\{\sum_{j=1}^{n-1} k^2(j/m)\|\boldsymbol{R}(j)\|^2\right\}\left\{\sum_{i=1}^{n-1} k^2(i/m)\|\boldsymbol{R}(i)\|^2\right\}\\
&= O(1)\left\{\sum_{j=1}^{n-1} k^2(j/m)\|\boldsymbol{R}(j)\|^2\right\}^2\\
&= O(1),
\end{aligned}$$

where the last equality makes use of the result in (B.15). This verifies $V_{1n}(k)^{1/2} = O(m^{1/2})$.

We now focus on the numerator. We have by Taylor's theorem

$$\hat{\rho}_n(j) = \rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)] + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)' f_1(\hat{\boldsymbol{\theta}}_T),\ 1 \leq j \leq n-1, \tag{B.16}$$

where $f_1(\hat{\boldsymbol{\theta}}_T)$ is such that $\lim_{\hat{\boldsymbol{\theta}}_T \to \boldsymbol{\theta}_0} f_1(\hat{\boldsymbol{\theta}}_T) = 0$. Given Assumption 2 that $(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) = O_p(T^{-1/2}) = O_p(n^{-1/2})$ and that $f_1(\hat{\boldsymbol{\theta}}_T)$ is bounded by $(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)$, we have

$$\hat{\rho}_n(j) = \rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)] + O_p(1/n). \tag{B.17}$$

Multiplying both sides by $\sqrt{n}$ we get

$$\sqrt{n}\hat{\rho}_n(j) = \sqrt{n}\rho_n(j) + \sqrt{n}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)] + O_p(n^{-1/2}). \tag{B.18}$$

Squaring both sides gives

$$\begin{aligned}
&n\hat{\rho}_n^2(j) - n\left\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\right\}^2\\
&= 2\{\sqrt{n}\rho_n(j) + \sqrt{n}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\}O_p(n^{-1/2}) + O_p(n^{-1})\\
&= O_p(n^{-1/2}).
\end{aligned} \tag{B.19}$$

The last equality makes use of (B.8) that for all $1 \leq j \leq n-1$, $\sqrt{n}\rho_n(j) + \sqrt{n}(\hat{\boldsymbol{\theta}}_T -$

$\boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)] = O_p(1)$. Therefore we have

$$\begin{aligned}
& n \sum_{j=1}^{n-1} k^2(j/m) \left\{ \hat{\rho}_n^2(j) - \left\{ \rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)] \right\}^2 \right\} \\
& \leq m \sup_{1\leq j\leq n-1} \left\{ n\hat{\rho}_n^2(j) - n \left\{ \rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)] \right\}^2 \right\} \left\{ m^{-1} \sum_{j=1}^{n-1} k^2(j/m) \right\} \\
& = O_p(mn^{-1/2}). \qquad \text{(B.20)}
\end{aligned}$$

The last equality follows by Assumption 4. The proof completes by observing that the expression of Lemma B.2 is $O_p(mn^{-1/2})/O_p(m^{1/2}) = O_p(m^{1/2}/n^{1/2}) = o_p(1)$, where the last equality follows by $m/n \to 0$.

□

Proof of Lemma B.3. Because $V_{1n}(k)^{1/2} = O(m^{1/2})$ as shown in the proof of Lemma B.2, it thus suffices to show

$$\sum_{j=1}^{n-1} k^2(j/m)[1 + \omega \hat{\boldsymbol{R}}(j)'\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{R}}(j)] - \sum_{j=1}^{n-1} k^2(j/m)[1 + \omega \boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)] = o_p(m^{1/2}). \qquad \text{(B.21)}$$

We first work out the consistency of $\hat{\boldsymbol{R}}(j)$ for $\boldsymbol{R}(j)$. In particular we shall show

$$\|\hat{\boldsymbol{R}}(j) - \boldsymbol{R}(j)\| = O_p(n^{-2/5}). \qquad \text{(B.22)}$$

By the triangle inequality we have

$$\begin{aligned}
\|\hat{\boldsymbol{R}}(j) - \boldsymbol{R}(j)\| &= \|\boldsymbol{R}_n(j, \hat{\boldsymbol{\theta}}_T) - \boldsymbol{R}(j)\| \\
&\leq \|\boldsymbol{R}_n(j, \hat{\boldsymbol{\theta}}_T) - \boldsymbol{R}_n(j, \boldsymbol{\theta}_0)\| + \|\boldsymbol{R}_n(j, \boldsymbol{\theta}_0) - \boldsymbol{R}(j)\|. \qquad \text{(B.23)}
\end{aligned}$$

We write for the first term

$$\begin{aligned}
\|\boldsymbol{R}_n(j, \hat{\boldsymbol{\theta}}_T) - \boldsymbol{R}_n(j, \boldsymbol{\theta}_0)\| &\leq \sup_{\tilde{\boldsymbol{\theta}}\in\boldsymbol{\Theta}_0} \left\| \nabla_{\boldsymbol{\theta}}[\boldsymbol{R}_n(j, \tilde{\boldsymbol{\theta}})] \right\| \times \left\| \hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0 \right\| \\
&= O_p(T^{-1/2}) = O_p(n^{-1/2}), \qquad \text{(B.24)}
\end{aligned}$$

which follows by the mean value theorem, Markov's inequality, Assumption 2 and Assumption 3(ii). Besides, with Assumption 3(iii) the second term in (B.23) is $O_p(n^{-2/5})$. This verifies (B.22).

We now verify (B.21). For $a, b = 1, 2, ..., p$, let $R_a(j)$ ($\hat{R}_a(j)$) denote the a-th element of $\boldsymbol{R}(j)$ ($\hat{\boldsymbol{R}}(j)$), and let S_{ab} ($\hat{S}_{ab}$) denote the (a,b)-th element of $\boldsymbol{\Sigma}$ ($\hat{\boldsymbol{\Sigma}}$). We write

$$\begin{aligned}
&\sum_{j=1}^{n-1} k^2(j/m)[1+\omega\hat{\boldsymbol{R}}(j)'\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{R}}(j)] - \sum_{j=1}^{n-1} k^2(j/m)[1+\omega\boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)] \\
&= \omega\sum_{j=1}^{n-1} k^2(j/m)\sum_{a=1}^{p}\sum_{b=1}^{p}\hat{R}_a(j)\hat{R}_b(j)\hat{S}_{ab} - R_a(j)R_b(j)S_{ab} \\
&= \omega\sum_{a=1}^{p}\sum_{b=1}^{p}\sum_{j=1}^{n-1} k^2(j/m)(\hat{R}_a(j)\hat{R}_b(j) - R_a(j)R_b(j))S_{ab} \\
&\quad + \omega\sum_{a=1}^{p}\sum_{b=1}^{p}\sum_{j=1}^{n-1} k^2(j/m)(\hat{S}_{ab} - S_{ab})R_a(j)R_b(j) \\
&\quad + \omega\sum_{a=1}^{p}\sum_{b=1}^{p}\sum_{j=1}^{n-1} k^2(j/m)(\hat{R}_a(j)\hat{R}_b(j) - R_a(j)R_b(j))(\hat{S}_{ab} - S_{ab}) \\
&= \omega\sum_{a=1}^{p}\sum_{b=1}^{p} S_{ab}\sum_{j=1}^{n-1} k^2(j/m)(\hat{R}_a(j)\hat{R}_b(j) - R_a(j)R_b(j)) \\
&\quad + \omega\sum_{a=1}^{p}\sum_{b=1}^{p}(\hat{S}_{ab} - S_{ab})\sum_{j=1}^{n-1} k^2(j/m)R_a(j)R_b(j) \\
&\quad + \omega\sum_{a=1}^{p}\sum_{b=1}^{p}(\hat{S}_{ab} - S_{ab})\sum_{j=1}^{n-1} k^2(j/m)(\hat{R}_a(j)\hat{R}_b(j) - R_a(j)R_b(j)) \\
&= \omega\sum_{a=1}^{p}\sum_{b=1}^{p}\mathcal{A}_1 + \omega\sum_{a=1}^{p}\sum_{b=1}^{p}\mathcal{A}_2 + \omega\sum_{a=1}^{p}\sum_{b=1}^{p}\mathcal{A}_3, \text{ say.}
\end{aligned} \tag{B.25}$$

To complete the proof we shall show $\mathcal{A}_i = o_p(m^{1/2})$ for $i = 1, 2, 3$. We begin with $\mathcal{A}_1$ and $\mathcal{A}_3$. Because by Assumption 2 we have $S_{ab} = O_p(1)$ and $\hat{S}_{ab} - S_{ab} = o_p(1) = O_p(1)$, it suffices to show $\sum_{j=1}^{n-1} k^2(j/m)(\hat{R}_a(j)\hat{R}_b(j) - R_a(j)R_b(j)) = o_p(m^{1/2})$. We have

$$\begin{aligned}
&\sum_{j=1}^{n-1} k^2(j/m)(\hat{R}_a(j)\hat{R}_b(j) - R_a(j)R_b(j)) \\
&= \sum_{j=1}^{n-1} k^2(j/m)(\hat{R}_a(j) - R_a(j))(\hat{R}_b(j) - R_b(j)) + \sum_{j=1}^{n-1} k^2(j/m)(\hat{R}_a(j) - R_a(j))R_b(j) \\
&\quad + \sum_{j=1}^{n-1} k^2(j/m)(\hat{R}_b(j) - R_b(j))R_a(j) \\
&= \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{13}, \text{ say.}
\end{aligned} \tag{B.26}$$

By Cauchy-Schwarz inequality, we write for $\mathcal{A}_{11}$

$$
\begin{aligned}
|\mathcal{A}_{11}| &\leq \left\{\sum_{j=1}^{n-1} k^2(j/m)(\hat{R}_a(j) - R_a(j))^2\right\}^{1/2} \left\{\sum_{j=1}^{n-1} k^2(j/m)(\hat{R}_b(j) - R_b(j))^2\right\}^{1/2} \\
&= O_p(m^{1/2}/n^{2/5}) \times O_p(m^{1/2}/n^{2/5}) \\
&= O_p(m/n^{4/5}) \\
&= O_p(m^{1/2}) \times O_p(m^{1/2}/n^{4/5}) = o_p(m^{1/2}), \qquad \text{(B.27)}
\end{aligned}
$$

which makes use of $m/n \to 0$, $\sum_{j=1}^{n-1} k^2(j/m) = O(m)$ and (B.22). We now focus on $\mathcal{A}_{12}$. By Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|\mathcal{A}_{12}| &\leq \left\{\sum_{j=1}^{n-1} k^2(j/m)(\hat{R}_a(j) - R_a(j))^2\right\}^{1/2} \left\{\sum_{j=1}^{n-1} k^2(j/m)R_b^2(j)\right\}^{1/2} \\
&= O_p(m^{1/2}/n^{2/5}) \times \left\{\sum_{j=1}^{n-1} R_b^2(j) + \sum_{j=1}^{n-1}[k^2(j/m) - 1]R_b^2(j)\right\}^{1/2} \\
&= O_p(m^{1/2}/n^{2/5}) \times O_p(1) \\
&= O_p(m^{2/5}/n^{2/5}) \times O_p(m^{1/10}) = o_p(m^{1/2}) \qquad \text{(B.28)}
\end{aligned}
$$

where the first equality makes use of the result in (B.27), the second equality follows by Assumption 3(i), dominated convergence theorem, $\lim_{m\to\infty} k^2(j/m) - 1 \to 0$ and $|k^2(j/m) - 1| \leq 1$. By the same reasonings, we also have $\mathcal{A}_{13} = o_p(m^{1/2})$.

We now focus on $\mathcal{A}_2$. Given $\hat{S}_{ab} - S_{ab} = o_p(1)$ it suffices to show $\sum_{j=1}^{n-1} k^2(j/m)R_a(j)R_b(j) = O_p(1)$. We have by Cauchy-Schwarz inequality

$$
\begin{aligned}
\sum_{j=1}^{n-1} k^2(j/m)R_a(j)R_b(j) &\leq \left\{\sum_{j=1}^{n-1} k^2(j/m)R_a^2(j)\right\}^{1/2} \left\{\sum_{j=1}^{n-1} k^2(j/m)R_b^2(j)\right\}^{1/2} \\
&= O_p(1), \qquad \text{(B.29)}
\end{aligned}
$$

which makes use of the result in (B.28) This completes the proof.

□

Proof of Lemma B.4. It suffices to show

$$\hat{\boldsymbol{R}}(j)'\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{R}}(j) \xrightarrow{p} \boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j).$$

By Assumption 2 we have $\hat{\boldsymbol{\Sigma}} \xrightarrow{p} \boldsymbol{\Sigma}$. The proof completes by the continuous mapping theorem and by noting that we have $\hat{\boldsymbol{R}}(j) \xrightarrow{p} \boldsymbol{R}(j)$ from the result in (B.22).

□

Appendix C Proof of Proposition 1

Recall the formula of Q_r

$$Q_r = \frac{n\sum_{j=1}^{n-1} k^2(j/m)\hat{\rho}_{nr}^2(j) - \sum_{j=1}^{n-1} k^2(j/m)}{\left\{2\sum_{j=1}^{n-1} k^4(j/m)\right\}^{1/2}}.$$

Recall the proposition

Proposition 1. Suppose the conditions of Theorem 1 hold. Then $Q_r \xrightarrow{d} \mathrm{N}(0,1)$ under $\mathbb{H}_0$.

Proof of Proposition 1. Recall that $\hat{\rho}_n(j)$ is the j-th element of $\hat{\boldsymbol{\rho}}_n$ and that $\hat{\rho}_{nr}(j)$ is the j-th element of $\hat{\boldsymbol{\Delta}}^{-1/2}\hat{\boldsymbol{\rho}}_n$. Let $\boldsymbol{r}_n$ collect $\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\}_{j=1}^{n-1}$. Let $\boldsymbol{r}_{nr} \equiv \boldsymbol{\Delta}^{-1/2}\boldsymbol{r}_n$, and let $r_{nr}(j)$ denote the j-th element of $\boldsymbol{r}_{nr}$. The proposition can be given by the following lemmas

Lemma C.1. Suppose the conditions of Proposition 1 hold. Then

$$\frac{n\sum_{j=1}^{n-1} k^2(j/m)r_{nr}^2(j) - \sum_{j=1}^{n-1} k^2(j/m)}{\left\{2\sum_{j=1}^{n-1} k^4(j/m)\right\}^{1/2}} \xrightarrow{d} \mathrm{N}(0,1).$$

Lemma C.2. Suppose the conditions of Proposition 1 hold. Then

$$\frac{n\sum_{j=1}^{n-1} k^2(j/m)\left\{\hat{\rho}_{nr}^2(j) - r_{nr}^2(j)\right\}}{\left\{2\sum_{j=1}^{n-1} k^4(j/m)\right\}^{1/2}} = o_p(1).$$

□

Proof of Lemma C.1. The result is immediate from the proof of Lemma B.1. The key is to note that we have from (B.8)–(B.10) $\sqrt{n}\boldsymbol{r}_n \xrightarrow{d} N(0,\boldsymbol{\Delta})$, where the (i,j)-th element of $\boldsymbol{\Delta}$ is given by $\boldsymbol{\Delta}(i,j) = \delta(i,j) + \omega\boldsymbol{R}(i)'\boldsymbol{\Sigma}\boldsymbol{R}(j)$. Repeating the rest of the proof of Lemma B.1 gives the desired result. □

Proof of Lemma C.2. Put $\{k^2(j/m)\}_{j=1}^{n-1}$ in the vector $\boldsymbol{K}_{n2}$. We have from the proof of Lemma B.2 that

$$m^{-1/2}n\hat{\boldsymbol{\rho}}_n'\boldsymbol{K}_{n2}\hat{\boldsymbol{\rho}}_n - m^{-1/2}n\boldsymbol{r}_n'\boldsymbol{K}_{n2}\boldsymbol{r}_n = o_p(1).$$

Next, the proof of Lemma B.4 implies that $\hat{\boldsymbol{\Delta}} - \boldsymbol{\Delta} = o_p(1)$. By the continuous mapping theorem

$$m^{-1/2} n \hat{\boldsymbol{\rho}}_n' \hat{\boldsymbol{\Delta}}^{-1/2} \boldsymbol{K}_{n2} \hat{\boldsymbol{\Delta}}^{-1/2} \hat{\boldsymbol{\rho}}_n - m^{-1/2} n \boldsymbol{r}_n' \boldsymbol{\Delta}^{-1/2} \boldsymbol{K}_{n2} \boldsymbol{\Delta}^{-1/2} \boldsymbol{r}_n = o_p(1).$$

Given $\sum_{j=1}^{n-1} k^4(j/m) = O(m)$, we have

$$\frac{n \hat{\boldsymbol{\rho}}_n' \hat{\boldsymbol{\Delta}}^{-1/2} \boldsymbol{K}_{n2} \hat{\boldsymbol{\Delta}}^{-1/2} \hat{\boldsymbol{\rho}}_n - n \boldsymbol{r}_n' \boldsymbol{\Delta}^{-1/2} \boldsymbol{K}_{n2} \boldsymbol{\Delta}^{-1/2} \boldsymbol{r}_n}{\{2 \sum_{j=1}^{n-1} k^4(j/m)\}^{1/2}} = o_p(1).$$

This completes the proof. □

Appendix D Proof of Theorem 2

Recall the result

Theorem 2. Suppose Assumptions 1–6 in the online appendix hold, and let $T \to \infty$, $n \to \infty$, $n/T \to \omega < \infty$, $m \to \infty$, $m/n \to 0$. Then

$$\frac{m^{1/2}}{n} Q \xrightarrow{p} \sum_{j=1}^{\infty} \rho^2(j) \Big/ \left[2 \int_0^{\infty} k^4(z) \mathrm{d}z \right]^{1/2}.$$

Proof of Theorem 2. Given Lemma B.4, to verify the theorem it suffices to show the following lemmas

Lemma D.1. Suppose the conditions of Theorem 2 hold. Then

$$\sum_{j=1}^{n-1} k^2(j/m) \left\{ \rho(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)' \nabla_{\boldsymbol{\theta}} [\rho(j)] \right\}^2 \xrightarrow{p} \sum_{j=1}^{\infty} \rho^2(j).$$

Lemma D.2. Suppose the conditions of Theorem 2 hold. Then

$$\sum_{j=1}^{n-1} k^2(j/m) \hat{\rho}_n^2(j) - \left\{ \rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)' \nabla_{\boldsymbol{\theta}} [\rho_n(j)] \right\}^2 = o_p(1).$$

Lemma D.3. Suppose the conditions of Theorem 2 hold. Then

$$\sum_{j=1}^{n-1} k^2(j/m) \left\{ \rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)' \nabla_{\boldsymbol{\theta}} [\rho_n(j)] \right\}^2 - \left\{ \rho(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)' \nabla_{\boldsymbol{\theta}} [\rho(j)] \right\}^2 = o_p(1).$$

Lemma D.4. Suppose the conditions of Theorem 2 hold. Then

$$n^{-1}\sum_{j=1}^{n-1} k^2(j/m)[1+\omega\boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)] = o_p(1).$$

Lemma D.5. Suppose the conditions of Theorem 2 hold. Then

$$m^{-1}V_{1n}(k) \xrightarrow{p} 2\int_0^\infty k^4(z)\mathrm{d}z.$$

□

Proof of Lemma D.1. We write

$$\begin{aligned}
&\sum_{j=1}^{n-1} k^2(j/m)\left\{\rho(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho(j)]\right\}^2 \\
&= \sum_{j=1}^{n-1} k^2(j/m)\rho^2(j) + \sum_{j=1}^{n-1} k^2(j/m)\{(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho(j)]\}^2 \\
&\quad + 2\sum_{j=1}^{n-1} k^2(j/m)\rho(j)(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho(j)] \\
&= \mathcal{B}_{1n} + \mathcal{B}_{2n} + \mathcal{B}_{3n}, \text{ say.}
\end{aligned}$$

To complete the proof we shall show $\mathcal{B}_{1n} \xrightarrow{p} \sum_{j=1}^{\infty}\rho^2(j)$, $\mathcal{B}_{2n} \xrightarrow{p} 0$ and $\mathcal{B}_{3n} \xrightarrow{p} 0$. We begin with $\mathcal{B}_{2n}$. By Cauchy-Schwarz inequality we have

$$\begin{aligned}
\mathcal{B}_{2n} &= \sum_{j=1}^{n-1} k^2(j/m)\{(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho(j)]\}^2 \\
&\le \|\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\|^2 \sum_{j=1}^{n-1} k^2(j/m)\|\nabla_{\boldsymbol{\theta}}[\rho(j)]\|^2 \\
&= O_p(m/T) = o_p(1).
\end{aligned}$$

The last equality follows by Markov's inequality, Assumptions 2, 4 and 5. We now focus on $\mathcal{B}_{1n}$

$$\begin{aligned}\mathcal{B}_{1n} &= \sum_{j=1}^{n-1} k^2(j/m)\rho^2(j) \\ &= \sum_{j=1}^{\infty} \rho^2(j) - \sum_{j=n}^{\infty} \rho^2(j) + \sum_{j=1}^{n-1} [k^2(j/m) - 1]\rho^2(j) \\ &\xrightarrow{p} \sum_{j=1}^{\infty} \rho^2(j).\end{aligned}$$

The result follows by noting the summability condition in Assumption 6, which implies $\sum_{j=1}^{\infty} \rho^2(j) = \gamma^{-2}(0) \sum_{j=1}^{\infty} \gamma^2(j) < \infty$. This implies that the second term goes to zero, and that together with dominated convergence theorem, $\lim_{m\to\infty} k^2(j/m) - 1 \to 0$ and $|k^2(j/m) - 1| \leq 1$, the third term goes to zero.

Finally, we write for $\mathcal{B}_{3n}$

$$\begin{aligned}\mathcal{B}_{3n} &= 2\sum_{j=1}^{n-1} k^2(j/m)\rho(j)(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho(j)] \\ &\leq 2\left\{\sum_{j=1}^{n-1} k^2(j/m)\rho^2(j)\right\}^{1/2} \times \left\{\sum_{j=1}^{n-1} k^2(j/m)\{(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho(j)]\}^2\right\}^{1/2} \\ &= 2\mathcal{B}_{1n}^{1/2} \times \mathcal{B}_{2n}^{1/2} \\ &= O(1) \times o_p(1) = o_p(1).\end{aligned}$$

This completes the proof.

□

Proof of Lemma D.2. By reasonings similar to that in the proof of Lemma B.2, we have for $1 \leq j \leq n-1$

$$\begin{aligned}& n\hat{\rho}_n^2(j) - n\left\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\right\}^2 \\ &= 2\{\sqrt{n}\rho_n(j) + \sqrt{n}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\}O_p(n^{-1/2}) + O_p(n^{-1}) \\ &= O_p(1), \end{aligned} \tag{D.1}$$

which implies

$$\hat{\rho}_n^2(j) - \left\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\right\}^2 = O_p(1/n). \tag{D.2}$$

Therefore we have

$$
\begin{aligned}
&\sum_{j=1}^{n-1} k^2(j/m)\left\{\hat{\rho}_n^2(j) - \left\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\right\}^2\right\} \\
&\leq m \sup_{1\leq j\leq n-1}\left\{\hat{\rho}_n^2(j) - \left\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\right\}^2\right\}\left\{m^{-1}\sum_{j=1}^{n-1} k^2(j/m)\right\} \\
&= O_p(m/n) = o_p(1). \qquad \text{(D.3)}
\end{aligned}
$$

This completes the proof.

□

Proof of Lemma D.3. We write

$$
\begin{aligned}
&\sum_{j=1}^{n-1} k^2(j/m)\left\{\rho_n(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho_n(j)]\right\}^2 - \left\{\rho(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho(j)]\right\}^2 \\
&= \sum_{j=1}^{n-1} k^2(j/m)[\rho_n^2(j) - \rho^2(j)] + \sum_{j=1}^{n-1} k^2(j/m)\{(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'[\nabla_{\boldsymbol{\theta}}[\rho_n(j)] - \nabla_{\boldsymbol{\theta}}[\rho(j)]]\}^2 \\
&\quad + 2\sum_{j=1}^{n-1} k^2(j/m)\rho(j)(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'[\nabla_{\boldsymbol{\theta}}[\rho_n(j)] - \nabla_{\boldsymbol{\theta}}[\rho(j)]] \\
&= \mathcal{C}_{1n} + \mathcal{C}_{2n} + \mathcal{C}_{3n}, \text{ say.}
\end{aligned}
$$

To complete the proof we shall show $\mathcal{C}_{in} \xrightarrow{p} 0$, for $i = 1, 2, 3$. We begin with $\mathcal{C}_{2n}$. By Cauchy-Schwarz inequality we have

$$
\begin{aligned}
\mathcal{C}_{2n} &= \sum_{j=1}^{n-1} k^2(j/m)\{(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'[\nabla_{\boldsymbol{\theta}}[\rho_n(j)] - \nabla_{\boldsymbol{\theta}}[\rho(j)]]\}^2 \\
&\leq \|\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\|^2 \sum_{j=1}^{n-1} k^2(j/m)\|\nabla_{\boldsymbol{\theta}}[\rho_n(j)] - \nabla_{\boldsymbol{\theta}}[\rho(j)]\|^2 \\
&= O_p(m/T) = o_p(1).
\end{aligned}
$$

The last equality follows by Markov's inequality, Assumptions 2, 4 and 5. We now focus on $\mathcal{C}_{1n}$

$$
\begin{aligned}
|\mathcal{C}_{1n}| &= \left|\sum_{j=1}^{n-1} k^2(j/m)\rho^2(j) - \sum_{j=1}^{n-1} k^2(j/m)\rho_n^2(j)\right| \\
&= \left|\gamma^{-2}(0)\sum_{j=1}^{n-1} k^2(j/m)\gamma^2(j) - \gamma_n^{-2}(0)\sum_{j=1}^{n-1} k^2(j/m)\gamma_n^2(j)\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left|\gamma^{-2}(0) - \gamma_n^{-2}(0)\right| \sum_{j=1}^{n-1} k^2(j/m)\gamma^2(j) + \gamma_n^{-2}(0) \left\{ \sum_{j=1}^{n-1} k^2(j/m) \left|\gamma^2(j) - \gamma_n^2(j)\right| \right\} \\
&= \left|\gamma^{-2}(0) - \gamma_n^{-2}(0)\right| \mathcal{C}_{11n} + \gamma_n^{-2}(0)\mathcal{C}_{12n}, \text{ say.}
\end{aligned}
$$

We have $|\gamma^{-2}(0) - \gamma_n^{-2}(0)| \mathcal{C}_{11n} = o_p(1)$, which follows by Markov's inequality, $\mathbb{E}[\gamma(0) - \gamma_n(0)] = 0$, and $\mathcal{C}_{11n} = \sum_{j=1}^{n-1} k^2(j/m)\gamma^2(j) \xrightarrow{p} \sum_{j=1}^{\infty} \gamma^2(j) \leq \infty$ as implied in the proof of Lemma D.1. Because $\gamma_n(0) = O(1)$, to show $\mathcal{C}_{1n} = o_p(1)$ it remains to show that $\mathcal{C}_{12n}$ vanishes in probability. Since $|\gamma^2(j) - \gamma_n^2(j)| = [\gamma_n(j) - \gamma(j)]^2 + 2\,|\gamma(j)|\,|\gamma_n(j) - \gamma(j)|$, we have by Cauchy-Schwarz inequality

$$
\begin{aligned}
\mathcal{C}_{12n} &= \sum_{j=1}^{n-1} k^2(j/m)[\gamma(j) - \gamma_n(j)]^2 + 2\sum_{j=1}^{n-1} k^2(j/m)\,|\gamma(j)|\,|\gamma_n(j) - \gamma(j)| \\
&\leq \sum_{j=1}^{n-1} k^2(j/m)[\gamma(j) - \gamma_n(j)]^2 + 2\mathcal{C}_{11n}^{1/2} \left\{ \sum_{j=1}^{n-1} k^2(j/m)[\gamma(j) - \gamma_n(j)]^2 \right\}^{1/2}.
\end{aligned}
$$

By observing that we have $\sup_{1\leq j\leq n-1} \mathbb{V}[\gamma_n(j)] \leq cn^{-1}$ given Assumption 6 (see, e.g., Hannan, 1970, p. 209), we therefore obtain $\mathcal{C}_{12n} = O_p(m^{1/2}/n^{1/2}) = o_p(1)$. Finally, given $\sum_{j=1}^{n-1} k^2(j/m)\rho^2(j) \xrightarrow{p} \sum_{j=1}^{\infty} \rho^2(j) \leq \infty$, and $\mathcal{C}_{2n} = o_p(1)$, we have $\mathcal{C}_{3n} = o_p(1)$, since

$$
\mathcal{C}_{3n} \leq 2\mathcal{C}_{2n}^{1/2} \left\{ \sum_{j=1}^{n-1} k^2(j/m)\rho^2(j) \right\}^{1/2}.
$$

This completes the proof.

□

Proof of Lemma D.4. Because $m/n \to 0$, it suffices to show $\sum_{j=1}^{n-1} k^2(j/m)[1+\omega \boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)] = O(m)$. We write

$$
\sum_{j=1}^{n-1} k^2(j/m)[1 + \omega \boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)] = \sum_{j=1}^{n-1} k^2(j/m) + \omega \sum_{j=1}^{n-1} k^2(j/m)\boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j). \qquad \text{(D.4)}
$$

The first term in (D.4) is $O(m)$ by Assumption 4. We write for the second term

$$
\begin{aligned}
&\omega \sum_{j=1}^{n-1} k^2(j/m)\boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j) \\
&\leq \omega \|\boldsymbol{\Sigma}\| \sum_{j=1}^{n-1} k^2(j/m)\|\boldsymbol{R}(j)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \omega \|\boldsymbol{\Sigma}\| \left\{ \sum_{j=1}^{n-1} \|\boldsymbol{R}(j)\|^2 + \sum_{j=1}^{n-1} [k^2(j/m) - 1] \|\boldsymbol{R}(j)\|^2 \right\} \\
&= O(1), \qquad \text{(D.5)}
\end{aligned}
$$

where we make use of Assumption 2, Assumption 3(i), dominated convergence theorem, $\lim_{m\to\infty} k^2(j/m) - 1 \to 0$ and $|k^2(j/m) - 1| \leq 1$. This completes the proof. □

Proof of Lemma D.5. We write

$$
\begin{aligned}
m^{-1} V_{1n}(k) = {} & 2m^{-1} \sum_{j=1}^{n-1} k^4(j/m) [1 + \omega \boldsymbol{R}(j)' \boldsymbol{\Sigma} \boldsymbol{R}(j)]^2 \\
& + 2m^{-1} \sum_{j=1}^{n-1} \sum_{\substack{i=1 \\ i \neq j}}^{n-1} k^2(i/m) k^2(j/m) [\omega \boldsymbol{R}(i)' \boldsymbol{\Sigma} \boldsymbol{R}(j)]^2. \qquad \text{(D.6)}
\end{aligned}
$$

To complete the proof we show that the first term converges to $2 \int_0^\infty k^4(z) \mathrm{d}z$, and that the second term vanishes. We write for the first term

$$
\begin{aligned}
& 2m^{-1} \sum_{j=1}^{n-1} k^4(j/m) [1 + \omega \boldsymbol{R}(j)' \boldsymbol{\Sigma} \boldsymbol{R}(j)]^2 \\
&= 2m^{-1} \sum_{j=1}^{n-1} k^4(j/m) + 4m^{-1} \omega \sum_{j=1}^{n-1} k^4(j/m) \boldsymbol{R}(j)' \boldsymbol{\Sigma} \boldsymbol{R}(j) + 2m^{-1} \omega^2 \sum_{j=1}^{n-1} k^4(j/m) [\boldsymbol{R}(j)' \boldsymbol{\Sigma} \boldsymbol{R}(j)]^2. \qquad \text{(D.7)}
\end{aligned}
$$

For the first term in (D.7), we have $2m^{-1} \sum_{j=1}^{n-1} k^4(j/m) \to 2 \int_0^\infty k^4(z) \mathrm{d}z$ given $m \to \infty$ as $n \to \infty$, $m/n \to 0$. In the following we show that the second and third terms in (D.7) are $o(1)$. We write for the second term

$$
\begin{aligned}
& 4m^{-1} \omega \sum_{j=1}^{n-1} k^4(j/m) \boldsymbol{R}(j)' \boldsymbol{\Sigma} \boldsymbol{R}(j) \\
& \leq 4m^{-1} \omega \|\boldsymbol{\Sigma}\| \sum_{j=1}^{n-1} k^4(j/m) \|\boldsymbol{R}(j)\|^2 \\
&= 4m^{-1} \omega \|\boldsymbol{\Sigma}\| \left\{ \sum_{j=1}^{n-1} \|\boldsymbol{R}(j)\|^2 + \sum_{j=1}^{n-1} [k^4(j/m) - 1] \|\boldsymbol{R}(j)\|^2 \right\} \\
&= O(m^{-1}) = o(1), \qquad \text{(D.8)}
\end{aligned}
$$

where we make use of Assumption 2, Assumption 3(i), dominated convergence theorem, $\lim_{m\to\infty} k^4(j/m) - 1 \to 0$ and $|k^4(j/m) - 1| \leq 1$. By Cauchy-Schwarz inequality we write for the third term in (D.7)

$$
\begin{aligned}
&2m^{-1}\omega^2\sum_{j=1}^{n-1}k^4(j/m)[\boldsymbol{R}(j)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2\\
&\le 2m^{-1}\omega^2\|\boldsymbol{\Sigma}\|^2\left\{\sum_{j=1}^{n-1}k^2(j/m)\|\boldsymbol{R}(j)\|^2\right\}\left\{\sum_{j=1}^{n-1}k^2(j/m)\|\boldsymbol{R}(j)\|^2\right\}\\
&\le O(m^{-1})\left\{\sum_{j=1}^{n-1}k^2(j/m)\|\boldsymbol{R}(j)\|^2\right\}^2\\
&= O(m^{-1})\left\{\left[\sum_{j=1}^{n-1}\|\boldsymbol{R}(j)\|^2+\sum_{j=1}^{n-1}[k^2(j/m)-1]\|\boldsymbol{R}(j)\|^2\right]\right\}^2\\
&= O(m^{-1})\times O(1)=o(1), \qquad \text{(D.9)}
\end{aligned}
$$

where we make use of Assumption 2, Assumption 3(i), dominated convergence theorem, $\lim_{m\to\infty}k^2(j/m)-1\to 0$ and $|k^2(j/m)-1|\le 1$.

To complete the proof it remains to show that the second term in (D.6) is $o(1)$. By Cauchy-Schwarz inequality, we write

$$
\begin{aligned}
&2m^{-1}\sum_{j=1}^{n-1}\sum_{\substack{i=1\\ i\neq j}}^{n-1}k^2(i/m)k^2(j/m)[\omega\boldsymbol{R}(i)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2\\
&\le 2m^{-1}\omega^2\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}k^2(i/m)k^2(j/m)[\boldsymbol{R}(i)'\boldsymbol{\Sigma}\boldsymbol{R}(j)]^2\\
&\le 2m^{-1}\omega^2\|\boldsymbol{\Sigma}\|\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}k^2(i/m)\|\boldsymbol{R}(i)\|^2k^2(j/m)\|\boldsymbol{R}(j)\|^2\\
&= 2m^{-1}\omega^2\|\boldsymbol{\Sigma}\|\left\{\sum_{j=1}^{n-1}k^2(j/m)\|\boldsymbol{R}(j)\|^2\right\}\left\{\sum_{i=1}^{n-1}k^2(i/m)\|\boldsymbol{R}(i)\|^2\right\}\\
&= O(m^{-1})\left\{\sum_{j=1}^{n-1}k^2(j/m)\|\boldsymbol{R}(j)\|^2\right\}^2\\
&= o(1),
\end{aligned}
$$

where the last equality makes use of the result in (D.9). This completes the proof.

□

Appendix E Proof of Proposition 2

Proof of Proposition 2. To ease the proof we introduce the following notations. Let $\boldsymbol{r}$ collect $\{\rho(j) + (\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho(j)]\}_{j=1}^{n-1}$. Let $\boldsymbol{\rho}$ collect $\{\rho(j)\}_{j=1}^{n-1}$, let $\boldsymbol{\rho}_{\boldsymbol{\nabla}}$ collect $\{(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0)'\nabla_{\boldsymbol{\theta}}[\rho(j)]\}_{j=1}^{n-1}$. Let $\boldsymbol{r}_r \equiv \boldsymbol{\Delta}^{-1/2}\boldsymbol{r}$, and let $r_r(j)$ denote the j-th element of $\boldsymbol{r}_r$. Let $\boldsymbol{\rho}_r \equiv \boldsymbol{\Delta}^{-1/2}\boldsymbol{\rho}$, and let $\rho_r(j)$ denote the j-th element of $\boldsymbol{\rho}_r$. Recall the result

Proposition 2. Suppose the conditions of Theorem 2 hold. Then

$$\frac{m^{1/2}}{n} Q_r \xrightarrow{p} \sum_{j=1}^{\infty} \rho_r^2(j) \Big/ \left[2\int_0^{\infty} k^4(z)\mathrm{d}z\right]^{1/2}.$$

The theorem is given by the following lemmas

Lemma E.1. Suppose the conditions of Proposition 2 hold. Then

$$\sum_{j=1}^{n-1} k^2(j/m) r_r^2(j) \xrightarrow{p} \sum_{j=1}^{\infty} \rho_r^2(j)$$

Lemma E.2. Suppose the conditions of Proposition 2 hold. Then

$$\sum_{j=1}^{n-1} k^2(j/m)\hat{\rho}_{nr}^2(j) - r_{nr}^2(j) = o_p(1)$$

Lemma E.3. Suppose the conditions of Proposition 2 hold. Then

$$\sum_{j=1}^{n-1} k^2(j/m) r_{nr}^2(j) - r_r^2(j) = o_p(1)$$

□

Proof of Lemma E.1. To begin we show $\sum_{j=1}^{n-1} k^2(j/m) r_r^2(j) \xrightarrow{p} \sum_{j=1}^{n-1} k^2(j/m)\rho_r^2(j)$. To that we note $\boldsymbol{\rho}_{\boldsymbol{\nabla}} = O_p(T^{-1/2})$ and $\boldsymbol{\Delta} = O(1)$, which implies $\boldsymbol{r}_r = \boldsymbol{\Delta}^{-1/2}\boldsymbol{\rho} + o_p(1)$. With this we have

$$\begin{aligned}
&\sum_{j=1}^{n-1} k^2(j/m) r_r^2(j) \\
&= \boldsymbol{r}_r' K_{n2} \boldsymbol{r}_r \\
&= [\boldsymbol{\Delta}^{-1/2}\boldsymbol{\rho} + \boldsymbol{\Delta}^{-1/2}\boldsymbol{\rho}_{\nabla}]' K_{n2} [\boldsymbol{\Delta}^{-1/2}\boldsymbol{\rho} + \boldsymbol{\Delta}^{-1/2}\boldsymbol{\rho}_{\nabla}] \\
&\xrightarrow{p} [\boldsymbol{\Delta}^{-1/2}\boldsymbol{\rho}]' K_{n2} [\boldsymbol{\Delta}^{-1/2}\boldsymbol{\rho}]
\end{aligned}$$

$$= \sum_{j=1}^{n-1} k^2(j/m)\rho_r^2(j).$$

In the following we shall show $\sum_{j=1}^{n-1} k^2(j/m)\rho_r^2(j) \xrightarrow{p} \sum_{j=1}^{\infty} \rho_r^2(j)$. We write

$$\begin{aligned}\sum_{j=1}^{n-1} k^2(j/m)\rho_r^2(j) &= \sum_{j=1}^{\infty} \rho_r^2(j) - \sum_{j=n}^{\infty} \rho_r^2(j) + \sum_{j=1}^{n-1}[k^2(j/m)-1]\rho_r^2(j) \\ &= \mathcal{D}_{1n} - \mathcal{D}_{2n} + \mathcal{D}_{3n}, \text{ say.}\end{aligned}$$

To complete the proof we shall show $\mathcal{D}_{2n} \to 0$ and $\mathcal{D}_{3n} \to 0$. We first show that $\mathcal{D}_{1n} = \sum_{j=1}^{\infty} \rho_r^2(j) < \infty$. Denote by λ_{max} the largest eigenvalues of $\boldsymbol{\Delta}^{-1}$, and we note that $0 < \lambda_{max} \leq 1$ because $\boldsymbol{\Delta}$ is given by the sum of an identity matrix and a positive semidefinite matrix. Therefore we have $\sum_{j=1}^{n-1} \rho_r^2(j) = \boldsymbol{\rho}'\boldsymbol{\Delta}^{-1}\boldsymbol{\rho} \leq \lambda_{max}\|\boldsymbol{\rho}\|^2 \leq \|\boldsymbol{\rho}\|^2 = \boldsymbol{\rho}'\boldsymbol{\rho} = \sum_{j=1}^{n-1}\rho^2(j)$, which, as $n \to \infty$, yields $\sum_{j=1}^{\infty}\rho_r^2(j) \leq \sum_{j=1}^{\infty}\rho^2(j) < \infty$. The last inequality follows from the result of Lemma D.1. This implies $\mathcal{D}_{2n} \to 0$. Finally, we have $\mathcal{D}_{3n} \to 0$ by dominated convergence theorem, $\lim_{m\to\infty} k^2(j/m)-1 \to 0$, $|k^2(j/m)-1| \leq 1$ and $\mathcal{D}_{1n} < \infty$. This completes the proof.

□

Proof of Lemma E.2. We have shown in Lemma D.2 that

$$\hat{\boldsymbol{\rho}}_n'\boldsymbol{K}_{n2}\hat{\boldsymbol{\rho}}_n - \boldsymbol{r}_n'\boldsymbol{K}_{n2}\boldsymbol{r}_n = o_p(1).$$

We also have $\hat{\boldsymbol{\Delta}} - \boldsymbol{\Delta} = o_p(1)$ from the proof of Lemma B.4. Therefore, by the continuous mapping theorem

$$\hat{\boldsymbol{\rho}}_n'\hat{\boldsymbol{\Delta}}^{-1/2}\boldsymbol{K}_{n2}\hat{\boldsymbol{\Delta}}^{-1/2}\hat{\boldsymbol{\rho}}_n - \boldsymbol{r}_n'\boldsymbol{\Delta}^{-1/2}\boldsymbol{K}_{n2}\boldsymbol{\Delta}^{-1/2}\boldsymbol{r}_n = o_p(1).$$

This completes the proof.

□

Proof of Lemma E.3. The proof is immediate by noting that we have from Lemma D.3

$$\boldsymbol{r}_n'\boldsymbol{K}_{n2}\boldsymbol{r}_n - \boldsymbol{r}'\boldsymbol{K}_{n2}\boldsymbol{r} = o_p(1),$$

which, given $\boldsymbol{\Delta} = O(1)$, implies

$$\boldsymbol{r}_n'\boldsymbol{\Delta}^{-1/2}\boldsymbol{K}_{n2}\boldsymbol{\Delta}^{-1/2}\boldsymbol{r}_n - \boldsymbol{r}'\boldsymbol{\Delta}^{-1/2}\boldsymbol{K}_{n2}\boldsymbol{\Delta}^{-1/2}\boldsymbol{r} = o_p(1).$$

□

Appendix F Computing $\hat{\boldsymbol{R}}(j)$

Recall that $\boldsymbol{R}(j)$ is given by

$$\boldsymbol{R}(j) = \frac{1}{\alpha(1/3 - \alpha/4)} \mathbb{E}\left\{(H_{t-j}(\alpha, \boldsymbol{\theta}_0) - \alpha/2)\nabla_{\boldsymbol{\theta}}[H_t(\alpha, \boldsymbol{\theta}_0)]\right\},$$

and that $H_t(\alpha, \boldsymbol{\theta}_0)$ is given by

$$\begin{aligned} H_t(\alpha, \boldsymbol{\theta}_0) &= \mathbb{1}[F_{Y_2}(Y_{2,t}|\Omega_{t-1}, \boldsymbol{\theta}_0) \leq \alpha] \times [1 - F_{Y_1|Y_2 \leq VaR_2(\alpha)}(Y_{1,t}|\Omega_{t-1}, \boldsymbol{\theta}_0)] \\ &= \mathbb{1}(u_{2,t}(\boldsymbol{\theta}_0) \leq \alpha) \times (1 - u_{12,t}(\boldsymbol{\theta}_0)). \end{aligned}$$

Because the indicator function $\mathbb{1}(u_{2,t}(\boldsymbol{\theta}_0) \leq \alpha)$ in $H_t(\alpha, \boldsymbol{\theta}_0)$ is not differentiable, we follow Banulescu-Radu et al. (2021) to evaluate $\nabla_{\boldsymbol{\theta}}[H_t(\alpha, \boldsymbol{\theta}_0)]$ using $\nabla_{\boldsymbol{\theta}}[H_t^{\oplus}(\alpha, \boldsymbol{\theta}_0)]$, where the quantity $H_t^{\oplus}(\alpha, \boldsymbol{\theta}_0)$ is analogous to $H_t(\alpha, \boldsymbol{\theta}_0)$, but with the indicator function $\mathbb{1}(\cdot)$ replaced by a continuously differentiable estimator. To ensure comparability we use the same consistent smoothing estimator as that given in Banulescu-Radu et al. (2021, Appendix E). Therefore, the consistent estimator for $\boldsymbol{R}(j)$ is given by

$$\hat{\boldsymbol{R}}(j) = \boldsymbol{R}_n(j, \hat{\boldsymbol{\theta}}_T) = \frac{1}{\alpha(1/3 - \alpha/4)} \frac{1}{n-j} \sum_{t=T+1+j}^{T+n} \left\{(H_{t-j}(\alpha, \hat{\boldsymbol{\theta}}_T) - \alpha/2)\nabla_{\boldsymbol{\theta}}[H_t^{\oplus}(\alpha, \hat{\boldsymbol{\theta}}_T)]\right\}.$$

Similarly, $\tilde{\boldsymbol{R}}(j)$ is the analogous counterpart of $\hat{\boldsymbol{R}}(j)$ with true parameter $\boldsymbol{\theta}_0$

$$\tilde{\boldsymbol{R}}(j) = \boldsymbol{R}_n(j, \boldsymbol{\theta}_0) = \frac{1}{\alpha(1/3 - \alpha/4)} \frac{1}{n-j} \sum_{t=T+1+j}^{T+n} \left\{(H_{t-j}(\alpha, \boldsymbol{\theta}_0) - \alpha/2)\nabla_{\boldsymbol{\theta}}[H_t^{\oplus}(\alpha, \boldsymbol{\theta}_0)]\right\}.$$

Finally, we allow as with Banulescu-Radu et al. (2021, Appendix F) a small sample factor $\sqrt{1 - j/n}$ for $\boldsymbol{R}(j)$ and its estimators. This does not affect the asymptotic properties of our backtests.

Appendix G Institution tickers and names

Ticker	Institution name	Ticker	Institution name
ABK	AMBAC FINANCIAL GROUP INC	HIG	HARTFORD FINANCIAL SVCS GRP INC
ACAS	AMERICAN CAPITAL STRATEGIES LTD	HUM	HUMANA INC
AET	AETNA INC NEW	ICE	INTERCONTINENTALEXCHANGE INC
AFL	AFLAC INC	KEY	KEYCORP NEW
AGE	EDWARDS A G INC	LEH	LEHMAN BROTHERS HOLDINGS INC
AIG	AMERICAN INTERNATIONAL GROUP INC	LM	LEGG MASON INC
AIZ	ASSURANT INC	LNC	LINCOLN NATIONAL CORP
ALL	ALLSTATE CORP	LTR	LOEWS CORP
AMP	AMERIPRISE FINANCIAL INC	MA	MASTERCARD INC
AMTD	AMERITRADE HOLDING CORP	MBI	M B I A INC
AOC	AON CORP	MER	MERRILL LYNCH & CO INC
ATH	ANTHEM INC	MET	METLIFE INC
AXP	AMERICAN EXPRESS CO	MI	MARSHALL & ILSLEY CORP
BAC	BANK OF AMERICA CORP	MMC	MARSH & MCLENNAN COS INC
BBT	B B & T CORP	MTB	M & T BANK CORP
BEN	FRANKLIN RESOURCES INC	MWD	MORGAN STANLEY DEAN WITTER & CO
BK	BANK NEW YORK INC	NCC	NATIONAL CITY CORP
BKLY	BERKLEY W R CORP	NMX	NYMEX HOLDINGS INC
BLK	BLACKROCK INC	NTRS	NORTHERN TRUST CORP
BOT	C B O T HOLDINGS INC	NYX	N Y S E GROUP INC
BRK	BERKSHIRE HATHAWAY INC DEL	PBCT	PEOPLES BANK BRIDGEPORT
BSC	BEAR STEARNS COMPANIES INC	PFG	PRINCIPAL FINANCIAL GROUP INC
C	CITIGROUP INC	PGR	PROGRESSIVE CORP OH
CB	CHUBB CORP	PNC	P N C BANK CORP
CBG	C B RICHARD ELLIS GROUP INC	PRU	PRUDENTIAL FINANCIAL INC
CBH	COMMERCE BANCORP INC NJ	QCSB	QUEENS COUNTY BANCORP INC
CBSS	COMPASS BANCSHARES INC	RGBK	REGIONS FINANCIAL CORP
CCR	COUNTRYWIDE CREDIT INDS INC	SAFC	SAFECO CORP
CI	C I G N A CORP	SCH	SCHWAB CHARLES CORP NEW
CINF	CINCINNATI FINANCIAL CORP	SEIC	S E I INVESTMENTS CO
CIT	C I T GROUP INC NEW	SLM	S L M HOLDING CORP
CMA	COMERICA INC	SNV	SYNOVUS FINANCIAL CORP
CMB	CHASE MANHATTAN CORP NEW	SPC	ST PAUL COS INC
CME	CHICAGO MERCANTILE EXCH HLDG INC	STI	SUNTRUST BANKS INC
CNA	C N A FINANCIAL CORP	STT	STATE STREET CORP
COF	CAPITAL ONE FINANCIAL CORP	SV	STILWELL FINANCIAL INC
CVTY	COVENTRY HEALTH CARE INC	SVRN	SOVEREIGN BANCORP INC
EGRP	E TRADE GROUP INC	TMK	TORCHMARK CORP
FHS	FOUNDATION HEALTH SYSTEMS INC	TROW	T ROWE PRICE ASSOC INC
FITB	FIFTH THIRD BANCORP	UB	UNIONBANCAL CORP
FNF	FIDELITY NATIONAL FINANCIAL INC	UNH	UNITED HEALTHCARE CORP
FNM	FEDERAL NATIONAL MORTGAGE ASSN	UNM	UNUMPROVIDENT CORP
FRE	FEDERAL HOME LOAN MORTGAGE CORP	USB	U S BANCORP DEL
FTU	FIRST UNION CORP	WFC	WELLS FARGO & CO NEW
GNW	GENWORTH FINANCIAL INC	WM	WASHINGTON MUTUAL INC
GS	GOLDMAN SACHS GROUP INC	WU	WESTERN UNION CO
HBAN	HUNTINGTON BANCSHARES INC	ZION	ZIONS BANCORPORATION
HCBK	HUDSON CITY BANCORP INC		

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