SupplementaryMaterial

for

Idiosyncratic Volatility and the ICAPM Covariance Risk

Appendix

A. Proof of Proposition 1

Assume the true stock excess return generating process for security *i*:

$$R_{i,t+1} = \beta_{iM,t} R_{M,t+1} + \beta_{iH,t} R_{H,t+1} + \varepsilon_{i,t+1}, \tag{A.1}$$

where $R_{M,t+1}$ is the market excess return, $R_{H,t+1}$ is the return of the hedge portfolio, $\varepsilon_{i,t+1}$ is the true idiosyncratic shock for security *i* (which has mean zero by definition) and $\beta_{iM,t}$ and $\beta_{iH,t}$ are the corresponding loadings:

$$\beta_{iM,t} = \frac{\sigma_{iM,t}\sigma_{H,t}^2 - \sigma_{iH,t}\sigma_{MH,t}}{\sigma_{H,t}^2\sigma_{M,t}^2 - \sigma_{MH,t}^2}, \quad \beta_{iH,t} = \frac{\sigma_{iH,t}\sigma_{M,t}^2 - \sigma_{iM,t}\sigma_{MH,t}}{\sigma_{H,t}^2\sigma_{M,t}^2 - \sigma_{MH,t}^2}. \tag{A.2}$$

Suppose that econometricians only use a simple (misspecified) CAPM model to estimate the idiosyncratic risk:

$$R_{i,t+1} = b_{iM,t} R_{M,t+1} + \eta_{i,t+1}, \tag{A.3}$$

where $b_{iM,t}$ is the market beta for security i defined as $b_{iM,t} = \frac{\sigma_{iM,t}}{\sigma_{M,t}^2}$ and $\eta_{i,t+1}$ is the misspecified idiosyncratic return. It can be shown that:

$$\beta_{iM,t} - b_{iM,t} = \frac{\sigma_{iM,t}\sigma_{H,t}^2 - \sigma_{iH,t}\sigma_{MH,t}}{\sigma_{H,t}^2\sigma_{M,t}^2 - \sigma_{MH,t}^2} - \frac{\sigma_{iM,t}}{\sigma_{M,t}^2} = -\beta_{iH,t}\frac{\sigma_{MH,t}}{\sigma_{M,t}^2}. \tag{A.4}$$

The conditional mean of the misspecified stock idiosyncratic return is given by:

$$(A.5)\qquad E_t\left(\eta_{i,t+1}\right) = \mu_{i,t} - b_{iM,t}\mu_{M,t}$$
$$= \beta_{iM,t}\mu_{M,t} + \beta_{iH,t}\mu_{H,t} - \left(\beta_{iM,t} + \beta_{iH,t}\frac{\sigma_{MH,t}}{\sigma_{M,t}^2}\right)\mu_{M,t}$$
$$= \beta_{iH,t}\mu_{H,t} - \beta_{iH,t}\frac{\sigma_{MH,t}}{\sigma_{M,t}^2}\mu_{M,t}.$$

Based on the ICAPM, the risk premia associated with the market portfolio and the hedge portfolio are given by:

$$(A.6)\qquad \begin{cases}\mu_{M,t} = \gamma_M\sigma_{M,t}^2 + \gamma_H\sigma_{MH,t} \\ \mu_{H,t} = \gamma_M\sigma_{MH,t} + \gamma_H\sigma_{H,t}^2\end{cases}.$$

Equations (A.5) and (A.6) imply the following relation:

$$(A.7)\qquad E_t\left(\eta_{i,t+1}\right) = \beta_{iH,t}\left(\gamma_M\sigma_{MH,t} + \gamma_H\sigma_{H,t}^2\right) - \beta_{iH,t}\frac{\sigma_{MH,t}}{\sigma_{M,t}^2}\left(\gamma_M\sigma_{M,t}^2 + \gamma_H\sigma_{MH,t}\right)$$
$$= \gamma_H\beta_{iH,t}\frac{\sigma_{H,t}^2\sigma_{M,t}^2 - \sigma_{MH,t}^2}{\sigma_{M,t}^2}$$
$$= \gamma_H\frac{\sigma_{iH,t}}{\sigma_{H,t}^2}\sigma_{H,t}^2 - \gamma_H\frac{\sigma_{iM,t}}{\sigma_{M,t}^2}\sigma_{MH,t}$$
$$= \gamma_H b_{iH,t}\sigma_{H,t}^2 - \gamma_H b_{iM,t}\sigma_{MH,t}\ .$$

Similarly, the conditional variance of the misspecified firm idiosyncratic return is given by:

$$(A.8)\qquad Var_t\left(\eta_{i,t+1}\right) = Var_t\left(R_{i,t+1} - b_{iM,t}R_{M,t+1}\right)$$
$$= \beta_{iH,t}^2\left(\frac{\sigma_{MH,t}}{\sigma_{M,t}^2}\right)^2\sigma_{M,t}^2 + \beta_{iH,t}^2\sigma_{H,t}^2 - 2\beta_{iH,t}\beta_{iH,t}\frac{\sigma_{MH,t}^2}{\sigma_{M,t}^2} + \sigma_{\varepsilon_i,t}^2$$
$$= \beta_{iH,t}\frac{\sigma_{iH,t}}{\sigma_{H,t}^2}\sigma_{H,t}^2 - \beta_{iH,t}\frac{\sigma_{iM,t}}{\sigma_{M,t}^2}\sigma_{MH,t} + \sigma_{\varepsilon_i,t}^2$$
$$= \beta_{iH,t}b_{iH,t}\sigma_{H,t}^2 - \beta_{iH,t}b_{iM,t}\sigma_{MH,t} + \sigma_{\varepsilon_i,t}^2\ .$$

This completes the proof.

B. Proof of Corollary 1.1

Based on Proposition 1, the average idiosyncratic variances IV_t^F and IV_t^S corresponding to two different sets of weights, $w_{i,t}^F$ and $w_{i,t}^S$, can be expressed as different combinations of $\sigma_{H,t}^2$ and $\sigma_{MH,t}$:

$$IV_t^F = \sum_{i=1}^{N_t} w_{i,t}^F Var_t(\eta_{i,t+1}) = A_t^F \sigma_{H,t}^2 - B_t^F \sigma_{MH,t} + \Omega_t^{\mathrm{F}}, \tag{A.9}$$

where $A_t^F = \sum_{i=1}^{N_t} w_{i,t}^F \beta_{iH,t} b_{iH,t}$, $B_t^F = \sum_{i=1}^{N_t} w_{i,t}^F \beta_{iH,t} b_{iM,t}$, $\Omega_t^{\mathrm{F}} = \sum_{i=1}^{N_t} w_{i,t}^F \sigma_{\varepsilon_i,t}^2$; and

$$IV_t^S = A_t^S \sigma_{H,t}^2 - B_t^S \sigma_{MH,t} + \Omega_t^S, \tag{A.10}$$

where $A_t^S = \sum_{i=1}^{N_t} w_{i,t}^S \beta_{iH,t} b_{iH,t}$, $B_t^S = \sum_{i=1}^{N_t} w_{i,t}^S \beta_{iH,t} b_{iM,t}$, $\Omega_t^S = \sum_{i=1}^{N_t} w_{i,t}^S \sigma_{\varepsilon_i,t}^2$. From (A.9) and (A.10), we can express $\sigma_{H,t}^2$ and $\sigma_{MH,t}$ as linear combinations of $\widetilde{IV}_t^F$ and $\widetilde{IV}_t^S$ defined as:

$$\begin{cases} \widetilde{IV}_t^F \equiv IV_t^F - \Omega_t^F = A_t^F \sigma_{H,t}^2 - B_t^F \sigma_{MH,t} \\ \widetilde{IV}_t^S \equiv IV_t^S - \Omega_t^S = A_t^S \sigma_{H,t}^2 - B_t^S \sigma_{MH,t} \end{cases}. \tag{A.11}$$

Specificially,

$$\begin{cases} \sigma_{H,t}^2 = \frac{B_t^S}{A_t^F B_t^S - A_t^S B_t^F} \widetilde{IV}_t^F - \frac{B_t^F}{A_t^F B_t^S - A_t^S B_t^F} \widetilde{IV}_t^S \\ \sigma_{MH,t} = \frac{A_t^S}{A_t^F B_t^S - A_t^S B_t^F} \widetilde{IV}_t^F - \frac{A_t^F}{A_t^F B_t^S - A_t^S B_t^F} \widetilde{IV}_t^S \end{cases}. \tag{A.12}$$

This completes the proof.

C. Proof of Proposition 2

The proposition can be derived from Corollary 1.1 and the ICAPM pricing relationships. Plugging the expression for $\sigma_{MH,t}$ in equation (12) to the right-hand side of equation (4), we obtain the following expression for the conditional equity risk premium:

$$(\text{A}.13) \qquad \mu_{M,t} = \gamma_M \sigma_{M,t}^2 + \gamma_H \sigma_{MH,t} = \gamma_M \times \sigma_{M,t}^2 + C_t^F \times \widetilde{IV}_t^F - C_t^S \times \widetilde{IV}_t^S,$$

where $C_t^F = \frac{\gamma_H A_t^S}{A_t^F B_t^S - A_t^S B_t^F}$ and $C_t^S = \frac{\gamma_H A_t^F}{A_t^F B_t^S - A_t^S B_t^F}$.

This completes the proof of the first equality in Proposition 2.

As to the second approximate equality, the proof makes use of the first order Taylor expansion of $ln(x)$:

$$(\text{A}.14) \qquad ln(x) = ln(x_0) + \frac{1}{x_0}(x - x_0) + O(x^2),$$

when x is close to x_0. We apply (A.14) to $x = C_t^F \times \widetilde{IV}_t^F$ or $x = C_t^S \times \widetilde{IV}_t^S$ around the following point ψ_0 as x_0:

$$(\text{A}.15) \qquad \psi_0 \equiv \frac{E(C_t^F \times \widetilde{IV}_t^F) + E(C_t^S \times \widetilde{IV}_t^S)}{2}.$$

When estimating C_t^F and C_t^S using an expanding window, we find that $C_t^F \times \widetilde{IV}_t^F$ is on average close to $C_t^S \times \widetilde{IV}_t^S$. For example, the average $C_t^F \times \widetilde{IV}_t^F$ over the sample period is -0.0146, while the average $C_t^S \times \widetilde{IV}_t^S$ is -0.0098. Their time-series standard deviations are 0.0061 and 0.0044 respectively. The Taylor expansion gives:

$$(\text{A}.16) \qquad \begin{cases} ln(C_t^F \times \widetilde{IV}_t^F) \approx ln(\psi_0) + \dfrac{1}{\psi_0} C_t^F \times \widetilde{IV}_t^F - 1 \\ ln(C_t^S \times \widetilde{IV}_t^S) \approx ln(\psi_0) + \dfrac{1}{\psi_0} C_t^S \times \widetilde{IV}_t^S - 1 \end{cases}.$$

Taking the difference between the two equations above:

$$C_t^F \times \widetilde{IV}_t^F - C_t^S \times \widetilde{IV}_t^S \approx \psi_0 \ln\left(\frac{C_t^F}{C_t^S}\right) + \psi_0 \ln\left(\frac{\widetilde{IV}_t^F}{\widetilde{IV}_t^S}\right). \tag{A.17}$$

Plugging (A.17) into (A.13), we obtain equation (13) stated in Proposition 2. This completes the proof.

D. Proof of Proposition 3

The tail index proposed by Kelly and Jiang (2014) is:

$$\lambda_{t+1}^{Hill} = \frac{1}{K_{t+1}} \sum_{k=1}^{K_{t+1}} ln\left(\frac{\eta_{k,t+1}}{u_{t+1}}\right), \tag{A.18}$$

where $\eta_{k,t+1}$ is the *k*th daily residual return that falls below an extreme value threshold u_{t+1} during month *t*+1, u_{t+1} is the 5th percentile of the cross-section of individual stock residual returns, and K_{t+1} is the total number of these exceedances within month *t*+1. The residual returns are obtained after removing the exposures individual stock returns to common return factors under a benchmark factor model such as the CAPM. Based on (A.1), (A.3), and Proposition 1, we have:

$$\begin{cases} \eta_{k,t+1} = E_t(\eta_{k,t+1}) + e_{k,t+1} = \gamma_H b_{kH,t} \sigma_{H,t}^2 - \gamma_H b_{kM,t} \sigma_{MH,t} + e_{k,t+1} \\ u_{t+1} = E_t(u_{t+1}) + e_{u,t+1} = \gamma_H b_{uH,t} \sigma_{H,t}^2 - \gamma_H b_{uM,t} \sigma_{MH,t} + e_{u,t+1} \end{cases}, \tag{A.19}$$

where $e_{k,t+1}$ is a random variable with mean zero. The mean of $\bar{\eta}_{t+1} = \frac{1}{K_{t+1}} \sum_{k=1}^{K_{t+1}} \eta_{k,t+1}$ and u_{t+1} are both linear in $\sigma_{H,t}^2$ and $\sigma_{MH,t}$:

(A.20)
$$\begin{cases} E_t(\bar{\eta}_{t+1}) = \dfrac{1}{K_{t+1}} \sum_{k=1}^{K_{t+1}} E_t(\eta_{k,t+1}) = D_t^{\bar{\eta}} \sigma_{H,t}^2 - G_t^{\bar{\eta}} \sigma_{MH,t} \\ E_t(u_{t+1}) = D_t^u \sigma_{H,t}^2 - G_t^u \sigma_{MH,t} \end{cases},$$

where $\begin{cases} D_t^{\bar{\eta}} = \frac{1}{K_{t+1}} \sum_{k=1}^{K_{t+1}} \gamma_H b_{kH,t} \\ D_t^u = \gamma_H b_{uH,t} \end{cases}$, $\begin{cases} G_t^{\bar{\eta}} = \frac{1}{K_{t+1}} \sum_{k=1}^{K_{t+1}} \gamma_H b_{kM,t} \\ G_t^u = \gamma_H b_{uM,t} \end{cases}$.

From the two equations in (A.20), the covariance risk $\sigma_{MH,t}$ can be identified from $E_t(\bar{\eta}_{t+1})$ and $E_t(u_{t+1})$:

(A.21)
$$\sigma_{MH,t} = J_t^{\bar{\eta}} E_t(\bar{\eta}_{t+1}) - J_t^u E_t(u_{t+1}) = J_t^{\bar{\eta}} \bar{\eta}_{t+1} - J_t^u u_{t+1} + \left(J_t^u e_{u,t+1} - J_t^{\bar{\eta}} e_{\bar{\eta},t+1}\right),$$

where $J_t^{\bar{\eta}} = \frac{D_t^u}{D_t^{\bar{\eta}} G_t^u - D_t^u G_t^{\bar{\eta}}}$ and $J_t^u = \frac{D_t^{\bar{\eta}}}{D_t^{\bar{\eta}} G_t^u - D_t^u G_t^{\bar{\eta}}}$. To link the right-hand side of (A.18) to (A.21), we apply the first order Taylor expansion of $ln(x)$ in (A.14) with $x = J_t^{\bar{\eta}} \bar{\eta}_{t+1}$ or $x = J_t^u u_{t+1}$ around the following point ψ_0 as x_0:

(A.22)
$$\psi_0 \equiv \frac{E\left(J_t^{\bar{\eta}} \bar{\eta}_{t+1}\right) + E(J_t^u u_{t+1})}{2}.$$

When estimating $J_t^{\bar{\eta}}$ and J_t^u using an expanding window, we find that, $J_t^{\bar{\eta}} \bar{\eta}_{t+1}$ is on average close to $J_t^u u_{t+1}$. For example, the average $J_t^{\bar{\eta}} \bar{\eta}_{t+1}$ over the sample period is 0.0328, while the average $J_t^u u_{t+1}$ is 0.0335. Their time-series standard deviations are 0.0320 and 0.0288 respectively. The Taylor expansion gives:

$$(A.23) \qquad \begin{cases} ln(J_t^{\bar{\eta}} \bar{\eta}_{t+1}) \approx ln(\psi_0) + \frac{1}{\psi_0} J_t^{\bar{\eta}} \bar{\eta}_{t+1} - 1 \\ ln(J_t^{u} u_{t+1}) \approx ln(\psi_0) + \frac{1}{\psi_0} J_t^{u} u_{t+1} - 1 \end{cases}.$$

Taking the difference between the two equations above gives:

$$(A.24) \qquad J_t^{\bar{\eta}} \bar{\eta}_{t+1} - J_t^{u} u_{t+1} \approx \psi_0 \, ln\left(\frac{J_t^{\bar{\eta}}}{J_t^{u}}\right) + \psi_0 \, ln\left(\frac{\bar{\eta}_{t+1}}{u_{t+1}}\right).$$

From (A.21) and (A.24), the conditional covariance risk can be approximated by:

$$(A.25) \qquad \begin{aligned} \sigma_{MH,t} &= J_t^{\bar{\eta}} \bar{\eta}_{t+1} - J_t^{u} u_{t+1} + \left(J_t^{u} e_{u,t+1} - J_t^{\bar{\eta}} e_{\bar{\eta},t+1}\right) \\ &\approx \psi_0 \, ln\left(\frac{J_t^{\bar{\eta}}}{J_t^{u}}\right) + \psi_0 \, ln\left(\frac{\bar{\eta}_{t+1}}{u_{t+1}}\right) + \left(J_t^{u} e_{u,t+1} - J_t^{\bar{\eta}} e_{\bar{\eta},t+1}\right). \end{aligned}$$

Similar Taylor expansion gives:

$$(A.26) \qquad ln\left(\frac{\eta_{k,t+1}}{u_{t+1}}\right) \approx ln(\phi_{0,t}) + \frac{1}{\phi_{0,t}}\left(\frac{\eta_{k,t+1}}{u_{t+1}} - \phi_{0,t}\right),$$

where $\phi_{0,t} = E_t\left(\frac{\eta_{k,t+1}}{u_{t+1}}\right)$. Thus, the tail index of Kelly and Jiang (2014) can be approximated as:

$$(A.27) \qquad \begin{aligned} \lambda_{t+1}^{Hill} &\approx \frac{1}{K_{t+1}} \sum_{k=1}^{K_{t+1}} \left(ln\left(\phi_{0,t}\right) + \frac{1}{\phi_{0,t}}\left(\frac{\eta_{k,t+1}}{u_{t+1}} - \phi_{0,t}\right) \right) \\ &= ln\left(\phi_{0,t}\right) + \frac{1}{\phi_{0,t}}\left(\frac{\overline{\eta}_{t+1}}{u_{t+1}} - \phi_{0,t}\right) \\ &\approx ln\left(\frac{\overline{\eta}_{t+1}}{u_{t+1}}\right). \end{aligned}$$

(A.25) and (A.27) lead to the following relationship between the conditional covariance $\sigma_{MH,t}$ and the tail index of Kelly and Jiang (2014):

$$\text{(A.28)} \qquad \sigma_{MH,t} \approx \psi_0 \, ln\left(\frac{J_t^{\overline{\eta}}}{J_t^u}\right) + \psi_0 \, ln\left(\frac{\overline{\eta}_{t+1}}{u_{t+1}}\right) + \left(J_t^u e_{u,t+1} - J_t^{\overline{\eta}} e_{\overline{\eta},t+1}\right)$$
$$\approx \psi_0 \, ln\left(\frac{J_t^{\overline{\eta}}}{J_t^u}\right) + \psi_0 \lambda_{t+1}^{Hill} + \left(J_t^u e_{u,t+1} - J_t^{\overline{\eta}} e_{\overline{\eta},t+1}\right).$$

This is equivalent to Proposition 3:

$$\text{(A.29)} \qquad \lambda_{t+1}^{Hill} \approx ln\left(\frac{J_t^u}{J_t^{\overline{\eta}}}\right) + \frac{1}{\psi_0}\sigma_{MH,t} + e_{t+1},$$

where e_{t+1} has mean zero. Thus, the tail index of Kelly and Jiang (2014) is proportional to the conditional covariance $\sigma_{MH,t}$ under the ICAPM. This completes the proof.