Online Appendix (not for publication): The New Keynesian Model and Bond Yields

Martin M. Andreasen

November 28, 2024

Contents

1 The New Keynesian DSGE Model

1.1 Household

The objective of the representative household is to

$$\max \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \mathcal{U}(c_{t+i} - h_{t+i}, l_{t+i})$$

subject to the real budget constraint

$$\mathbb{E}_t \left[\frac{M_{t,t+1} X_{t+1}}{P_t} \right] + c_t = \frac{X_t}{P_t} + w_t^* l_t + Div_t + T_t,$$

where c_t is real consumption, h_t is habit stock (taken to be exogenous to the household), l_t is labor supply, w_t^* is the real frictionless wage, P_t is the price level, Div_t is a real dividends from firms, and T_t denotes lump-sum transfers. Letting $x_{t+1}^{real} = X_{t+1}/P_t$ denote the real value of state contingent claims, this can be written as

$$\mathbb{E}_t \left[M_{t,t+1} x_{t+1}^{real} \right] + c_t = \frac{X_t}{P_t} \frac{P_{t-1}}{P_{t-1}} + w_t^* l_t + Div_t + T_t,$$

$\Updownarrow$

$$\mathbb{E}_t \left[M_{t,t+1} x_{t+1}^{real} \right] + c_t = \frac{x_t^{real}}{P_t} \frac{P_{t-1}}{1} + w_t^* l_t + Div_t + T_t,$$

$\Updownarrow$

$$\mathbb{E}_t \left[M_{t,t+1} x_{t+1}^{real} \right] + c_t = \frac{x_t^{real}}{\pi_t} + w_t^* l_t + Div_t + T_t,$$

where $\pi_t = P_{t+1}/P_t$ is the gross inflation rate. Following Rudebusch & Swanson (2012), recursive Epstein-Zin-Weil preferences implies that the value function of the representative household is given by

$$V_t = \mathcal{U}(c_t - h_t, l_t) + \beta \left(\mathbb{E}_t \left[V_{t+1}^{1-\alpha} \right] \right)^{\frac{1}{1-\alpha}},$$

whenever $\mathcal{U}(c_t, l_t) > 0$ (The arguments are similar for the reverse case). The maximization problem can be formulated as a Lagrangean, where the household chooses state-contingent plans for consumption, labor, and asset holdings, i.e. $\left(c_t, l_t, x_{t+1}^{real}\right)$, to maximize V_0 subject to the infinite sequence of state-contingent constraints. I.e.,

$$\mathcal{L} = V_0 + \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \gamma_{t+i} \left[\mathcal{U}\left(c_{t+i} - h_{t+i}, l_{t+i}\right) + \beta \left(\mathbb{E}_{t+i} \left[V_{t+1+i}^{1-\alpha} \right] \right)^{\frac{1}{1-\alpha}} - V_{t+i} \right]$$

$$+ \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \lambda_{t+i} \left[\frac{x_{t+i}^{real}}{\pi_{t+i}} + l_{t+i} w_{t+i}^* + Div_{t+i} + T_{t+i} - \mathbb{E}_t \left[M_{t+i,t+i+1} x_{t+1+i}^{real} \right] - c_{t+i} \right],$$

where γ_t and λ_t are lagrange multipliers. The first order conditions are then given by

$$\frac{\partial \mathcal{L}}{\partial c_t} = \gamma_t \mathcal{U}_c \left(c_t - h_t, l_t\right) - \lambda_t = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial l_t} = \gamma_t \mathcal{U}_l \left(c_t - h_t, l_t\right) + \lambda_t w_t^* = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial x_{t+1}^{real}} = \mathbb{P}(s) \left[\beta \lambda_{t+1}(s) \frac{1}{\pi_{t+1}(s)} - \lambda_t M_{t,t+1}(s) \right] = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial V_{t+1}} = \gamma_t \beta \mathbb{E}_t \left[V_{t+1}^{1-\alpha} \right]^{\frac{\alpha}{1-\alpha}} \mathbb{P}(s) V_{t+1}^{-\alpha} - \gamma_{t+1} \mathbb{P}(s) \beta = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = \frac{x_t^{real}}{\pi_t} + l_t w_t + Div_t - \mathbb{E}_t \left[M_{t,t+1} x_{t+1}^{real} \right] - c_t = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial \gamma_t} = \mathcal{U}(c_t - h_t, l_t) + \beta \left(\mathbb{E}_t \left[V_{t+1}^{1-\alpha} \right] \right)^{\frac{1}{1-\alpha}} - V_t = 0, \tag{6}$$

where s indexes the (discrete) states in period $t+1$ and $\mathbb{P}(s)$ denotes the attached probability. Rewriting (1) as

$$\gamma_t \mathcal{U}_c (c_t - h_t, l_t) = \lambda_t$$

and substituting into (2) yields

$$\gamma_t \mathcal{U}_l (c_t - h_t, l_t) + \gamma_t \mathcal{U}_c (c_t - h_t, l_t) w_t^* = 0$$

$\Updownarrow$

$$\frac{1}{w_t^*} = -\frac{\mathcal{U}_c (c_t - h_t, l_t)}{\mathcal{U}_l (c_t - h_t, l_t)}, \tag{7}$$

which can be interpreted as an intratemporal optimality condition, as it relates consumption and labor hours within a given period t. Further, rewriting (3) as

$$\beta \frac{\lambda_{t+1}(s)}{\lambda_t} \frac{1}{\pi_{t+1}(s)} = M_{t,t+1}(s).$$

Inserting the expressions for λ_t (and λ_{t+1}) from (1) yields

$$M_{t,t+1} \equiv \beta \frac{\gamma_{t+1}}{\gamma_t} \frac{\mathcal{U}_c (c_{t+1} - h_{t+1}, l_{t+1})}{\mathcal{U}_c (c_t - h_t, l_t)} \frac{1}{\pi_{t+1}}$$

for all states. Now, (4) implies that

$$\left(\mathbb{E}_t \left[V_{t+1}^{1-\alpha} \right] \right)^{\frac{\alpha}{1-\alpha}} V_{t+1}^{-\alpha} = \frac{\gamma_{t+1}}{\gamma_t}.$$

Hence, we get

$$M_{t,t+1} = \beta \left(\frac{\left(\mathbb{E}_t \left[V_{t+1}^{1-\alpha} \right] \right)^{\frac{1}{1-\alpha}}}{V_{t+1}} \right)^{\alpha} \frac{\mathcal{U}_c (c_{t+1} - h_{t+1}, l_{t+1})}{\mathcal{U}_c (c_t - h_t, l_t)} \frac{1}{\pi_{t+1}}. \tag{8}$$

Given the stochastic discount factor, we have

$$\mathbb{E}_t \left[M_{t,t+1} R_t \right] = 1 \tag{9}$$

where R_t is the gross one-period risk-free short rate. From (6) we have that

$$V_t = \mathcal{U}(c_t - h_t, l_t) + \beta \left(\mathbb{E}_t \left[V_{t+1}^{1-\alpha} \right] \right)^{\frac{1}{1-\alpha}}. \tag{10}$$

1.1.1 Separable utility

The considered utility function is given by

$$\mathcal{U}(c_t - h_t, l_t) = u_0 z_t^{(1-\chi)(1-\chi_0)} + d_t \left[\frac{1}{1-\chi} \left(\frac{c_t - b c_{t-1}}{(\widetilde{c}_{ss} z_t)^{\chi_0}} \right)^{1-\chi} + u_0^d \times z_t^{(1-\chi)(1-\chi_0)} + z_t^{(1-\chi)(1-\chi_0)} n_t \varphi_0 \frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}} \right]$$

using the external habits $h_t = b c_{t-1}$. Here, d_t denotes preference shocks, which we specify below, and $\widetilde{c}_{ss} z_t$ is the level of consumption along the balanced growth path. The variable n_t is an exogenous labor supply shock. Hence, for $\chi_0 = 0$, we get

$$\mathcal{U}(c_t - h_t, l_t) = u_0 z_t^{1-\chi} + d_t \left[\frac{1}{1-\chi} (c_t - b c_{t-1})^{1-\chi} + u_0^d z_t^{1-\chi} + z_t^{1-\chi} n_t \varphi_0 \frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}} \right]$$

as in Rudebusch & Swanson (2012), provided $u_0^d, u_0 = 0$, and $b = 0$. For $\chi_0 = 1$, we get

$$\mathcal{U}\left(c_t - h_t, l_t\right) = u_0 + d_t\left[\frac{1}{1-\chi}\left(\frac{c_t - bc_{t-1}}{\widetilde{c}_{ss}z_t}\right)^{1-\chi} + u_0^d + n_t\varphi_0\frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}\right]$$

which is very similar to the specification in Andreasen, Fernandez-Villaverde & Rubio-Ramirez (2018). Note that we throughout the present paper impose $\chi_0 = 0$ and $u_0^d = 0$, although both parameters are incorporated in this Online Appendix.

Hence, equation (7) is given by

$$\frac{1}{w_t^*} = -\frac{\mathcal{U}_c\left(c_t - h_t, l_t\right)}{\mathcal{U}_l\left(c_t - h_t, l_t\right)}$$

$\Updownarrow$

$$-\mathcal{U}_l\left(c_t - h_t, l_t\right) = \mathcal{U}_c\left(c_t - h_t, l_t\right) w_t^*$$

$\Updownarrow$

$$z_t^{(1-\chi)(1-\chi_0)} d_t n_t \varphi_0 \left(1 - l_t\right)^{-\frac{1}{\varphi}} = d_t\left(\frac{c_t - bc_{t-1}}{\left(\widetilde{c}_{ss}z_t\right)^{\chi_0}}\right)^{-\chi}\frac{w_t^*}{\left(\widetilde{c}_{ss}z_t\right)^{\chi_0}}$$

$\Updownarrow$

$$w_t^* = z_t^{(1-\chi)(1-\chi_0)} n_t \varphi_0 \left(1 - l_t\right)^{-\frac{1}{\varphi}}\left(\frac{c_t - bc_{t-1}}{\left(\widetilde{c}_{ss}z_t\right)^{\chi_0}}\right)^{\chi}\left(\widetilde{c}_{ss}z_t\right)^{\chi_0}$$

Equation (8) becomes

$$M_{t,t+1} = \beta\left(\frac{\left(\mathbb{E}_t\left[V_{t+1}^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}}{V_{t+1}}\right)^{\alpha}\frac{d_{t+1}}{d_t}\frac{\left(\frac{c_{t+1}-bc_t}{\left(\widetilde{c}_{ss}z_{t+1}\right)^{\chi_0}}\right)^{-\chi} z_{t+1}^{-\chi_0}}{\left(\frac{c_t-bc_{t-1}}{\left(\widetilde{c}_{ss}z_t\right)^{\chi_0}}\right)^{-\chi} z_t^{-\chi_0}}\frac{1}{\pi_{t+1}}.$$

Equation (9) is not changed and finally equation (10) is

$$\begin{aligned} V_t \quad = \quad & u_0 z_t^{(1-\chi)(1-\chi_0)} + d_t\left[\frac{1}{1-\chi}\left(\frac{c_t - bc_{t-1}}{\left(\widetilde{c}_{ss}z_t\right)^{\chi_0}}\right)^{1-\chi} + u_0^d \times z_t^{(1-\chi)(1-\chi_0)} + z_t^{(1-\chi)(1-\chi_0)} n_t\varphi_0\frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}\right] \\ & +\beta\left(\mathbb{E}_t\left[V_{t+1}^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}. \end{aligned}$$

1.2 Firms

Firms are modeled by a two-layer structure. The perfect competitive final good producer bundles together a continuum of intermediates good $y_t(i)$ indexed by $i \in [0, 1]$. Taking prices as given, the final good producer maximizes profits, i.e., solves the problem

$$\max_{y_t(i)} \Pi = P_t y_t - \int_0^1 P_t(i) y_t(i) di,$$

subject to the production function $y_t = \left(\int_0^1 y_t(i)^{\frac{\eta-1}{\eta}} di\right)^{\frac{\eta}{\eta-1}}$, where y_t denotes final output and $\eta > 1$. The Lagrangian for solving this problem reads

$$\mathcal{L} = P_t y_t - \int_0^1 P_t(i) y_t(i) di + \Psi\left(\left(\int_0^1 y_t(i)^{\frac{\eta-1}{\eta}} di\right)^{\frac{\eta}{\eta-1}} - y_t\right).$$

First order conditions are

$$\frac{\partial \mathcal{L}}{\partial y_t(i)} = -P_t(i) + \Psi \frac{\eta}{\eta-1}\left(\int_0^1 y_t(i)^{\frac{\eta-1}{\eta}} di\right)^{\frac{\eta}{\eta-1}-1} \frac{\eta-1}{\eta} y_t(i)^{\frac{\eta-1}{\eta}-1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \Psi} = \left(\int_0^1 y_t(i)^{\frac{\eta-1}{\eta}} di\right)^{\frac{\eta}{\eta-1}} - y_t = 0.$$

From the first equation,

$$P_t(i) = \Psi \left(\int_0^1 y_t(i)^{\frac{\eta-1}{\eta}} di\right)^{\frac{1}{\eta-1}} y_t(i)^{\frac{-1}{\eta}}$$

$\Updownarrow$

$$\left(\frac{P_t(i)}{\Psi}\right)^{\eta} = \left(\int_0^1 y_t(i)^{\frac{\eta-1}{\eta}} di\right)^{\frac{\eta}{\eta-1}} y_t(i)^{-1} = y_t y_t(i)^{-1}$$

$\Updownarrow$

$$y_t(i) = y_t \left(\frac{P_t(i)}{\Psi}\right)^{-\eta}.$$

Substituting this into the second first order condition gives

$$y_t = \left(\int_0^1 y_t(i)^{\frac{\eta-1}{\eta}} di\right)^{\frac{\eta}{\eta-1}} = \left(\int_0^1 \left(y_t \left(\frac{P_t(i)}{\Psi}\right)^{-\eta}\right)^{\frac{\eta-1}{\eta}} di\right)^{\frac{\eta}{\eta-1}} = y_t \Psi^{\eta} \left(\int_0^1 P_t(i)^{1-\eta} di\right)^{\frac{\eta}{\eta-1}}$$

$\Updownarrow$

$$\Psi = \left(\int_0^1 P_t(i)^{1-\eta} di\right)^{\frac{1}{1-\eta}} \equiv P_t,$$

where P_t is the overall price level. Hence, demand for the i'th intermediate good is

$$y_t(i) = y_t \left(\frac{P_t(i)}{P_t}\right)^{-\eta}.$$

Intermediate goods producers maximize profits, i.e., the dividend transfers to households. We have for the ith intermediate firm that

$$\begin{aligned} Div_t(i) &= \left(\frac{P_t(i)}{P_t}\right) y_t(i) - w_t l_t(i) - \frac{\xi}{2}\left(\frac{P_t(i)}{P_{t-1}(i)}\frac{1}{\pi_{ss}^{\nu}} - 1\right)^2 y_t - z_t \delta k_{ss} \\ &= \left(\frac{P_t(i)}{P_t}\right)^{1-\eta} y_t - w_t l_t(i) - \frac{\xi}{2}\left(\frac{P_t(i)}{P_{t-1}(i)}\frac{1}{\pi_{ss}^{\nu}} - 1\right)^2 y_t - z_t \delta k_{ss} \end{aligned}$$

where w_t is the real wage payed by the firms to the workers. Intermediate producers have technology given by $y_t(i) = z_t a_t k_{ss}^{\theta} l_t(i)^{1-\theta}$ available, where z_t and a_t are exogeneous technology processes presented below. Price stickiness is modeled by the Rotemberg scheme. Hence, the ith intermediate firm solves

$$\max_{L_t(i), P_t(i)} \mathbb{E}_t \sum_{k=0}^{\infty} M_{t,t+k}^{\text{real}} \left(\left(\frac{P_{t+k}(i)}{P_{t+k}}\right)^{1-\eta} y_{t+k} - w_{t+k} l_{t+k}(i) - \frac{\xi}{2}\left(\frac{P_{t+k}(i)}{P_{t+k-1}(i)}\frac{1}{\pi_{ss}^{\nu}} - 1\right)^2 y_{t+k}\right) - z_{t+k}\delta k_{ss},$$

subject to $y_t(i) = y_t \left(\frac{P_t(i)}{P_t}\right)^{-\eta}$ and $y_t(i) = z_t a_t k_{ss}^{\theta} l_t(i)^{1-\theta}$. Here, $M_{t,t+k}^{\text{real}}$ denotes the real stochastic discount factor. The Lagrangian reads

$$\mathcal{L} = \mathbb{E}_t \sum_{k=0}^{\infty} M_{t,t+k}^{\text{real}} \left(\left(\frac{P_t(i)}{P_t}\right)^{1-\eta} y_{t+k} - w_{t+k} l_{t+k}(i) - \frac{\xi}{2}\left(\frac{P_{t+k}(i)}{P_{t+k-1}(i)}\frac{1}{\pi_{ss}^{\nu}} - 1\right)^2 y_{t+k}\right) - z_{t+k}\delta k_{ss} +$$

$$+\mathbb{E}_t \sum_{k=0}^{\infty} M^{\text{real}}_{t,t+k} mc_{t+k}(i) \left(z_{t+k} a_{t+k} k_{ss}^{\theta} l_{t+k}(i)^{1-\theta} - \left(\frac{P_{t+k}(i)}{P_{t+k}} \right)^{-\eta} y_{t+k} \right)$$

where $mc_t(i)$ is the lagrange multiplier and can be interpreted as the marginal cost of production. The first order condition is

$$\frac{\partial \mathcal{L}}{\partial l_t(i)} = -w_t + mc_t(i)\,(1-\theta)\, z_t a_t k_{ss}^{\theta} l_t(i)^{-\theta} = 0,$$

or simply

$$w_t = mc_t(i)\,(1-\theta)\, z_t a_t k_{ss}^{\theta} l_t(i)^{-\theta}.$$

The first-order condition with respect to $P_t(i)$ is

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial P_t(i)} &= (1-\eta)\left(\frac{P_t(i)}{P_t}\right)^{-\eta} \frac{1}{P_t} y_t - \xi \left(\frac{P_t\,(i)}{P_{t-1}\,(i)} \frac{1}{\pi_{ss}^{\nu}} - 1 \right) \frac{y_t}{P_{t-1}\,(i)} \frac{1}{\pi_{ss}^{\nu}} \\
&\quad -\mathbb{E}_t \left[\xi M^{\text{real}}_{t,t+1} \left(\frac{P_{t+1}\,(i)}{P_t\,(i)} \frac{1}{\pi_{ss}^{\nu}} - 1 \right) \left(\frac{-P_{t+1}\,(i)}{P_t\,(i)^2} \right) \frac{1}{\pi_{ss}^{\nu}} y_{t+1} \right] \\
&\quad +\eta mc_t\,(i) \left(\frac{P_t(i)}{P_t}\right)^{-\eta-1} \frac{y_t}{P_t} \\
&= 0
\end{aligned}$$

$\Updownarrow$

$$\begin{aligned}
&(1-\eta)\left(\frac{P_t(i)}{P_t}\right)^{-\eta} \frac{1}{P_t} y_t \\
&+\mathbb{E}_t \left[\xi M^{\text{real}}_{t,t+1} \left(\frac{P_{t+1}\,(i)}{P_t\,(i)} \frac{1}{\pi_{ss}^{\nu}} - 1 \right) \frac{P_{t+1}\,(i)}{P_t\,(i)^2} \frac{1}{\pi_{ss}^{\nu}} y_{t+1} \right] \\
&+\eta mc_t\,(i) \left(\frac{P_t(i)}{P_t}\right)^{-\eta-1} \frac{y_t}{P_t} \\
=\; &\xi \left(\frac{P_t\,(i)}{P_{t-1}\,(i)} \frac{1}{\pi_{ss}^{\nu}} - 1 \right) \frac{y_t}{P_{t-1}\,(i)} \frac{1}{\pi_{ss}^{\nu}}
\end{aligned}$$

All firms are identical, so we can drop the index i. Hence,

$$\begin{aligned}
&(1-\eta)\left(\frac{P_t}{P_t}\right)^{-\eta} \frac{1}{P_t} y_t \\
&+\mathbb{E}_t \left[\xi M^{\text{real}}_{t,t+1} \left(\frac{P_{t+1}}{P_t} \frac{1}{\pi_{ss}^{\nu}} - 1 \right) \frac{P_{t+1}}{P_t^2} \frac{1}{\pi_{ss}^{\nu}} y_{t+1} \right] \\
&+\eta mc_t \left(\frac{P_t}{P_t}\right)^{-\eta-1} \frac{y_t}{P_t} \\
=\; &\xi \left(\frac{P_t}{P_{t-1}} \frac{1}{\pi_{ss}^{\nu}} - 1 \right) \frac{y_t}{P_{t-1}} \frac{1}{\pi_{ss}^{\nu}}
\end{aligned}$$

$\Updownarrow$

$$
\begin{aligned}
& (1-\eta)\frac{1}{P_t}y_t \\
& +\mathbb{E}_t\left[\xi M^{\text{real}}_{t,t+1}\left(\frac{\pi_{t+1}}{\pi^{\nu}_{ss}}-1\right)\frac{\pi_{t+1}}{P_t}\frac{1}{\pi^{\nu}_{ss}}y_{t+1}\right] \\
& +\eta mc_t\frac{y_t}{P_t} \\
= \quad & \xi\left(\frac{\pi_t}{\pi^{\nu}_{ss}}-1\right)\frac{y_t}{P_{t-1}}\frac{1}{\pi^{\nu}_{ss}}
\end{aligned}
$$

$\Updownarrow$

$$
\begin{aligned}
& (1-\eta)y_t \\
& +\mathbb{E}_t\left[\xi M^{\text{real}}_{t,t+1}\left(\frac{\pi_{t+1}}{\pi^{\nu}_{ss}}-1\right)\frac{\pi_{t+1}}{P_t}\frac{P_t}{\pi^{\nu}_{ss}}y_{t+1}\right] \\
& +\eta mc_t\frac{y_t}{P_t}P_t \\
= \quad & \xi\left(\frac{\pi_t}{\pi^{\nu}_{ss}}-1\right)\frac{y_t}{P_{t-1}}\frac{1}{\pi^{\nu}_{ss}}P_t
\end{aligned}
$$

$\Updownarrow$

$$(1-\eta)y_t+\mathbb{E}_t\left[\xi M^{\text{real}}_{t,t+1}\left(\frac{\pi_{t+1}}{\pi^{\nu}_{ss}}-1\right)\frac{\pi_{t+1}}{\pi^{\nu}_{ss}}y_{t+1}\right]+\eta mc_t y_t=\xi\left(\frac{\pi_t}{\pi^{\nu}_{ss}}-1\right)y_t\frac{\pi_t}{\pi^{\nu}_{ss}}$$

Or equivalently

$$(1-\eta)y_t+\mathbb{E}_t\left[\xi M^{\text{real}}_{t,t+1}\underbrace{\frac{\pi_{t+1}}{\pi_{t+1}}}_{1}\left(\frac{\pi_{t+1}}{\pi^{\nu}_{ss}}-1\right)\frac{\pi_{t+1}}{\pi^{\nu}_{ss}}y_{t+1}\right]+\eta mc_t y_t=\xi\left(\frac{\pi_t}{\pi^{\nu}_{ss}}-1\right)y_t\frac{\pi_t}{\pi^{\nu}_{ss}}$$

$\Updownarrow$

$$(1-\eta)y_t+\mathbb{E}_t\left[\xi M_{t,t+1}\pi_{t+1}\left(\frac{\pi_{t+1}}{\pi^{\nu}_{ss}}-1\right)\frac{\pi_{t+1}}{\pi^{\nu}_{ss}}y_{t+1}\right]+\eta mc_t y_t=\xi\left(\frac{\pi_t}{\pi^{\nu}_{ss}}-1\right)y_t\frac{\pi_t}{\pi^{\nu}_{ss}}$$

Note, that without sticky prices, i.e. $\xi=0$, we have $(1-\eta)y_t+\eta mc_t y_t=0$, or $mc_t=(\eta-1)/\eta$. Thus, we have that the markup is $1/mc_t$.

1.3 Central Bank

We assume a Taylor rule of the form

$$R_t=R_{ss}\mathbb{E}_t\left[\exp\left\{\phi_\pi\log\left(\frac{\pi_{t+1}}{\pi_{ss}\pi^*_{t+1}}\right)+\phi_{\Delta c}\left(\Delta c_{t+1}-\Delta c_{ss}\right)\right\}\right].$$

1.4 Aggregation and Market Clearing

For the labor market, we follow Blanchard & Gali (2005), Rudebusch & Swanson (2008), among others and introduce a simple wage bargaining friction in the labor market. Specifically, we assume that the real market wage faced by the firms w_t is given by

$$w_t=\kappa_w\left(\widetilde{w}_{ss}z_t\right)+\left(1-\kappa_w\right)w^*_t,$$

where the parameter $\kappa_w \in [0,1)$ captures the notion of wage stickiness by smoothing the real frictionless wage w_t^* in relation to the long-term equilibrium real wage $\widetilde{w}_{ss} z_t$. Inserting for w_t^* we get

$$w_t = \kappa_w \left(w_{ss} z_t\right) + (1-\kappa_w) \left[z_t^{(1-\chi)(1-\chi_0)} n_t \varphi_0 \left(1-l_t\right)^{-\frac{1}{\varphi}} \left(\frac{c_t - b c_{t-1}}{\left(\widetilde{c}_{ss} z_t\right)^{\chi_0}} \right)^{\chi} \left(\widetilde{c}_{ss} z_t\right)^{\chi_0} \right]$$

Although this simple rule does not explicitly introduce Nash bargaining between workers and firms, Blanchard & Gali (2005) argue that it is a simple way to capture the essential features of real wage bargaining.

From the household budget constraint, workers receive the frictionless wage w_t^* and not the actual wage w_t paid by the firms. To eliminate any resource costs linked to wage stickiness, we let the real transfers to the household be

$$\begin{aligned} T_t &= l_t w_t - l w_t^* \\ &= l_t \left(w_t - w_t^*\right) \\ &= l_t \left(\kappa_w \left(\widetilde{w}_{ss} z_t\right) + (1-\kappa_w) w_t^* - w_t^*\right) \\ &= l_t \left(\kappa_w \left(\widetilde{w}_{ss} z_t\right) - \kappa_w w_t^*\right) \\ &= \kappa_w l_t \left(\left(\widetilde{w}_{ss} z_t\right) - w_t^*\right), \end{aligned}$$

to ensure that the wage bill $w_t l_t$ paid by the firm is also the wage bill received by the household. From the budget restriction we have

$$\mathbb{E}_t \left[M_{t,t+1} x_{t+1}^{real}\right] + c_t = \frac{x_t^{real}}{\pi_t} + w_t^* l_t + Div_t + T_t$$

$\Downarrow$

$$0 + c_t = 0 + w_t^* l_t + Div_t + l_t w_t - l w_t^*$$

$\Updownarrow$

$$Div_t = c_t - w_t l_t$$

given that the amount of state contigent claims x_t are in zero net supply. The expression for dividends is

$$Div_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{1-\eta} y_t - w_t l_t(i) - \frac{\xi}{2} \left(\frac{P_t(i)}{P_{t-1}(i)} \frac{1}{\pi_{ss}^{\nu}} - 1 \right)^2 y_t - z_t \delta k_{ss}$$

$\Downarrow$

$$\begin{aligned} Div_t &= y_t - w_t l_t - \frac{\xi}{2} \left(\frac{\pi_t}{\pi_{ss}^{\nu}} - 1 \right)^2 y_t - z_t \delta k_{ss} \\ &= \left(1 - \frac{\xi}{2} \left(\frac{\pi_t}{\pi_{ss}^{\nu}} - 1 \right)^2 \right) y_t - w_t l_t - z_t \delta k_{ss} \end{aligned}$$

Combining the two expressions for dividends, we get

$$c_t - w_t l_t = \left(1 - \frac{\xi}{2} \left(\frac{\pi_t}{\pi_{ss}^{\nu}} - 1 \right)^2 \right) y_t - w_t l_t - z_t \delta k_{ss}$$

$\Updownarrow$

$$c_t + z_t \delta k_{ss} = \left(1 - \frac{\xi}{2} \left(\frac{\pi_t}{\pi_{ss}^{\nu}} - 1 \right)^2 \right) y_t.$$

1.5 Exogeneous Shocks

The exogenous shocks are given by

$$\log\left(\frac{\mu_{z,t+1}}{\mu_{z,ss}}\right) = \rho_{\mu_z}\log\left(\frac{\mu_{z,t}}{\mu_{z,ss}}\right) + \sigma_{\mu_z}\epsilon_{\mu_z,t+1}$$

$$\log d_{t+1} = \rho_d \log d_t + \sigma_d \epsilon_{d,t+1}$$

$$\log n_{t+1} = \rho_n \log n_t + \sigma_n \epsilon_{n,t+1}$$

$$\log \pi^*_{t+1} = \rho_{\pi^*} \log \pi^*_t + \sigma_{\pi^*} \epsilon_{\pi^*,t+1}$$

$$\log a_{t+1} = \rho_a \log a_t + \sigma_a \epsilon_{a,t+1}$$

1.6 Model Equations

The equations for the baseline implementation are summarized in the table below.

	Household
1	$V_t = u_0 z_t^{(1-\chi)(1-\chi_0)} + d_t\left[\frac{1}{1-\chi}\left(\frac{c_t - bc_{t-1}}{(\widetilde{c}_{ss} z_t)^{\chi_0}}\right)^{1-\chi} + u_0^d \times z_t^{(1-\chi)(1-\chi_0)} + z_t^{(1-\chi)(1-\chi_0)} n_t \varphi_0 \frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}\right] + \beta\left(\mathbb{E}_t\left[V_{t+1}^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}$
2	$w_t = \kappa_w\left(\widetilde{w}_{ss} z_t\right) + (1-\kappa_w)\left[z_t^{(1-\chi)(1-\chi_0)} n_t \varphi_0 (1-l_t)^{-\frac{1}{\varphi}} \left(\frac{c_t - bc_{t-1}}{(\widetilde{c}_{ss} z_t)^{\chi_0}}\right)^{\chi} (\widetilde{c}_{ss} z_t)^{\chi_0}\right]$
3	$\mathbb{E}_t\left[M_{t,t+1} R_t\right] = 1$
	$M_{t,t+1} = \beta\left(\frac{\left[\mathbb{E}_t\left[V_{t+1}^{1-\alpha}\right]\right]^{\frac{1}{1-\alpha}}}{V_{t+1}}\right)^{\alpha} \frac{d_{t+1}}{d_t} \frac{\left(\frac{c_{t+1} - bc_t}{(\widetilde{c}_{ss} z_{t+1})^{\chi_0}}\right)^{-\chi} z_{t+1}^{-\chi_0}}{\left(\frac{c_t - bc_{t-1}}{(\widetilde{c}_{ss} z_t)^{\chi_0}}\right)^{-\chi} z_t^{-\chi_0}} \frac{1}{\pi_{t+1}}$
	Firm
4	$w_t = mc_t (1-\theta) z_t a_t k_{ss}^{\theta} l_t^{-\theta}$
5	$(1-\eta) y_t + \mathbb{E}_t\left[\xi M_{t,t+1}^{\text{real}}\left(\frac{\pi_{t+1}}{\pi_{ss}^{\nu}} - 1\right)\frac{\pi_{t+1}}{\pi_{ss}^{\nu}} y_{t+1}\right] + \eta mc_t y_t = \xi\left(\frac{\pi_t}{\pi_{ss}^{\nu}} - 1\right) y_t \frac{\pi_t}{\pi_{ss}^{\nu}}$
	Central bank
6	$R_t = R_{ss}\mathbb{E}_t\left[\exp\left\{\phi_\pi \log\left(\frac{\pi_{t+1}}{\pi_{ss}\pi^*_{t+1}}\right) + \phi_{\Delta c}\left(\Delta c_{t+1} - \Delta c_{ss}\right)\right\}\right]$
	Aggregation
7	$c_t + z_t \delta k_{ss} = \left(1 - \frac{\xi}{2}\left(\frac{\pi_t}{\pi_{ss}^{\nu}} - 1\right)^2\right) y_t$
8	$y_t = z_t a_t k_{ss}^{\theta} l_t^{1-\theta}$
	Links
9	$(c_{t-1})_{t+1} = c_t$
10-	**Shocks**

To these equations, we add the recursive equations for bond prices, i.e. $B_t^{(k)} = \mathbb{E}_t\left[M_{t,t+1} B_{t+1}^{(k-1)}\right]$.

1.7 Detrending

1.7.1 Separable case

Define the stationary variables $\widetilde{c}_t = \frac{c_t}{z_t}$, $\tilde{V}_t = \frac{V_t}{z_t^{(1-\chi)(1-\chi_0)}}$, $\widetilde{w}_t = \frac{w_t}{z_t}$, $\widetilde{y}_t = \frac{y_t}{z_t}$

EQ 1

$$V_t = u_0 z_t^{(1-\chi)(1-\chi_0)} + d_t\left[\frac{1}{1-\chi}\left(\frac{c_t - bc_{t-1}}{(\widetilde{c}_{ss} z_t)^{\chi_0}}\right)^{1-\chi} + u_0^d \times z_t^{(1-\chi)(1-\chi_0)} + z_t^{(1-\chi)(1-\chi_0)} n_t \varphi_0 \frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}\right] + \beta\left(\mathbb{E}_t\left[V_{t+1}^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}$$

$\Updownarrow$

$$\frac{V_t}{z_t^{(1-\chi)(1-\chi_0)}} = u_0 + d_t \left[\frac{1}{1-\chi}\left(\frac{c_t - bc_{t-1}}{\widetilde{c}_{ss}^{\chi_0} z_t^{\chi_0}}\right)^{1-\chi}\frac{1}{z_t^{(1-\chi)(1-\chi_0)}} + u_0^d + n_t\varphi_0 \frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}\right] + \frac{1}{z_t^{(1-\chi)(1-\chi_0)}}\beta\left(\mathbb{E}_t\left[V_{t+1}^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}$$

$\Updownarrow$

$$\frac{V_t}{z_t^{(1-\chi)(1-\chi_0)}} = u_0 + d_t \left[\frac{1}{1-\chi}\left(\frac{c_t - bc_{t-1}}{\widetilde{c}_{ss}^{\chi_0} z_t^{\chi_0} z_t^{1-\chi_0}}\right)^{1-\chi} + u_0^d + n_t\varphi_0 \frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}\right] + \frac{1}{z_t^{(1-\chi)(1-\chi_0)}}\beta\left(\mathbb{E}_t\left[V_{t+1}^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}$$

$\Updownarrow$

$$\frac{V_t}{z_t^{(1-\chi)(1-\chi_0)}} = u_0 + d_t \left[\frac{1}{1-\chi}\left(\frac{c_t - bc_{t-1}}{\widetilde{c}_{ss}^{\chi_0} z_t}\right)^{1-\chi} + u_0^d + n_t\varphi_0 \frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}\right] + \frac{1}{z_t^{(1-\chi)(1-\chi_0)}}\beta\left(\mathbb{E}_t\left[V_{t+1}^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}$$

$\Updownarrow$

$$\tilde{V}_t = u_0 + d_t \left[\frac{1}{1-\chi}\left(\frac{\widetilde{c}_t - b\widetilde{c}_{t-1}\mu_{z,t}^{-1}}{\widetilde{c}_{ss}^{\chi_0}}\right)^{1-\chi} + u_0^d + n_t\varphi_0 \frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}\right] + \beta\left(\mathbb{E}_t\left[\left(\frac{V_{t+1}}{z_{t+1}^{(1-\chi)(1-\chi_0)}}\frac{z_{t+1}^{(1-\chi)(1-\chi_0)}}{z_t^{(1-\chi)(1-\chi_0)}}\right)^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}$$

$\Updownarrow$

$$\tilde{V}_t = u_0 + d_t \left[\frac{1}{1-\chi}\left(\frac{\widetilde{c}_t - b\widetilde{c}_{t-1}\mu_{z,t}^{-1}}{\widetilde{c}_{ss}^{\chi_0}}\right)^{1-\chi} + u_0^d + n_t\varphi_0 \frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}\right] + \beta\left(\mathbb{E}_t\left[\left(\tilde{V}_{t+1}\mu_{z,t+1}^{(1-\chi)(1-\chi_0)}\right)^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}$$

EQ 2

$$w_t = \kappa_w\left(\widetilde{w}_{ss} z_t\right) + (1-\kappa_w)\left[z_t^{(1-\chi)(1-\chi_0)} n_t\varphi_0 (1-l_t)^{-\frac{1}{\varphi}}\left(\frac{c_t - bc_{t-1}}{(\widetilde{c}_{ss} z_t)^{\chi_0}}\right)^{\chi}(\widetilde{c}_{ss} z_t)^{\chi_0}\right]$$

$\Updownarrow$

$$\frac{w_t}{z_t} = \kappa_w\widetilde{w}_{ss} + (1-\kappa_w)\left[z_t^{(1-\chi)(1-\chi_0)} n_t\varphi_0 (1-l_t)^{-\frac{1}{\varphi}}\left(\frac{c_t - bc_{t-1}}{(\widetilde{c}_{ss} z_t)^{\chi_0}}\right)^{\chi}(\widetilde{c}_{ss} z_t)^{\chi_0}\frac{1}{z_t}\right]$$

$\Updownarrow$

$$\frac{w_t}{z_t} = \kappa_w\widetilde{w}_{ss} + (1-\kappa_w)\left[z_t^{1-\chi_0-\chi+\chi\chi_0-1+\chi_0-\chi\chi_0} n_t\varphi_0 (1-l_t)^{-\frac{1}{\varphi}}\left(\frac{c_t - bc_{t-1}}{(\widetilde{c}_{ss})^{\chi_0}}\right)^{\chi}(\widetilde{c}_{ss})^{\chi_0}\right]$$

$\Updownarrow$

$$\widetilde{w}_t = \kappa_w\widetilde{w}_{ss} + (1-\kappa_w)\left[z_t^{-\chi} n_t\varphi_0 (1-l_t)^{-\frac{1}{\varphi}}\left(\frac{c_t - bc_{t-1}}{(\widetilde{c}_{ss})^{\chi_0}}\right)^{\chi}(\widetilde{c}_{ss})^{\chi_0}\right]$$

$\Updownarrow$

$$\widetilde{w}_t = \kappa_w\widetilde{w}_{ss} + (1-\kappa_w)\left[n_t\varphi_0 (1-l_t)^{-\frac{1}{\varphi}}\left(\frac{\frac{c_t}{z_t} - b\frac{z_{t-1}}{z_t}\frac{c_{t-1}}{z_{t-1}}}{(\widetilde{c}_{ss})^{\chi_0}}\right)^{\chi}(\widetilde{c}_{ss})^{\chi_0}\right]$$

$\Updownarrow$

$$\widetilde{w}_t = \kappa_w\widetilde{w}_{ss} + (1-\kappa_w)\left[n_t\varphi_0 (1-l_t)^{-\frac{1}{\varphi}}\left(\frac{\frac{c_t}{z_t} - b\frac{z_{t-1}}{z_t}\frac{c_{t-1}}{z_{t-1}}}{(\widetilde{c}_{ss})^{\chi_0}}\right)^{\chi}(\widetilde{c}_{ss})^{\chi_0}\right]$$

$\Updownarrow$

$$\widetilde{w}_t = \kappa_w\widetilde{w}_{ss} + (1-\kappa_w)\left[n_t\varphi_0 (1-l_t)^{-\frac{1}{\varphi}}\left(\frac{\widetilde{c}_t - b\widetilde{c}_{t-1}\mu_{z,t}^{-1}}{(\widetilde{c}_{ss})^{\chi_0}}\right)^{\chi}(\widetilde{c}_{ss})^{\chi_0}\right]$$

EQ 3

$\mathbb{E}_t\left[M_{t,t+1}R_t\right] = 1$

where

$$M_{t,t+1} = \beta\left(\frac{\left[\mathbb{E}_t\left[V_{t+1}^{1-\alpha}\right]\right]^{\frac{1}{1-\alpha}}}{V_{t+1}}\right)^{\alpha}\frac{d_{t+1}}{d_t}\frac{\left(\frac{c_{t+1}-bc_t}{\widetilde{c}_{ss}^{\chi_0} z_{t+1}^{\chi_0}}\right)^{-\chi} z_{t+1}^{-\chi_0}}{\left(\frac{c_t - bc_{t-1}}{\widetilde{c}_{ss}^{\chi_0} z_t^{\chi_0}}\right)^{-\chi} z_t^{-\chi_0}}\frac{1}{\pi_{t+1}}$$

$\Updownarrow$

$$M_{t,t+1} = \beta\left(\frac{\left[\mathbb{E}_t\left[\left(V_{t+1}\frac{z_{t+1}^{(1-\chi)(1-\chi_0)}}{z_{t+1}^{(1-\chi)(1-\chi_0)}}\right)^{1-\alpha}\right]\right]^{\frac{1}{1-\alpha}}}{V_{t+1}\frac{z_{t+1}^{(1-\chi)(1-\chi_0)}}{z_{t+1}^{(1-\chi)(1-\chi_0)}}}\right)^{\alpha}\frac{d_{t+1}}{d_t}\frac{\left(\frac{c_{t+1}-bc_t}{z_{t+1}^{\chi_0}}\frac{z_{t+1}^{1-\chi_0}}{z_{t+1}^{1-\chi_0}}\right)^{-\chi}}{\left(\frac{c_t - bc_{t-1}}{z_t^{\chi_0}}\frac{z_t^{1-\chi_0}}{z_t^{1-\chi_0}}\right)^{-\chi}}\left(\frac{z_{t+1}}{z_t}\right)^{-\chi_0}\frac{1}{\pi_{t+1}}$$

$\Updownarrow$

$$M_{t,t+1}=\beta\left(\frac{\left[\mathbb{E}_t\left[\left(\tilde{V}_{t+1}z_{t+1}^{(1-\chi)(1-\chi_0)}\right)^{1-\alpha}\right]\right]^{\frac{1}{1-\alpha}}}{\tilde{V}_{t+1}z_{t+1}^{(1-\chi)(1-\chi_0)}}\right)^{\alpha}\frac{d_{t+1}}{d_t}\frac{\left(\frac{c_{t+1}-bc_t}{z_{t+1}}\frac{z_{t+1}^{1-\chi_0}}{1}\right)^{-\chi}}{\left(\frac{c_t-bc_{t-1}}{z_t}\frac{z_t^{1-\chi_0}}{1}\right)^{-\chi}}\left(\frac{z_{t+1}}{z_t}\right)^{-\chi_0}\frac{1}{\pi_{t+1}}$$

$\Updownarrow$

$$M_{t,t+1}=\beta\left(\frac{\left[\mathbb{E}_t\left[\left(\tilde{V}_{t+1}\frac{z_{t+1}^{(1-\chi)(1-\chi_0)}}{z_t^{(1-\chi)(1-\chi_0)}}\right)^{1-\alpha}\right]\right]^{\frac{1}{1-\alpha}}}{\tilde{V}_{t+1}\frac{z_{t+1}^{(1-\chi)(1-\chi_0)}}{z_t^{(1-\chi)(1-\chi_0)}}}\right)^{\alpha}\frac{d_{t+1}}{d_t}\frac{\left(\widetilde{c}_{t+1}-b\mu_{z,t}^{-1}\widetilde{c}_t\right)^{-\chi}}{\left(\widetilde{c}_t-b\mu_{z,t-1}^{-1}\widetilde{c}_{t-1}\right)^{-\chi}}\frac{z_{t+1}^{-\chi(1-\chi_0)}}{z_t^{-\chi(1-\chi_0)}}\left(\frac{z_{t+1}}{z_t}\right)^{-\chi_0}\frac{1}{\pi_{t+1}}$$

$\Updownarrow$

$$M_{t,t+1}=\beta\left(\frac{\left[\mathbb{E}_t\left[\left(\tilde{V}_{t+1}\mu_{z,t+1}^{(1-\chi)(1-\chi_0)}\right)^{1-\alpha}\right]\right]^{\frac{1}{1-\alpha}}}{\tilde{V}_{t+1}\mu_{z,t+1}^{(1-\chi)(1-\chi_0)}}\right)^{\alpha}\frac{d_{t+1}}{d_t}\frac{\left(\widetilde{c}_{t+1}-b\mu_{z,t}^{-1}\widetilde{c}_t\right)^{-\chi}}{\left(\widetilde{c}_t-b\mu_{z,t-1}^{-1}\widetilde{c}_{t-1}\right)^{-\chi}}\mu_{z,t+1}^{-\chi(1-\chi_0)}\mu_{z,t+1}^{-\chi_0}\frac{1}{\pi_{t+1}}$$

$\Updownarrow$

$$M_{t,t+1}=\beta\left(\frac{\left[\mathbb{E}_t\left[\left(\tilde{V}_{t+1}\mu_{z,t+1}^{(1-\chi)(1-\chi_0)}\right)^{1-\alpha}\right]\right]^{\frac{1}{1-\alpha}}}{\tilde{V}_{t+1}\mu_{z,t+1}^{(1-\chi)(1-\chi_0)}}\right)^{\alpha}\frac{d_{t+1}}{d_t}\frac{\left(\widetilde{c}_{t+1}-b\mu_{z,t}^{-1}\widetilde{c}_t\right)^{-\chi}}{\left(\widetilde{c}_t-b\mu_{z,t-1}^{-1}\widetilde{c}_{t-1}\right)^{-\chi}}\mu_{z,t+1}^{-\chi(1-\chi_0)-\chi_0}\frac{1}{\pi_{t+1}}$$

EQ 4

$w_t=mc_t\left(1-\theta\right)z_ta_tk_{ss}^{\theta}l_t{}^{-\theta}$

$\Updownarrow$

$\frac{w_t}{z_t}=mc_t\left(1-\theta\right)a_tk_{ss}^{\theta}l_t^{-\theta}$

$\Updownarrow$

$\widetilde{w}_t=mc_t\left(1-\theta\right)a_tk_{ss}^{\theta}l_t^{-\theta}$

EQ 5

$\left(1-\eta\right)y_t+\mathbb{E}_t\left[\xi M_{t,t+1}^{\text{real}}\left(\frac{\pi_{t+1}}{\pi_{ss}^{\nu}}-1\right)\frac{\pi_{t+1}}{\pi_{ss}^{\nu}}y_{t+1}\right]+\eta mc_ty_t=\xi\left(\frac{\pi_t}{\pi_{ss}^{\nu}}-1\right)y_t\frac{\pi_t}{\pi_{ss}^{\nu}}$

$\Updownarrow$

$\left(1-\eta\right)\frac{y_t}{z_t}+\mathbb{E}_t\left[\xi M_{t,t+1}^{\text{real}}\left(\frac{\pi_{t+1}}{\pi_{ss}^{\nu}}-1\right)\frac{\pi_{t+1}}{\pi_{ss}^{\nu}}\frac{y_{t+1}}{z_{t+1}}\frac{z_{t+1}}{z_t}\right]+\eta mc_t\frac{y_t}{z_t}=\xi\left(\frac{\pi_t}{\pi_{ss}^{\nu}}-1\right)\frac{y_t}{z_t}\frac{\pi_t}{\pi_{ss}^{\nu}}$

$\Updownarrow$

$\left(1-\eta\right)\widetilde{y}_t+\mathbb{E}_t\left[\xi M_{t,t+1}^{\text{real}}\left(\frac{\pi_{t+1}}{\pi_{ss}^{\nu}}-1\right)\frac{\pi_{t+1}}{\pi_{ss}^{\nu}}\widetilde{y}_{t+1}\mu_{z,t+1}\right]+\eta mc_t\widetilde{y}_t=\xi\left(\frac{\pi_t}{\pi_{ss}^{\nu}}-1\right)\widetilde{y}_t\frac{\pi_t}{\pi_{ss}^{\nu}}$

or as in the Appendix

$\left(1-\eta\right)+\mathbb{E}_t\left[\xi M_{t,t+1}\pi_{t+1}\left(\frac{\pi_{t+1}}{\pi_{ss}^{\nu}}-1\right)\frac{\pi_{t+1}}{\pi_{ss}^{\nu}}\frac{\widetilde{y}_{t+1}}{\widetilde{y}_t}\mu_{z,t+1}\right]+\eta mc_t=\xi\left(\frac{\pi_t}{\pi_{ss}^{\nu}}-1\right)\frac{\pi_t}{\pi_{ss}^{\nu}}$

$\Updownarrow$

$\eta mc_t=\xi\left(\frac{\pi_t}{\pi_{ss}^{\nu}}-1\right)\frac{\pi_t}{\pi_{ss}^{\nu}}-\left(1-\eta\right)-\mathbb{E}_t\left[\xi M_{t,t+1}\pi_{t+1}\left(\frac{\pi_{t+1}}{\pi_{ss}^{\nu}}-1\right)\frac{\pi_{t+1}}{\pi_{ss}^{\nu}}\frac{\widetilde{y}_{t+1}}{\widetilde{y}_t}\mu_{z,t+1}\right]$

$\Updownarrow$

$1 = \frac{1}{\eta mc_t} \left\{ \xi \left(\frac{\pi_t}{\pi_{ss}^{\nu}} - 1 \right) \frac{\pi_t}{\pi_{ss}^{\nu}} - (1-\eta) - \mathbb{E}_t \left[\xi M_{t,t+1} \pi_{t+1} \left(\frac{\pi_{t+1}}{\pi_{ss}^{\nu}} - 1 \right) \frac{\pi_{t+1}}{\pi_{ss}^{\nu}} \frac{\widetilde{y}_{t+1}}{\widetilde{y}_t} \mu_{z,t+1} \right] \right\}$

EQ 6
$R_t = R_{ss} \mathbb{E}_t \left[\exp \left\{ \phi_\pi \log \left(\frac{\pi_{t+1}}{\pi_{ss} \pi_{t+1}^*} \right) + \phi_{\Delta c} \left(\Delta c_{t+1} - \Delta c_{ss} \right) \right\} \right]$
Note that

$$\begin{aligned} \Delta c_t &= \log \left(c_t / c_{t-1} \right) \\ &= \log \left(\frac{\widetilde{c}_t z_t}{\widetilde{c}_{t-1} z_{t-1}} \right) \\ &= \log \left(\frac{\widetilde{c}_t}{\widetilde{c}_{t-1}} \right) + \log \mu_{z,t} \end{aligned}$$

so no transformation of Δc_t is needed. Also

$$\Delta c_{t+1} = \log \left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t} \right) + \log \mu_{z,t+1}$$

So we get

$$R_t = R_{ss} \mathbb{E}_t \left[\exp \left\{ \phi_\pi \log \left(\frac{\pi_{t+1}}{\pi_{ss} \pi_{t+1}^*} \right) + \phi_{\Delta c} \left(\log \left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t} \right) + \log \mu_{z,t+1} - \log \mu_{z,ss} \right) \right\} \right]$$

as $\Delta c_{ss} = \log \mu_{z,ss}$.

EQ 7
$c_t + z_t \delta k_{ss} = \left(1 - \frac{\xi}{2} \left(\frac{\pi_t}{\pi_{ss}^{\nu}} - 1 \right)^2 \right) y_t$
$\Updownarrow$
$\widetilde{c}_t + \delta k_{ss} = \left(1 - \frac{\xi}{2} \left(\frac{\pi_t}{\pi_{ss}^{\nu}} - 1 \right)^2 \right) \widetilde{y}_t$

EQ 8
$y_t = z_t a_t k_{ss}^{\theta} l_t^{1-\theta}$
$\Updownarrow$
$\widetilde{y}_t = a_t k_{ss}^{\theta} l_t^{1-\theta}$

The remaining equations in the model do not need to be detrended.

	Household
1	$\tilde{V}_t = u_0 + d_t \left[\frac{1}{1-\chi} \left(\frac{\widetilde{c}_t - b\widetilde{c}_{t-1}\mu_{z,t}^{-1}}{\widetilde{c}_{ss}^{\chi_0}} \right)^{1-\chi} + u_0^d + n_t \varphi_0 \frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}} \right] + \beta \left(\mathbb{E}_t \left[\left(\tilde{V}_{t+1} \mu_{z,t+1}^{(1-\chi)(1-\chi_0)} \right)^{1-\alpha} \right] \right)^{\frac{1}{1-\alpha}}$
2	$\widetilde{w}_t = \kappa_w \widetilde{w}_{ss} + (1-\kappa_w) \left[n_t \varphi_0 (1-l_t)^{-\frac{1}{\varphi}} \left(\frac{\widetilde{c}_t - b\widetilde{c}_{t-1}\mu_{z,t}^{-1}}{(\widetilde{c}_{ss})^{\chi_0}} \right)^{\chi} (\widetilde{c}_{ss})^{\chi_0} \right]$
3	$\mathbb{E}_t [M_{t,t+1} R_t] = 1$
	$M_{t,t+1} = \beta \left(\frac{\left[\mathbb{E}_t \left[\left(\tilde{V}_{t+1} \mu_{z,t+1}^{(1-\chi)(1-\chi_0)} \right)^{1-\alpha} \right] \right]^{\frac{1}{1-\alpha}}}{\tilde{V}_{t+1} \mu_{z,t+1}^{(1-\chi)(1-\chi_0)}} \right)^{\alpha} \frac{d_{t+1}}{d_t} \frac{\left(\widetilde{c}_{t+1} - b\mu_{z,t}^{-1} \widetilde{c}_t \right)^{-\chi}}{\left(\widetilde{c}_t - b\mu_{z,t-1}^{-1} \widetilde{c}_{t-1} \right)^{-\chi}} \mu_{z,t+1}^{-\chi(1-\chi_0)-\chi_0} \frac{1}{\pi_{t+1}}$
	Firm
4	$\widetilde{w}_t = mc_t (1-\theta) a_t k_{ss}^{\theta} l_t^{-\theta}$
5	$(1-\eta) \widetilde{y}_t + \mathbb{E}_t \left[\xi M_{t,t+1}^{\text{real}} \left(\frac{\pi_{t+1}}{\pi_{ss}^{\nu}} - 1 \right) \frac{\pi_{t+1}}{\pi_{ss}^{\nu}} \widetilde{y}_{t+1} \mu_{z,t+1} \right] + \eta mc_t \widetilde{y}_t = \xi \left(\frac{\pi_t}{\pi_{ss}^{\nu}} - 1 \right) \widetilde{y}_t \frac{\pi_t}{\pi_{ss}^{\nu}}$
	Central bank
6	$R_t = R_{ss} \mathbb{E}_t \left[\exp \left\{ \phi_{\pi} \log \left(\frac{\pi_{t+1}}{\pi_{ss} \pi_{t+1}^{*}} \right) + \phi_{\Delta c} \left(\log \left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t} \right) + \log \mu_{z,t+1} - \log \mu_{z,ss} \right) \right\} \right]$
	Aggregation
7	$\widetilde{c}_t + \delta k_{ss} = \left(1 - \frac{\xi}{2} \left(\frac{\pi_t}{\pi_{ss}^{\nu}} - 1 \right)^2 \right) \widetilde{y}_t$
8	$\widetilde{y}_t = a_t k_{ss}^{\theta} l_t^{1-\theta}$
	Links
9	$(\widetilde{c}_{t-1})_{t+1} = \widetilde{c}_t$
10-	**Shocks**

To these equations, we add the recursive equations for bond prices, i.e. $B_t^{(k)} = \mathbb{E}_t \left[M_{t,t+1} B_{t+1}^{(k-1)} \right]$.

1.8 Steady state

Some steady state values are calibrated to certain values, i.e. $KoY = \frac{k_{ss}}{4y_{ss}}$, l_{ss}, π_{ss}, $\mu_{z,ss}$.

1.8.1 Separable utility case

Notice, that in steady state we have

$$M_{t,t+1} = \beta \left(\frac{\left[\mathbb{E}_t \left[\left(\tilde{V}_{t+1} \mu_{z,t+1}^{(1-\chi)(1-\chi_0)} \right)^{1-\alpha} \right] \right]^{\frac{1}{1-\alpha}}}{\tilde{V}_{t+1} \mu_{z,t+1}^{(1-\chi)(1-\chi_0)}} \right)^{\alpha} \frac{d_{t+1}}{d_t} \frac{\left(\widetilde{c}_{t+1} - b\mu_{z,t}^{-1} \widetilde{c}_t \right)^{-\chi}}{\left(\widetilde{c}_t - b\mu_{z,t-1}^{-1} \widetilde{c}_{t-1} \right)^{-\chi}} \mu_{z,t+1}^{-\chi(1-\chi_0)-\chi_0} \frac{1}{\pi_{t+1}}$$

$\Downarrow$

$$M_{ss,ss} = \beta \frac{\mu_{z,ss}^{-\chi(1-\chi_0)-\chi_0}}{\pi_{ss}}$$

Then it follows from equation (3) that

$$\mathbb{E}_t [M_{t,t+1} R_t] = 1$$

$\Downarrow$

$$R_{ss} = \frac{1}{M_{ss}}$$

From equation (8), we have that $\widetilde{y}_{ss} = a_{ss} k_{ss}^{\theta} l_{ss}^{1-\theta}$. Dividing by $k_{ss} \neq 0$ and multiplying by 4

$$\frac{4\widetilde{y}_{ss}}{k_{ss}} = \frac{1}{KoY} = 4a_{ss} k_{ss}^{\theta-1} l_{ss}^{1-\theta}.$$

Then, as $a_{ss} = 1$, we get

$$\frac{1}{k_{ss}^{\theta-1}} = 4KoY l_{ss}^{1-\theta}$$

$\Updownarrow$

$$k_{ss}^{1-\theta} = 4KoY l_{ss}^{1-\theta}$$

$\Updownarrow$

$$k_{ss} = l_{ss} \left(4 \cdot KoY\right)^{\frac{1}{1-\theta}}.$$

From equation (8) then

$$\widetilde{y}_{ss} = k_{ss}^{\theta} l_{ss}^{1-\theta},$$

and from equation (7)

$$\widetilde{c}_{ss} + \delta k_{ss} = \left(1 - \frac{\xi}{2}\left(\frac{\pi_{ss}}{\pi_{ss}^{\nu}} - 1\right)^2\right)\widetilde{y}_{ss}$$

$\Updownarrow$

$$\widetilde{c}_{ss} = \left(1 - \frac{\xi}{2}\left(\frac{\pi_{ss}}{\pi_{ss}^{\nu}} - 1\right)^2\right)\widetilde{y}_{ss} - \delta k_{ss}$$

From equation (5) we have

$$(1-\eta)\,\widetilde{y}_{ss} + \xi M_{ss}^{\text{real}}\left(\frac{\pi_{ss}}{\pi_{ss}^{\nu}} - 1\right)\frac{\pi_{ss}}{\pi_{ss}^{\nu}}\widetilde{y}_{ss}\mu_{z,ss} + \eta mc_{ss}\widetilde{y}_{ss} = \xi\left(\frac{\pi_{ss}}{\pi_{ss}^{\nu}} - 1\right)\widetilde{y}_{ss}\frac{\pi_{ss}}{\pi_{ss}^{\nu}}$$

$\Updownarrow$

$$\eta mc_{ss}\widetilde{y}_{ss} = \xi\left(\frac{\pi_{ss}}{\pi_{ss}^{\nu}} - 1\right)\widetilde{y}_{ss}\frac{\pi_{ss}}{\pi_{ss}^{\nu}} - (1-\eta)\,\widetilde{y}_{ss} - \xi M_{ss}^{\text{real}}\left(\frac{\pi_{ss}}{\pi_{ss}^{\nu}} - 1\right)\frac{\pi_{ss}}{\pi_{ss}^{\nu}}\widetilde{y}_{ss}\mu_{z,ss}$$

$\Updownarrow$

$$mc_{ss} = \frac{1}{\eta\widetilde{y}_{ss}}\left[\xi\left(\frac{\pi_{ss}}{\pi_{ss}^{\nu}} - 1\right)\widetilde{y}_{ss}\frac{\pi_{ss}}{\pi_{ss}^{\nu}} - (1-\eta)\,\widetilde{y}_{ss} - \xi M_{ss}^{\text{real}}\left(\frac{\pi_{ss}}{\pi_{ss}^{\nu}} - 1\right)\frac{\pi_{ss}}{\pi_{ss}^{\nu}}\widetilde{y}_{ss}\mu_{z,ss}\right]$$

Inserted in Equation (4) we then get

$$\widetilde{w}_{ss} = mc_{ss}\left(1-\theta\right)a_t k_{ss}^{\theta} l_t^{-\theta}$$

The parameter φ_0 is set to imply the choosen steady state for labor, i.e. l_{ss}. Hence, from equation (2) we get

$$\widetilde{w}_{ss} = \kappa_w \widetilde{w}_{ss} + (1-\kappa_w)\left[\varphi_0\left(1-l_{ss}\right)^{-\frac{1}{\varphi}}\left(\frac{\widetilde{c}_{ss} - b\widetilde{c}_{ss}\mu_{z,ss}^{-1}}{\left(\widetilde{c}_{ss}\right)^{\chi_0}}\right)^{\chi}\left(\widetilde{c}_{ss}\right)^{\chi_0}\right]$$

$\Updownarrow$

$$\widetilde{w}_{ss} = \varphi_0\left(1-l_{ss}\right)^{-\frac{1}{\varphi}}\left(\frac{\widetilde{c}_{ss} - b\widetilde{c}_{ss}\mu_{z,ss}^{-1}}{\left(\widetilde{c}_{ss}\right)^{\chi_0}}\right)^{\chi}\left(\widetilde{c}_{ss}\right)^{\chi_0}$$

$\Updownarrow$

$$\frac{\widetilde{w}_{ss}}{\left(\widetilde{c}_{ss}\right)^{\chi_0}}\left(\frac{\widetilde{c}_{ss} - b\widetilde{c}_{ss}\mu_{z,ss}^{-1}}{\left(\widetilde{c}_{ss}\right)^{\chi_0}}\right)^{-\chi} = \varphi_0\left(1-l_{ss}\right)^{-\frac{1}{\varphi}}$$

$\Updownarrow\Downarrow$

$$\varphi_0 \left(1-l_{ss}\right)^{-\frac{1}{\varphi}} = \left(\frac{\widetilde{c}_{ss} - b\widetilde{c}_{ss}\mu_{z,ss}^{-1}}{\widetilde{c}_{ss}^{\chi_0}}\right)^{-\chi} \frac{\widetilde{w}_{ss}}{\widetilde{c}_{ss}^{\chi_0}}$$

$\Updownarrow$

$$\varphi_0 = \frac{\left(\frac{\widetilde{c}_{ss} - b\widetilde{c}_{ss}\mu_{z,ss}^{-1}}{\widetilde{c}_{ss}^{\chi_0}}\right)^{-\chi} \widetilde{w}_{ss}}{\left(1-l_{ss}\right)^{-\frac{1}{\varphi}} \widetilde{c}_{ss}^{\chi_0}}$$

Finally, from equation (1)

$$\tilde{V}_{ss} = u_0 + \frac{1}{1-\chi}\left(\frac{\widetilde{c}_{ss} - b\widetilde{c}_{ss}\mu_{z,ss}^{-1}}{\widetilde{c}_{ss}^{\chi_0}}\right)^{1-\chi} + u_0^d + \varphi_0 \frac{\left(1-l_{ss}\right)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}} + \beta\left(\tilde{V}_{ss}\mu_{z,ss}^{(1-\chi)(1-\chi_0)}\right)$$

$\Updownarrow$

$$V_{ss} = \frac{1}{1-\beta\mu_{z,ss}^{(1-\chi)(1-\chi_0)}}\left[u_0 + u_0^d + \frac{1}{1-\chi}\left(\frac{\widetilde{c}_{ss} - b\widetilde{c}_{ss}\mu_{z,ss}^{-1}}{\widetilde{c}_{ss}^{\chi_0}}\right)^{1-\chi} + \varphi_0 \frac{\left(1-l_{ss}\right)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}\right].$$

2 Digression on The New Keynesian Model

2.1 Risk Aversion at the Steady State

We follow Swanson (2012) and compute two measures of relative risk aversion in our model. With recursive Epstein-Zin preferences controlled by α, there are two measures of relative risk aversion. The first measure RRA^c applies when there is no upper bound for labor and therefore total household wealth A_t equals the present discounted value of consumption. The other measure RRA^{cl} applies when the upper bound for the household's time endowment is well-specified, meaning that total household wealth $\widetilde{A}_t$ equals the present discounted value of leisure plus consumption.

Throughout this section we use the notational convention in Swanson (2012) where a variable in the steady state is denoted without a subscript. For instance c is the steady state value of c_t. Moreover, $u = u(c, l)$.

The general formulas with external habit formation are (see Swanson (2012), page 24, eq 53 and eq 54)

$$RRA^{cl} = \frac{-u_{11} + \lambda u_{12}}{u_1}\frac{c + w(1-l)}{1 + w\lambda} + \alpha\frac{\left(c + w(1-l)\right)u_1}{u}$$

$$RRA^{c} = c\left(\frac{-u_{11} + \lambda u_{12}}{u_1}\frac{1}{1 + w\lambda} + \alpha\frac{u_1}{u}\right)$$

where

$$w = -\frac{u_2}{u_1}$$

$$\lambda = \frac{wu_{11} + u_{12}}{u_{22} + wu_{12}}$$

Note that

$$RRA^{cl} = \frac{-u_{11}+\lambda u_{12}}{u_1}\frac{c+w(1-l)}{1+w\lambda} + \alpha\frac{(c+w(1-l))u_1}{u}$$

$$= \frac{c+w(1-l)}{c}c\left[\frac{-u_{11}+\lambda u_{12}}{u_1}\frac{1}{1+w\lambda} + \alpha\frac{u_1}{u}\right]$$

$\Updownarrow$

$$RRA^{cl} = \left(1 + \frac{w}{c}(1-l)\right)RRA^{c}.$$

Here, we use the notation that

- $u =$ the utility index
- $u_1 =$ the partial derivative of u with respect to consumption
- $u_2 =$ the partial derivative of u with respect to hours worked
- $w =$ the steady state wage level
- $c =$ the steady state consumption level
- $l =$ hours worked

Note that these formulas also apply in our case even though we have wage strickiness. The reason is that the household derives its FOC in a frictionless setting, and the steady state of wage is not affected by the presence of wage stickiness, i.e., when using the notation above we have $w_t^*/z_t = w_t/z_t$ in the steady state. Also, Swanson (2012) shows that the above formulas also hold with balanced growth (see Swanson (2012) page 48). Recall that our utility function reads (ignoring d_t and n_t as $d_{ss} = n_{ss} = 1$)

$$u\left(c_t - bc_{t-1}, 1 - l_t\right) = \left(u_0^d + u_0\right) z_t^{(1-\chi)(1-\chi_0)} + \frac{1}{1-\chi}\left(\frac{c_t - bc_{t-1}}{\widetilde{c}_{ss}^{\chi_0} z_t^{\chi_0}}\right)^{1-\chi} + z_t^{(1-\chi)(1-\chi_0)} \varphi_0 \frac{\left(1 - l_t\right)^{1-\frac{1}{\varphi}}}{1 - \frac{1}{\varphi}}$$

Hence, from now on all steady state variables refer to those in the normalized economy without trends. Hence, in the steady state we have

$u = \left(u_0^d + u_0\right) + \frac{1}{1-\chi}\left(\frac{c_t - bc\mu_{z,t}^{-1}}{c^{\chi_0}}\right)^{1-\chi} + \varphi_0 \frac{(1-l)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}} = \left(u_0^d + u_0\right) + \frac{c^{\chi_0(\chi-1)}}{1-\chi}\left(c - bc\mu_z^{-1}\right)^{1-\chi} + \varphi_0 \frac{(1-l)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}$

$u_1 = c^{\chi_0(\chi-1)}\left(c - bc\mu_z^{-1}\right)^{-\chi}$

$u_{11} = -\chi c^{\chi_0(\chi-1)}\left(c - bc\mu_z^{-1}\right)^{-\chi-1}$

$u_2 = -\phi_0\left(1 - l\right)^{-\frac{1}{\varphi}}$

$u_{22} = -\frac{1}{\varphi}\left(-\phi_0\right)\left(1 - l\right)^{-\frac{1}{\varphi}-1}(-1) = -\frac{1}{\varphi}\phi_0\left(1 - l\right)^{-\frac{1}{\varphi}-1}$

$u_{12} = 0$

Thus

$w = -\frac{u_2}{u_1} = -\frac{-\phi_0(1-l)^{-\frac{1}{\varphi}}}{c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}} = \frac{\phi_0(1-l)^{-\frac{1}{\varphi}}}{c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}$

$\Updownarrow$

$\phi_0 = \frac{c^{\chi_0(\chi-1)} w\left(c-bc\mu_z^{-1}\right)^{-\chi}}{(1-l)^{-\frac{1}{\varphi}}}$

And

$\lambda = \frac{wu_{11}+u_{12}}{u_{22}+wu_{12}} = \frac{w\left[-\chi c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi-1}\right]}{-\frac{1}{\varphi}\phi_0(1-l)^{-\frac{1}{\varphi}-1}}$

$= \frac{w\chi c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi-1}}{\frac{1}{\varphi}\frac{c^{\chi_0(\chi-1)} w\left(c-bc\mu_z^{-1}\right)^{-\chi}}{(1-l)^{-\frac{1}{\varphi}}}(1-l)^{-\frac{1}{\varphi}-1}}$

$= \frac{\chi\left(c-bc\mu_z^{-1}\right)^{-1}}{\frac{1}{\varphi}(1-l)^{-1}}$

$= \frac{\chi(1-l)}{\frac{1}{\varphi}\left(c-bc\mu_z^{-1}\right)}$

Note also that

$$w\lambda = \frac{\phi_0 (1-l)^{-\frac{1}{\varphi}}}{c^{\chi_0(\chi-1)}\left(c - bc\mu_z^{-1}\right)^{-\chi}} \frac{\chi(1-l)}{\frac{1}{\varphi}\left(c-bc\mu_z^{-1}\right)} = \frac{\chi\phi_0(1-l)^{1-\frac{1}{\varphi}}}{\frac{1}{\varphi}c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{1-\chi}}$$

Hence, the first measure of relative risk-aversion is:

$$RRA^c = c\left(\frac{-u_{11}}{u_1}\frac{1}{1+w\lambda} + \alpha\frac{u_1}{u}\right)$$

$$= c\frac{\chi c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi-1}}{c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}} \frac{1}{1+\frac{\chi\phi_0(1-l)^{1-\frac{1}{\varphi}}}{\frac{1}{\varphi}c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{1-\chi}}} + c\alpha\frac{c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}{\left(u_0^d+u_0\right)+\frac{c^{\chi_0(\chi-1)}}{1-\chi}\left(c-bc\mu_z^{-1}\right)^{1-\chi}+\varphi_0\frac{(1-l)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}}$$

$$= c\left[\chi\left(c-bc\mu_z^{-1}\right)^{-1}\right] \frac{1}{1+\frac{\chi\phi_0(1-l)^{1-\frac{1}{\varphi}}}{\frac{1}{\varphi}c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{1-\chi}}} + c\alpha\frac{c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}{\left(u_0^d+u_0\right)+\frac{c^{\chi_0(\chi-1)}}{1-\chi}\left(c-bc\mu_z^{-1}\right)^{1-\chi}+\varphi_0\frac{(1-l)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}}$$

$$= \frac{c\chi}{c-bc\mu_z^{-1}} \frac{1}{1+\frac{\chi\phi_0(1-l)^{1-\frac{1}{\varphi}}}{\frac{1}{\varphi}c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{1-\chi}}} + c\alpha\frac{c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}{\left(u_0^d+u_0\right)+\frac{c^{\chi_0(\chi-1)}}{1-\chi}\left(c-bc\mu_z^{-1}\right)^{1-\chi}+\varphi_0\frac{(1-l)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}}$$

$$= \frac{c\chi}{c-bc\mu_z^{-1}+\frac{\chi\phi_0(1-l)^{1-\frac{1}{\varphi}}}{\frac{1}{\varphi}c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}} + c\alpha\frac{c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}{\left(u_0^d+u_0\right)+\frac{c^{\chi_0(\chi-1)}}{1-\chi}\left(c-bc\mu_z^{-1}\right)^{1-\chi}+\varphi_0\frac{(1-l)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}}$$

$$= \frac{c\chi\left(c-bc\mu_z^{-1}\right)^{-\chi}}{\left(c-bc\mu_z^{-1}\right)^{1-\chi}+\frac{\chi\phi_0(1-l)^{1-\frac{1}{\varphi}}}{c^{\chi_0(\chi-1)}\frac{1}{\varphi}}} + c\alpha\frac{c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}{\left(u_0^d+u_0\right)+\frac{c^{\chi_0(\chi-1)}}{1-\chi}\left(c-bc\mu_z^{-1}\right)^{1-\chi}+\varphi_0\frac{(1-l)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}}$$

$$= \frac{c\chi\left(c-bc\mu_z^{-1}\right)^{-\chi}}{\left(c-bc\mu_z^{-1}\right)^{1-\chi}+\frac{wc^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}{(1-l)^{-\frac{1}{\varphi}}}\frac{\chi(1-l)^{1-\frac{1}{\varphi}}}{c^{\chi_0(\chi-1)}\frac{1}{\varphi}}} + c\alpha\frac{c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}{\left(u_0^d+u_0\right)+\frac{c^{\chi_0(\chi-1)}}{1-\chi}\left(c-bc\mu_z^{-1}\right)^{1-\chi}+\frac{wc^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}{(1-l)^{-\frac{1}{\varphi}}}\frac{(1-l)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}}$$

using $\phi_0 = \frac{wc^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}{(1-l)^{-\frac{1}{\varphi}}}$

$$= \frac{c\chi}{\left(c-bc\mu_z^{-1}\right)+\frac{w}{1}\frac{\chi(1-l)}{\frac{1}{\varphi}}} + c\alpha\frac{c^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}{\left(u_0^d+u_0\right)+\frac{c^{\chi_0(\chi-1)}}{1-\chi}\left(c-bc\mu_z^{-1}\right)^{1-\chi}+\frac{wc^{\chi_0(\chi-1)}\left(c-bc\mu_z^{-1}\right)^{-\chi}}{1}\frac{(1-l)}{1-\frac{1}{\varphi}}}$$

$$= \frac{c\chi}{\left(c-bc\mu_z^{-1}\right)+\frac{w}{1}\frac{\chi(1-l)}{\frac{1}{\varphi}}} + c\alpha\frac{\left(c-bc\mu_z^{-1}\right)^{-\chi}}{\frac{\left(u_0^d+u_0\right)}{c^{\chi_0(\chi-1)}}+\frac{1}{1-\chi}\left(c-bc\mu_z^{-1}\right)^{1-\chi}+\frac{w\left(c-bc\mu_z^{-1}\right)^{-\chi}}{1}\frac{(1-l)}{1-\frac{1}{\varphi}}}$$

$$= \frac{\chi}{\left(1-b\mu_z^{-1}\right)+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}} + c\alpha\frac{1}{\frac{\left(u_0^d+u_0\right)}{c^{\chi_0(\chi-1)}}\left(c-bc\mu_z^{-1}\right)^{\chi}+\frac{1}{1-\chi}\left(c-bc\mu_z^{-1}\right)+\frac{w}{1}\frac{(1-l)}{1-\frac{1}{\varphi}}}$$

$$= \frac{\chi}{\left(1-b\mu_z^{-1}\right)+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}} + \alpha\frac{1}{\frac{\left(u_0^d+u_0\right)}{c^{\chi_0(\chi-1)}c}\left(c-bc\mu_z^{-1}\right)^{\chi}+\frac{1}{1-\chi}\left(1-b\mu_z^{-1}\right)+\frac{w}{c}\frac{(1-l)}{1-\frac{1}{\varphi}}}$$

$$= \frac{\chi}{\left(1-b\mu_z^{-1}\right)+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}} + \alpha\frac{1-\chi}{(1-\chi)\frac{\left(u_0^d+u_0\right)}{c^{\chi_0(\chi-1)+1}}\left(c-bc\mu_z^{-1}\right)^{\chi}+\left(1-b\mu_z^{-1}\right)+\frac{w(1-l)}{c}\frac{(1-\chi)}{1-\frac{1}{\varphi}}}$$

Note when $b = 0, \chi_0 = 0$, and $u_0 = 0$, we get

$$RRA^c = \frac{\chi}{1+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}} + \alpha\frac{1-\chi}{(1-\chi)\frac{u_0^d}{c^{\chi_0(\chi-1)+1}}c^{\chi}+1+\frac{w(1-l)}{c}\frac{(1-\chi)}{1-\frac{1}{\varphi}}}$$

$$= \frac{\chi}{1+\chi\varphi\frac{w(1-l)}{c}} + \alpha\frac{1-\chi}{(1-\chi)u_0^d c^{\chi-1}+1+\frac{(1-\chi)}{1-\frac{1}{\varphi}}\frac{w(1-l)}{c}}$$

Also, when $u_0^d = u_0 = 0$ (standard EZ preferences) we get

$$RRA^c = \frac{\chi}{\left(1-b\mu_z^{-1}\right)+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}} + \alpha\frac{1-\chi}{(1-\chi)\frac{\left(u_0^d+u_0\right)}{c^{\chi_0(\chi-1)+1}}\left(c-bc\mu_z^{-1}\right)^{\chi}+\left(1-b\mu_z^{-1}\right)+\frac{w(1-l)}{c}\frac{(1-\chi)}{1-\frac{1}{\varphi}}}$$

$$= \frac{\chi}{1-b\mu_z^{-1}+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}} + \alpha\frac{1-\chi}{1-b\mu_z^{-1}+\frac{w(1-l)}{c}\frac{(1-\chi)}{1-\frac{1}{\varphi}}}$$

Thus, to back out α from RRA^c we have

$$RRA^c = \frac{\chi}{\left(1-b\mu_z^{-1}\right)+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}} + \alpha\frac{1-\chi}{(1-\chi)\frac{\left(u_0^d+u_0\right)}{c^{\chi_0(\chi-1)+1}}\left(c-bc\mu_z^{-1}\right)^{\chi}+\left(1-b\mu_z^{-1}\right)+\frac{w(1-l)}{c}\frac{(1-\chi)}{1-\frac{1}{\varphi}}}$$

$\Updownarrow$

$$\alpha = \frac{RRA^c - \frac{\chi}{\left(1-b\mu_z^{-1}\right)+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}}}{\frac{1-\chi}{(1-\chi)\frac{\left(u_0^d+u_0\right)}{c^{\chi_0(\chi-1)+1}}\left(c-bc\mu_z^{-1}\right)^{\chi}+\left(1-b\mu_z^{-1}\right)+\frac{w(1-l)}{c}\frac{(1-\chi)}{1-\frac{1}{\varphi}}}}$$

Note also that

$$RRA^c = c\left(\frac{-u_{11}}{u_1}\frac{1}{1+w\lambda}+\alpha\frac{u_1}{u}\right)$$

$\Updownarrow$

$$\frac{RRA^c}{c}+\frac{u_{11}}{u_1}\frac{1}{1+w\lambda} = \alpha\frac{u_1}{u}$$

$\Updownarrow$

$$\alpha = \left(\frac{RRA^c}{c}+\frac{u_{11}}{u_1}\frac{1}{1+w\lambda}\right)\frac{u}{u_1}$$

And therefore

$$RRA^{cl} = \left(1+\frac{w}{c}(1-l)\right)RRA^c = \left(1+\frac{w}{c}(1-l)\right)RRA^c$$

If we condition on a given value of α (for instance, based on reasons motivated by accuracy of the model solution), then we can alternatively back out the value of u_0 to get a given RRA^c. That is, we get

$$RRA^c = \frac{\chi}{\left(1-b\mu_z^{-1}\right)+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}} + \alpha\frac{1-\chi}{(1-\chi)\frac{\left(u_0^d+u_0\right)}{c^{\chi_0(\chi-1)+1}}\left(c-bc\mu_z^{-1}\right)^{\chi}+\left(1-b\mu_z^{-1}\right)+\frac{w(1-l)}{c}\frac{(1-\chi)}{1-\frac{1}{\varphi}}}$$

$\Updownarrow$

$$\frac{1}{\alpha(1-\chi)}\left[RRA^c - \frac{\chi}{\left(1-b\mu_z^{-1}\right)+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}}\right] = \frac{1}{(1-\chi)\frac{\left(u_0^d+u_0\right)}{c^{\chi_0(\chi-1)+1}}\left(c-bc\mu_z^{-1}\right)^{\chi}+\left(1-b\mu_z^{-1}\right)+\frac{w(1-l)}{c}\frac{(1-\chi)}{1-\frac{1}{\varphi}}}$$

$\Updownarrow$

$$(1-\chi)\frac{\left(u_0^d+u_0\right)}{c^{\chi_0(\chi-1)+1}}\left(c-bc\mu_z^{-1}\right)^{\chi}+\left(1-b\mu_z^{-1}\right)+\frac{w\,(1-l)}{c}\frac{(1-\chi)}{1-\frac{1}{\varphi}}=\frac{\alpha\,(1-\chi)}{\left(RRA^c-\frac{\chi}{\left(1-b\mu_z^{-1}\right)+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}}\right)}$$

$\Updownarrow$

$$(1-\chi)\frac{\left(u_0^d+u_0\right)}{c^{\chi_0(\chi-1)+1}}\left(c-bc\mu_z^{-1}\right)^{\chi}=\frac{\alpha\,(1-\chi)}{\left(RRA^c-\frac{\chi}{\left(1-b\mu_z^{-1}\right)+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}}\right)}-\left(1-b\mu_z^{-1}\right)-\frac{w\,(1-l)}{c}\frac{(1-\chi)}{1-\frac{1}{\varphi}}$$

$\Updownarrow$

$$u_0^d+u_0=\frac{c^{\chi_0(\chi-1)+1}}{\left(c-bc\mu_z^{-1}\right)^{\chi}(1-\chi)}\left[\frac{\alpha\,(1-\chi)}{\left(RRA^c-\frac{\chi}{\left(1-b\mu_z^{-1}\right)+\frac{w}{c}\frac{\chi(1-l)}{\frac{1}{\varphi}}}\right)}-\left(1-b\mu_z^{-1}\right)-\frac{w\,(1-l)}{c}\frac{(1-\chi)}{1-\frac{1}{\varphi}}\right]$$

where either u_0^d or u_0 are zero.

2.2 The Frisch Labor Supply Elasticity

Recall that this elasticity is given by

$elas_F=\frac{u_l}{l\left(u_{ll}-\frac{u_{cl}}{u_{cc}}\right)}$

$\varphi_0\frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}$

In our case (given that the utility of leisure is $\varphi_0\frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}$)

$u_l=-\varphi_0\,(1-l_t)^{-\frac{1}{\varphi}}$

$u_{ll}=-\varphi_0\left(-\frac{1}{\varphi}\right)(1-l_t)^{-\frac{1}{\varphi}-1}(-1)=-\varphi_0\frac{1}{\varphi}(1-l_t)^{-\frac{1}{\varphi}-1}$

$u_{cl}=0$

So, in the steady state we have

$elas_F=\frac{u_l}{l_{ss}\left(u_{ll}-\frac{u_{cl}}{u_{cc}}\right)}$

$=\frac{-\varphi_0(1-l_{ss})^{-\frac{1}{\varphi}}}{l_{ss}\left(-\varphi_0\frac{1}{\varphi}(1-l_{ss})^{-\frac{1}{\varphi}-1}-0\right)}$

$=\frac{\varphi}{l_{ss}(1-l_{ss})^{-1}}$

$=\varphi\frac{1-l_{ss}}{l_{ss}}$

$=\varphi\left(\frac{1}{l_{ss}}-1\right)$

When $l_{ss}=\frac{1}{3}$, then

$elas_F=\varphi\left(\frac{3}{1}-1\right)=2\varphi$

2.3 Intertemporal Elasticity of Substitution

The intertemporal elasticity of substitution (IES) is given by

$$\begin{aligned}
IES_t &= -\frac{\mathcal{U}_c}{\mathcal{U}_{cc}c_t} \\
&= \frac{-\widetilde{c}_{ss}^{\chi_0(\chi-1)}\left(\frac{c_t-bc_{t-1}}{z_t^{\chi_0}}\right)^{-\chi}\frac{1}{\widetilde{c}_{ss}^{\chi_0}z_t^{\chi_0}}}{-\chi\widetilde{c}_{ss}^{\chi_0(\chi-1)}\left(\frac{c_t-bc_{t-1}}{z_t^{\chi_0}}\right)^{-\chi-1}\frac{c_t}{z_t^{2\chi_0}}} \\
&= \frac{-\left(\frac{c_t-bc_{t-1}}{z_t^{\chi_0}}\frac{z_t^{1-\chi_0}}{z_t^{1-\chi_0}}\right)^{-\chi}}{-\chi\left(\frac{c_t-bc_{t-1}}{z_t^{\chi_0}}\frac{z_t^{1-\chi_0}}{z_t^{1-\chi_0}}\right)^{-\chi-1}\frac{c_t}{z_t^{\chi_0}}} \\
&= \frac{-\left(\frac{c_t-bc_{t-1}}{z_t}\frac{z_t^{1-\chi_0}}{1}\right)^{-\chi}}{-\chi\left(\frac{c_t-bc_{t-1}}{z_t}\frac{z_t^{1-\chi_0}}{1}\right)^{-\chi-1}\frac{c_t}{z_t^{\chi_0}}} \\
&= \frac{\left(\tilde{c}_t-b\tilde{c}_{t-1}\mu_{z,t}^{-1}\right)^{-\chi}}{\chi\left(\tilde{c}_t-b\tilde{c}_{t-1}\mu_{z,t}^{-1}\right)^{-\chi-1}\frac{c_t}{z_t^{\chi_0}}}\frac{1}{z_t^{-(1-\chi_0)}} \\
&= \frac{\left(\tilde{c}_t-b\tilde{c}_{t-1}\mu_{z,t}^{-1}\right)^{-\chi}}{\chi\left(\tilde{c}_t-b\tilde{c}_{t-1}\mu_{z,t}^{-1}\right)^{-\chi-1}c_tz_t^{-\chi_0}}\frac{1}{z_t^{-1+\chi_0}} \\
&= \frac{\tilde{c}_t-b\tilde{c}_{t-1}\mu_{z,t}^{-1}}{\chi\tilde{c}_t}
\end{aligned}$$

In the steady state we get

$$IES_{ss} = \frac{1}{\chi}\left(1-b\mu_{z,ss}^{-1}\right)$$

2.4 The Equity Price

We have

$$P_t^m = \mathbb{E}_t\left[M_{t,t+1}^{\text{real}}\left(Div_{t+1}+P_{t+1}^m\right)\right],$$

where $Div_t = c_t$, i.e. a claim on consumption.

Inducing stationarity

$$\frac{P_t^m}{z_t} = \mathbb{E}_t\left[M_{t,t+1}^{\text{real}}\left(\left(\frac{Div_{t+1}}{z_{t+1}}\right)+\left(\frac{P_{t+1}^m}{z_{t+1}}\right)\right)\left(\frac{z_{t+1}}{z_t}\right)\right]$$

$\Updownarrow$

$$\tilde{P}_t^m = \mathbb{E}_t\left[M_{t,t+1}^{\text{real}}\left(\widetilde{Div}_{t+1}+\tilde{P}_{t+1}^m\right)\mu_{z,t+1}\right]$$

Hence, in the steady state:

$$\tilde{P}_{ss}^m = M_{ss,ss+1}^{\text{real}}\left(\widetilde{Div}_{ss}+\tilde{P}_{ss}^m\right)\mu_{z,ss}$$

$\Updownarrow$

$$\tilde{P}_{ss}^m = M_{ss,ss+1}^{\text{real}}\mu_{z,ss}\widetilde{Div}_{ss}+M_{ss}^{\text{real}}\mu_{z,ss}\tilde{P}_{ss}^m$$

$\Updownarrow$

$$\tilde{P}_{ss}^m = \frac{M_{ss,ss+1}^{\text{real}}\mu_{z,ss}\widetilde{Div}_{ss}}{1-M_{ss}^{\text{real}}\mu_{z,ss}}$$

Recall that $M_{ss,ss} = \beta \frac{\mu_{z,ss}^{-\chi(1-\chi_0)-\chi_0}}{\pi_{ss}}$, so $M_{ss,ss}^{\text{real}} = \beta \mu_{z,ss}^{-\chi(1-\chi_0)-\chi_0}$, and we therefore get

$$\tilde{P}_{ss}^{m} = \frac{\beta \mu_{z,ss}^{1-\chi(1-\chi_0)-\chi_0} \widetilde{Div}_{ss}}{1 - \beta \mu_{z,ss}^{1-\chi(1-\chi_0)-\chi_0}}$$

where $\widetilde{Div}_{ss} = \tilde{c}_{ss}$. Note also that

$$\begin{aligned} P_t^m &= \tilde{P}_t^m z_t \\ &= \exp\left\{\log \tilde{P}_t^m + \log z_t\right\} \end{aligned}$$

and similarly for dividends.

The requity return is

$$\begin{aligned} \exp\left\{r_{t+1}^m\right\} &\equiv \frac{\left(Div_{t+1} + P_{t+1}^m\right)}{\tilde{P}_t^m} \\ &= \frac{\left(\widetilde{Div}_{t+1}^{\omega} + \tilde{P}_{t+1}^m\right)\mu_{z,t+1}}{\tilde{P}_t^m} \end{aligned}$$

⇕

$$r_{t+1}^m = \log\left(\widetilde{Div}_{t+1} + \tilde{P}_{t+1}^m\right) - \log\left(\tilde{P}_t^m\right) + \log \mu_{z,t+1}.$$

2.5 The Timing Premium

We extend the definition of the timing premium to accommodate leisure in the utility function. Here, we apply the approach suggested in Andreasen & Jørgensen (2020). The basic idea is to impose that the household is on the equilibrium path for the leisure and consumption trade-off. That is, we use the first order condition

$$z_t^{(1-\chi)(1-\chi_0)} n_t \varphi_0 \left(1 - l_t\right)^{-\frac{1}{\varphi}} = \left(\frac{c_t - bc_{t-1}}{\left(\widetilde{c}_{ss} z_t\right)^{\chi_0}}\right)^{-\chi} \frac{w_t^*}{\left(\widetilde{c}_{ss} z_t\right)^{\chi_0}}$$

⇕

$$\frac{1}{\left(1 - l_t\right)^{\frac{1}{\varphi}}} = \frac{\left(\frac{c_t - bc_{t-1}}{\left(\widetilde{c}_{ss} z_t\right)^{\chi_0}}\right)^{-\chi} \frac{w_t^*}{\left(\widetilde{c}_{ss} z_t\right)^{\chi_0}}}{z_t^{(1-\chi)(1-\chi_0)} n_t \varphi_0}$$

⇕

$$\left(1 - l_t\right)^{\frac{1}{\varphi}} = \frac{z_t^{(1-\chi)(1-\chi_0)} n_t \varphi_0}{\left(\frac{c_t - bc_{t-1}}{\left(\widetilde{c}_{ss} z_t\right)^{\chi_0}}\right)^{-\chi} \frac{w_t^*}{\left(\widetilde{c}_{ss} z_t\right)^{\chi_0}}}$$

⇕

$$\left(1 - l_t\right) = \left[\frac{z_t^{(1-\chi)(1-\chi_0)} n_t \varphi_0}{\left(\frac{c_t - bc_{t-1}}{\left(\widetilde{c}_{ss} z_t\right)^{\chi_0}}\right)^{-\chi} \frac{w_t^*}{\left(\widetilde{c}_{ss} z_t\right)^{\chi_0}}}\right]^{\varphi}$$

This implies that

$$\begin{aligned}
n_t\varphi_0\frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}} &= n_t\varphi_0\frac{\left[\frac{z_t^{(1-\chi)(1-\chi_0)}n_t\varphi_0}{\left(\frac{c_t-bc_{t-1}}{(\widetilde{c}_{ss}z_t)^{\chi_0}}\right)^{-\chi}\frac{w_t^*}{(\widetilde{c}_{ss}z_t)^{\chi_0}}}\right]^{\varphi\left(1-\frac{1}{\varphi}\right)}}{1-\frac{1}{\varphi}} \\
&= n_t^{1+\varphi\left(1-\frac{1}{\varphi}\right)}\frac{\varphi_0^{1+\varphi\left(1-\frac{1}{\varphi}\right)}}{1-\frac{1}{\varphi}}\frac{z_t^{(1-\chi)(1-\chi_0)\varphi\left(1-\frac{1}{\varphi}\right)}}{\left(\frac{c_t-bc_{t-1}}{(\widetilde{c}_{ss}z_t)^{\chi_0}}\right)^{-\chi\varphi\left(1-\frac{1}{\varphi}\right)}\frac{(w_t^*)^{\varphi\left(1-\frac{1}{\varphi}\right)}}{(\widetilde{c}_{ss}z_t)^{\chi_0\varphi\left(1-\frac{1}{\varphi}\right)}}} \\
&= n_t^{1+\varphi-1}\frac{\varphi_0^{1+\varphi-1}}{1-\frac{1}{\varphi}}\frac{z_t^{(1-\chi)(1-\chi_0)\varphi\left(1-\frac{1}{\varphi}\right)}}{\left(\frac{c_t-bc_{t-1}}{(\widetilde{c}_{ss}z_t)^{\chi_0}}\right)^{-\chi\varphi\left(1-\frac{1}{\varphi}\right)}\frac{(w_t^*)^{\varphi\left(1-\frac{1}{\varphi}\right)}}{(\widetilde{c}_{ss}z_t)^{\chi_0\varphi\left(1-\frac{1}{\varphi}\right)}}} \\
&= \frac{(n_t\varphi_0)^{\varphi}}{1-\frac{1}{\varphi}}\frac{z_t^{(1-\chi)(1-\chi_0)\varphi\left(1-\frac{1}{\varphi}\right)}(\widetilde{c}_{ss}z_t)^{\chi_0\varphi\left(1-\frac{1}{\varphi}\right)}\left(\frac{c_t-bc_{t-1}}{(\widetilde{c}_{ss}z_t)^{\chi_0}}\right)^{\chi\varphi\left(1-\frac{1}{\varphi}\right)}}{(w_t^*)_t^{\varphi\left(1-\frac{1}{\varphi}\right)}} \\
&= \frac{(n_t\varphi_0)^{\varphi}}{1-\frac{1}{\varphi}}\frac{z_t^{(1-\chi)(1-\chi_0)(\varphi-1)}(c_t-bc_{t-1})^{\chi(\varphi-1)}(\widetilde{c}_{ss}z_t)^{\chi_0(\varphi-1)(1-\chi)}}{(w_t^*)^{(\varphi-1)}}
\end{aligned}$$

Hence, we can write the value function as (for negative value function) as

$$\begin{aligned}
V_t &= u_0z_t^{(1-\chi)(1-\chi_0)}+d_t\left[\frac{1}{1-\chi}\left(\frac{c_t-bc_{t-1}}{(\widetilde{c}_{ss}z_t)^{\chi_0}}\right)^{1-\chi}+u_0^d\times z_t^{(1-\chi)(1-\chi_0)}+z_t^{(1-\chi)(1-\chi_0)}n_t\varphi_0\frac{(1-l_t)^{1-\frac{1}{\varphi}}}{1-\frac{1}{\varphi}}\right] \\
&\quad -\beta\left(\mathbb{E}_t\left[(-V_{t+1})^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

$\Updownarrow$

$$\begin{aligned}
V_t &= u_0z_t^{(1-\chi)(1-\chi_0)}+d_t\left[\begin{array}{c}\frac{1}{1-\chi}\left(\frac{c_t-bc_{t-1}}{(\widetilde{c}_{ss}z_t)^{\chi_0}}\right)^{1-\chi}+u_0^d\times z_t^{(1-\chi)(1-\chi_0)} \\ +z_t^{(1-\chi)(1-\chi_0)}\frac{(n_t\varphi_0)^{\varphi}}{1-\frac{1}{\varphi}}\frac{z_t^{(1-\chi)(1-\chi_0)(\varphi-1)}(c_t-bc_{t-1})^{\chi(\varphi-1)}(\widetilde{c}_{ss}z_t)^{\chi_0(\varphi-1)(1-\chi)}}{(w_t^*)^{(\varphi-1)}}\end{array}\right] \\
&\quad -\beta\left(\mathbb{E}_t\left[(-V_{t+1})^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

$\Updownarrow$

$$\begin{aligned}
V_t &= u_0z_t^{(1-\chi)(1-\chi_0)}+d_t\left[\begin{array}{c}\frac{1}{1-\chi}\left(\frac{c_t-bc_{t-1}}{(\widetilde{c}_{ss}z_t)^{\chi_0}}\right)^{1-\chi}+u_0^d\times z_t^{(1-\chi)(1-\chi_0)} \\ +\frac{(n_t\varphi_0)^{\varphi}}{1-\frac{1}{\varphi}}\frac{z_t^{(1-\chi)(1-\chi_0)\varphi}(c_t-bc_{t-1})^{\chi(\varphi-1)}(\widetilde{c}_{ss}z_t)^{\chi_0(\varphi-1)(1-\chi)}}{(w_t^*)^{(\varphi-1)}}\end{array}\right] \\
&\quad -\beta\left(\mathbb{E}_t\left[(-V_{t+1})^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

We define the timing premium Π_t implicitely as

$$\begin{aligned} V_t &= u_0 \left(z_t \left(1-\Pi_t\right)\right)^{(1-\chi)(1-\chi_0)} \\ &+ d_t \left[\frac{1}{1-\chi}\left(\frac{c_t - bc_{t-1}}{\left(\widetilde{c}_{ss} z_t\right)^{\chi_0}}\left(1-\Pi_t\right)\right)^{1-\chi} + u_0^d \times \left(z_t\left(1-\Pi_t\right)\right)^{(1-\chi)(1-\chi_0)}\right] \\ &+ d_t \frac{\left(n_t \varphi_0\right)^{\varphi}}{1-\frac{1}{\varphi}} \frac{z_t^{(1-\chi)(1-\chi_0)\varphi}\left(\left(c_t - bc_{t-1}\right)\left(1-\Pi_t\right)\right)^{\chi(\varphi-1)}}{\left(w_t^*\right)^{(\varphi-1)}} \\ &- \beta\left(\mathbb{E}_t\left[\left(\begin{array}{cc} -N_{0,t+1}\left(1-\Pi_t\right)^{(1-\chi)(1-\chi_0)} - N_{c,t+1}\left(1-\Pi_t\right)^{(1-\chi)} \\ -N_{d,t+1}\left(1-\Pi_t\right)^{(1-\chi)(1-\chi_0)} - N_{l,t+1}\left(1-\Pi_t\right)^{\chi(\varphi-1)} \end{array}\right)^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

$\Updownarrow$

$$\begin{aligned} V_t &= u_0 \left(z_t \left(1-\Pi_t\right)\right)^{(1-\chi)(1-\chi_0)} \\ &+ \left[\frac{d_t}{1-\chi}\left(\frac{\left(c_t - bc_{t-1}\right)}{\left(\widetilde{c}_{ss} z_t\right)^{\chi_0}}\left(1-\Pi_t\right)\right)^{1-\chi} + d_t u_0^d \times \left(z_t\left(1-\Pi_t\right)\right)^{(1-\chi)(1-\chi_0)}\right] \\ &+ d_t \frac{\left(n_t \varphi_0\right)^{\varphi}}{1-\frac{1}{\varphi}} \frac{z_t^{(1-\chi)(1-\chi_0)\varphi}\left(\left(c_t - bc_{t-1}\right)\left(1-\Pi_t\right)\right)^{\chi(\varphi-1)}\left(\widetilde{c}_{ss} z_t\right)^{\chi_0(\varphi-1)(1-\chi)}}{\left(w_t^*\right)^{(\varphi-1)}} \\ &- \beta\left(\mathbb{E}_t\left[\left(\begin{array}{cc} -N_{0,t+1}\left(1-\Pi_t\right)^{(1-\chi)(1-\chi_0)} - N_{c,t+1}\left(1-\Pi_t\right)^{(1-\chi)} \\ -N_{d,t+1}\left(1-\Pi_t\right)^{(1-\chi)(1-\chi_0)} - N_{l,t+1}\left(1-\Pi_t\right)^{\chi(\varphi-1)} \end{array}\right)^{1-\alpha}\right]\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

Here, $N_{0,t}$ is the continuation value of $u_0 z_t^{(1-\chi)(1-\chi_0)}$ if uncertainty is resolved in the next period. The variable $N_{c,t}$ measures the continuation value of $\frac{d_t}{1-\chi}\left(\frac{c_t - bc_{t-1}}{(\widetilde{c}_{ss} z_t)^{\chi_0}}\right)^{1-\chi}$, while $N_{d,t}$ measures the continuation value of $d_t u_0^d z_t^{(1-\chi)(1-\chi_0)}$. Finally, $N_{l,t}$ is the continuation value of leisure. Note that

$$\begin{aligned} N_{0,t+1} &= u_0 z_{t+1}^{(1-\chi)(1-\chi_0)} + \beta u_0 z_{t+2}^{(1-\chi)(1-\chi_0)} + \beta^2 u_0 z_{t+3}^{(1-\chi)(1-\chi_0)} + \ldots \\ &= u_0 \sum_{i=1}^{\infty} \beta^{i-1} z_{t+i}^{(1-\chi)(1-\chi_0)} \end{aligned}$$

$$N_{c,t+1} = \sum_{i=1}^{\infty} \beta^{i-1} \frac{d_{t+i}}{1-\chi}\left(\frac{\left(c_{t+i} - bc_{t-1+i}\right)}{\left(\widetilde{c}_{ss} z_{t+i}\right)^{\chi_0}}\right)^{1-\chi}$$

$$N_{d,t+1} = u_0^d \sum_{i=1}^{\infty} \beta^{i-1} d_{t+i} z_{t+i}^{(1-\chi)(1-\chi_0)}$$

and

$$N_{l,t+1} = \sum_{i=1}^{\infty} \beta^{i-1} d_{t+i} \frac{\left(n_{t+i}\varphi_0\right)^{\varphi}}{1-\frac{1}{\varphi}} \frac{z_{t+i}^{(1-\chi)(1-\chi_0)\varphi}\left(c_{t+i} - bc_{t-1+i}\right)^{\chi(\varphi-1)}\left(\widetilde{c}_{ss} z_{t+i}\right)^{\chi_0(\varphi-1)(1-\chi)}}{\left(w_{t+i}^*\right)^{\varphi-1}}$$

where we recall that

$$\log\left(\frac{\mu_{z,t+1}}{\mu_{z,ss}}\right) = \rho_{\mu_z}\log\left(\frac{\mu_{z,t}}{\mu_{z,ss}}\right) + \sigma_{\mu_z}\epsilon_{\mu_z,t+1}$$

$$\log d_{t+1} = \rho_d \log d_t + \sigma_d \epsilon_{d,t+1}$$

$$\log n_{t+1} = \rho_n \log n_t + \sigma_n \epsilon_{n,t+1}$$

$$z_t = z_{t-1}\mu_{z,t}$$

and

$$w_t = \kappa_w \left(\widetilde{w}_{ss} z_t\right) + \left(1 - \kappa_w\right) w_t^*$$

$\Updownarrow$

$$w_t^* = \frac{w_t - \kappa_w \left(\widetilde{w}_{ss} z_t\right)}{1 - \kappa_w}$$

We start the simulations of all these sums at the steady state, i.e. where $z_t = 1$, $n_t = 1$, and $d_t = 1$. Also the timing premium is computed for the economy at the steady state.

3 Data

3.1 Hours

Hours is calculated based on total nonfarm payrolls, which is detrended using the regression filter of Hamilton (2018).

3.2 Wages

For real wages, we use the real hourly compensation for all employed persons in the nonfarm business sector, Index 2012=100, and seasonally adjusted. This data series is then detrended using the regression filter of Hamilton (2018).

3.3 Consumption

Quarterly consumption is for real per capita non-durables and service expenditures, which are available from the Federal Reserve Bank of St. Louis.

3.4 Inflation

Inflation is calculated as the year-on-year growth rate in the seasonal adjusted Consumer Price Index (excluding food and energy) for all urban consumers.

3.5 Bond Yields

We use the 3-month nominal risk-free rate r_t from the secondary market. The 1-year, 3-year, 5-year, 7-year, and 10-year bond yields are from the Gurkaynak, Sack and Wright dataset. This data series is available from 1961Q2.

3.6 Short Rate Expectations from Surveys

Expected future short rates 1,2,3, and 4 quarters ahead are taken from the survey of Professionel Forecasts for the 3-month T bill. Here, we use the median from panel of forecasts in the survey Professionel Forecasts. These survey data are not used in the paper, but they are included in the matlab codes linked to the paper.

3.7 Stock Market Data

Dividend and market return series are constructed from two CRSP series, the value-weighted index including distributions (VWRD) and the value-weighted index excluding distributions (VWRX). A price index is constructed as

$$P_{s+1} = P_s \left(1 + VWRX_{s+1}\right), \tag{11}$$

with $P_0 = 1$, where s denotes a monthly time index. The related dividends are then calculated as

$$D_{s+1} = P_s \left(VWRD_{s+1} - VWRX_{s+1}\right). \tag{12}$$

To remove seasonality effects in monthly dividends, we construct aggregate dividends $D_s^a = \sum_{i=0}^{11} D_{s-i}$. Using this series, dividend growth is calculated as

$$\Delta d_s = \log D_s^a - \log D_{s-1}^a, \tag{13}$$

and then summing these monthly observations over the particular quarter to get quarterly dividends. The market return each month is

$$r_s^m = \log\left(P_s + D_s\right) - \log P_{s-1}, \tag{14}$$

which we annualize. Finally the price-dividend ratio is

$$p_s - d_s = \log P_s - \log D_s^a. \tag{15}$$

We then obtain quarterly time series of the price-dividend ratio by using values at the end of each quarter.

3.8 Related Model Variables

1. Hours

We have

$$\hat{l}_t^{Data} = \hat{l}_t^{Model},$$

where we use the standard notation that a "hat" denotes deviation from the steady state, i.e. $\hat{l}_t = \log\left(\frac{l_t}{l_{ss}}\right)$.

2. Wages

We have

$$\hat{w}_t^{Data} = \widehat{\widetilde{w}}_t^{Model}$$

3. Consumption growth

The expressions for real quarterly consumption growth is given by

$$\begin{aligned} \Delta c_t &\equiv \log \frac{c_t}{c_{t-1}} = \log \frac{\widetilde{c}_t z_t}{\widetilde{c}_{t-1} z_{t-1}} = \log \frac{\widetilde{c}_t}{\widetilde{c}_{ss}} - \log \frac{\widetilde{c}_{t-1}}{\widetilde{c}_{ss}} + \log \frac{z_t}{z_{t-1}} \\ &= \widehat{\widetilde{c}}_t - \widehat{\widetilde{c}}_{t-1} + \log\left(\mu_{z,t}\right) \end{aligned}$$

because $\widehat{\widetilde{c}}_t \equiv \log \frac{\widetilde{c}_t}{\widetilde{c}_{ss}}$.

4. Inflation

The quarterly inflation rate is given by

$$\log \pi_t = \log \pi_{ss} + \hat{\pi}_t$$

where $\hat{\pi}_t = \log\left(\frac{\pi_t}{\pi_{ss}}\right)$

5. The Nominal Yield Curve

All yields are given by

$$r_t^{(k)} = r_{ss} + \hat{r}_t^{(k)},$$

where k denotes the maturity. Note that $r_t^{(1)} \equiv r_t$.

6. Surveys of Expected future short rates

$$\mathbb{E}_t\left[r_{t+i}\right] = r_{ss} + \widehat{\mathbb{E}_t\left[r_{t+i}\right]}$$

where $i = \{1, 2, 3, 4\}$.

Note finally, that all variables are annualized through a multiplication by 4, except for hours and wages.

4 Estimation Methodology: Filtering with Shrinkage

This section presents the estimation methodology used in the paper.

4.1 The Estimator

To describe the considered estimator, let $\boldsymbol{\theta}$ denote the structural parameters of dimension $n_\theta \times 1$. We would like to estimate $\boldsymbol{\theta}$ using the observables $\mathbf{y}_t^{obs}$ with dimension $n_y \times 1$. Given the presence of latent variables $\mathbf{x}_t$ of dimension $n_x \times 1$ in the model, a log-likelihood function $\sum_{t=1}^{T} \mathcal{L}_t^{CDKF}$ is evaluated by Kalman filtering. Due to the nonlinear structure of our state space system, we evaluate a quasi-log likelihood function by the central difference Kalman filter of Norgaard, Poulsen & Ravn (2000) as studied in Andreasen (2013) within the context of nonlinear DSGE models.

To robustify the estimation of $\boldsymbol{\theta}$ from a quasi log-likelihood function, we propose to shrink the estimation of $\boldsymbol{\theta}$ towards a set of unconditonal moments. The empirical moments are denoted by $\frac{1}{T}\sum_{t=1}^{T}\mathbf{m}_t$ and the model-implied moments by $\mathbb{E}\left[\mathbf{m}\left(\boldsymbol{\theta}\right)\right]$, which both have dimension $n_m \times 1$. Hence, the contribution from these shrinkage moments are given by

$$\begin{aligned} Q &= \left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{m}_t - \mathbb{E}\left[\mathbf{m}\left(\boldsymbol{\theta}\right)\right]\right)' \mathbf{W} \left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{m}_t - \mathbb{E}\left[\mathbf{m}\left(\boldsymbol{\theta}\right)\right]\right) \\ &= \mathbf{g}_{1:T}\left(\boldsymbol{\theta}\right)' \mathbf{W} \mathbf{g}_{1:T}\left(\boldsymbol{\theta}\right) \end{aligned}$$

where $\mathbf{g}_{1:T}\left(\boldsymbol{\theta}\right) \equiv \frac{1}{T}\sum_{t=1}^{T}\mathbf{g}_t\left(\boldsymbol{\theta}\right)$ with $\mathbf{g}_t\left(\boldsymbol{\theta}\right) \equiv \left(\mathbf{m}_t - \mathbb{E}\left[\mathbf{m}\left(\boldsymbol{\theta}\right)\right]\right)$ and $\mathbf{W}$ is a weigthing matrix. Here, we consider the case where $\mathbf{W}$ is diagonal and its elements are given by the inverse of the standard error attached to each of the moments in $\mathbf{m}_T$.

Hence, the considered estimator is given by

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}}{Max} \ \frac{1}{T}\sum_{t=1}^{T}\mathcal{L}_t^{CDKF} - \lambda \mathbf{g}_{1:T}\left(\boldsymbol{\theta}\right)' \mathbf{W} \mathbf{g}_{1:T}\left(\boldsymbol{\theta}\right) \tag{16}$$

where $\lambda \in \mathbb{R}_+$ controls the weight given to the skrinkage moments. For the implemention in Matlab, where we use minimization routines, we use the equivalent formulation

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}}{Min} \ -\frac{1}{T}\mathcal{L}_t^{CDKF} + \lambda \mathbf{g}_{1:T}\left(\boldsymbol{\theta}\right)' \mathbf{W} \mathbf{g}_{1:T}\left(\boldsymbol{\theta}\right).$$

4.2 The Unconditional Shrinkage Moments

To describe the unconditional moments included in the estimation, consider the following nine variables:

$$\underbrace{\begin{bmatrix} \hat{l}_t \\ \hat{w}_t \\ \Delta c_t \\ \log \pi_t \\ r_t^{(i)} \end{bmatrix}}_{\mathbf{y}_t^{mom}}$$

where $i = \{1, 4, 12, 20, 28, 40\}$. The first set of moments we include contains the first and second uncentered moments for $\mathbf{y}_t^{mom}$, that is

$$\frac{1}{T}\sum_{t=1}^{T}\mathbf{m}_{1,t} = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}_t^{mom} \\ \frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{y}_t^{mom}\right)^2 \end{bmatrix}.$$

To describe the second set of moments, consider the Campbell-Shiller regression

$$r_{t+m}^{(k-m)} - r_t^{(k)} = \alpha_k + \beta_k \frac{m}{k-m}\left(r_t^{(k)} - r_t^{(m)}\right) + u_t^{(k)}$$

where m indicates the forecast horizon. Here, we apply $m = 4$ for forecasting four quarters ahead. The population value for β_k is given by

$$
\begin{aligned}
\beta_k &= \frac{Cov\left(r_{t+m}^{(k-m)} - r_t^{(k)}, \frac{m}{k-m}\left(r_t^{(k)} - r_t^{(m)}\right)\right)}{Var\left(\frac{m}{k-m}\left(r_t^{(k)} - r_t^{(m)}\right)\right)} \\
&= \frac{\frac{m}{k-m} Cov\left(r_{t+m}^{(k-m)} - r_t^{(k)}, r_t^{(k)} - r_t^{(m)}\right)}{\left(\frac{m}{k-m}\right)^2 Var\left(r_t^{(k)} - r_t^{(m)}\right)} \\
&= \frac{Cov\left(r_{t+m}^{(k-m)} - r_t^{(k)}, r_t^{(k)} - r_t^{(m)}\right)}{\frac{m}{k-m} Var\left(r_t^{(k)} - r_t^{(m)}\right)}
\end{aligned}
$$

We represent β_k by including moments for the covariance and the variance determining β_k. That is, we target

$$
\frac{1}{T}\sum_{t=1}^{T} \mathbf{m}_{2,t} = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T} \mathbf{m}_{2,t}^{Cov} \\ \frac{1}{T}\sum_{t=1}^{T} \mathbf{m}_{2,t}^{VAR} \end{bmatrix}
$$

where

$$
\frac{1}{T}\sum_{t=1}^{T} \mathbf{m}_{2,t}^{Cov} = \left\{ \frac{1}{T}\sum_{t=1}^{T-m} \left(r_{t+m}^{(k-m)} - r_t^{(k)}\right)\left(\left(r_t^{(k)} - r_t^{(m)}\right) - \overline{slope_t}^{(k)}\right)\right\}_{k=\{8,16,\ldots,40\}}'
$$

and

$$
\frac{1}{T}\sum_{t=1}^{T} \mathbf{m}_{2,t}^{VAR} = \left\{ \frac{1}{T}\sum_{t=1}^{T-m} \left(r_t^{(k)} - r_t^{(m)}\right)\left(\left(r_t^{(k)} - r_t^{(m)}\right) - \overline{slope_t}^{(k)}\right)\right\}_{k=\{8,16,\ldots,40\}}',
$$

with $\overline{slope_t}^{(k)} \equiv \frac{1}{T}\sum_{t=1}^{T}\left(r_t^{(k)} - r_t^{(m)}\right)$ for $k = \{8, 16, ..., 40\}$.

Thus, we consider the following unconditional moments for the skrinkage part of our estimator

$$
\frac{1}{T}\sum_{t=1}^{T} \mathbf{m}_t \equiv \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T} \mathbf{m}_{1,t} \\ \frac{1}{T}\sum_{t=1}^{T} \mathbf{m}_{2,t} \end{bmatrix}.
$$

4.3 Computing the Shrinkage Moments

In general, $\mathbb{E}[\mathbf{m}(\boldsymbol{\theta})]$ and hence Q must be computed by simulation. However, for the pruned approximation (and a linearized approximation), it is possible to compute $\mathbb{E}[\mathbf{m}(\boldsymbol{\theta})]$ in closed form. When the standard perturbation approximation is used, we can easily compute the closed-form expression for $\mathbb{E}[\mathbf{m}(\boldsymbol{\theta})]$ using the results in Andreasen et al. (2018). For the second-order projection approximation applied in the paper, we extend the results of Andreasen et al. (2018) to this approximation, such that it is also possible to obtain a closed form expression for $\mathbb{E}[\mathbf{m}(\boldsymbol{\theta})]$ in this case.

The value of $\mathbb{E}[\mathbf{m}_1(\boldsymbol{\theta})]$ follows directly from the unconditional mean of a control variable $\mathbf{y}_t$ and its covariance matrix. For $\mathbb{E}\left[\mathbf{m}_2(\boldsymbol{\theta})^{Cov}\right]$, we first observe for the kth element of $\mathbf{m}_2(\boldsymbol{\theta})^{Cov}$ that

$$
\begin{aligned}
\mathbb{E}\left[m_2(\boldsymbol{\theta})_k^{Cov}\right] &= \mathbb{E}\left[\left(r_{t+m}^{(k-m)} - r_t^{(k)}\right)\left(\left(r_t^{(k)} - r_t^{(m)}\right) - \mathbb{E}\left[slope_t^{(k)}\right]\right)\right] \\
&= \mathbb{E}\left[\left(\left(r_{t+m}^{(k-m)} - r_t^{(k)}\right) - \mathbb{E}\left[r_{t+m}^{(k-m)} - r_t^{(k)}\right]\right)\left(slope_t^{(k)} - \mathbb{E}\left[slope_t^{(k)}\right]\right)\right] \\
&= Cov\left[r_{t+m}^{(k-m)} - r_t^{(k)}, slope_t^{(k)}\right] \\
&= Cov\left[r_{t+m}^{(k-m)}, slope_t^{(k)}\right] - Cov\left[r_t^{(k)}, slope_t^{(k)}\right]
\end{aligned}
$$

Hence, $Cov\left[r_t^{(k)}, slope_t^{(k)}\right]$ can be computed directly using the expression for unconditional second moments for control variables, and $Cov\left[r_{t+m}^{(k-m)}, slope_t^{(k)}\right]$ follows from the auto-covariance matrix for control variables. As for $\mathbb{E}\left[\mathbf{m}_2(\boldsymbol{\theta})^{VAR}\right]$ we have

$$\begin{aligned}
\mathbb{E}\left[m_2(\boldsymbol{\theta})_k^{VAR}\right] &= \mathbb{E}\left[\left(r_t^{(k)} - r_t^{(m)}\right)\left(\left(r_t^{(k)} - r_t^{(m)}\right) - \mathbb{E}\left[slope_t^{(k)}\right]\right)\right] \\
&= \mathbb{E}\left[\left(\left(r_t^{(k)} - r_t^{(m)}\right) - \mathbb{E}\left[slope_t^{(k)}\right]\right)\left(\left(r_t^{(k)} - r_t^{(m)}\right) - \mathbb{E}\left[slope_t^{(k)}\right]\right)\right] \\
&= Var\left[\left(r_t^{(k)} - r_t^{(m)}\right)\right],
\end{aligned}$$

which also follows directly using the expression for unconditional second moments for control variables when applied to the slope of the yield curve.

4.4 The Link Between Campbell-Shiller and Return Regressions

We first introduce some notation. Let $B_t^{(k)}$ denote the price of a zero-coupon bond with maturity k at time t. The m-period holding period return on a k-period bond is

$$\begin{aligned}
\widetilde{hpr}_{t+m}^{(k)} &\equiv \log\left(\frac{B_{t+m}^{(k-m)}}{B_t^{(k)}}\right) \\
&= \log B_{t+m}^{(k-m)} - \log B_t^{(k)} \\
&= -(k-m)\, r_{t+m}^{(k-m)} + k r_t^{(k)},
\end{aligned}$$

because $r_{t,}^{(k)} = -\frac{1}{k}\log B_t^{(k)}$. One period in our model corresponds to one quarter, so it is natural to express everything in quarterly returns. That is, we get quarterly returns as

$$hpr_{t+m}^{(k)} = -\frac{k-m}{4} r_{t+m}^{(k-m)} + \frac{k}{4} r_t^{(k)}$$

The Campbell-Shiller regression is given by

$$r_{t+m}^{(k-m)} - r_t^{(k)} = \alpha_k + \beta_k \frac{m}{k-m}\left(r_t^{(k)} - r_t^{(m)}\right) + u_{t+m,k},$$

The classic return regression is given by

$$hpr_{t+m}^{(k)} - \frac{m}{4} r_t^{(m)} = \tilde{\alpha}_k + \tilde{\beta}_k\left(r_t^{(k)} - r_t^{(m)}\right) + \varepsilon_{t+m,k}$$

To see the link between these two regressions, consider the quarterly holding period returns

$$hpr_{t+m}^{(k)} = -\frac{k-m}{4} r_{t+m}^{(k-m)} + \frac{k}{4} r_t^{(k)}.$$

Now subtract $\frac{m}{4} r_t^{(m)}$ on both sides to obtain

$$hpr_{t+m}^{(k)} - \frac{m}{4} r_t^{(m)} = -\frac{k-m}{4} r_{t+m}^{(k-m)} + \frac{k}{4} r_t^{(k)} - \frac{m}{4} r_t^{(m)}.$$

Then add and subtract $\frac{m}{4} r_t^{(k)}$ on the right hand side

$$\begin{aligned}
hpr_{t+m}^{(k)} - \frac{m}{4} r_t^{(m)} &= -\frac{k-m}{4} r_{t+m}^{(k-m)} + \frac{k}{4} r_t^{(k)} - \frac{m}{4} r_t^{(m)} + \frac{m}{4} r_t^{(k)} - \frac{m}{4} r_t^{(k)} \\
&= -\frac{k-m}{4} r_{t+m}^{(k-m)} + \frac{k}{4} r_t^{(k)} + \frac{m}{4}\left(r_t^{(k)} - r_t^{(m)}\right) - \frac{m}{4} r_t^{(k)} \\
&= -\frac{k-m}{4} r_{t+m}^{(k-m)} + \frac{k-m}{4} r_t^{(k)} + \frac{m}{4}\left(r_t^{(k)} - r_t^{(m)}\right) \\
&= -\frac{k-m}{4}\left(r_{t+m}^{(k-m)} - r_t^{(k)}\right) + \frac{m}{4}\left(r_t^{(k)} - r_t^{(m)}\right).
\end{aligned}$$

Multiplying both sides by $\frac{4}{k-m}$ and rearranging, we get

$$\frac{4}{k-m}\left(hpr_{t+m}^{(k)}-\frac{m}{4}r_t^{(m)}\right)=\frac{4}{k-m}\left(-\frac{k-m}{4}\left(r_{t+m}^{(k-m)}-r_t^{(k)}\right)+\frac{m}{4}\left(r_t^{(k)}-r_t^{(m)}\right)\right)$$

$\Updownarrow$

$$\frac{4}{k-m}\left(hpr_{t+m}^{(k)}-\frac{m}{4}r_t^{(m)}\right)=\left(-\frac{k-m}{k-m}\left(r_{t+m}^{(k-m)}-r_t^{(k)}\right)+\frac{m}{k-m}\left(r_t^{(k)}-r_t^{(m)}\right)\right)$$

$\Updownarrow$

$$\frac{4}{k-m}\left(hpr_{t+m}^{(k)}-\frac{m}{4}r_t^{(m)}\right)=\frac{m}{k-m}\left(r_t^{(k)}-r_t^{(m)}\right)-\left(r_{t+m}^{(k-m)}-r_t^{(k)}\right). \tag{17}$$

Regressing each of the three terms in (17) on a constant and $\frac{m}{k-m}\left(r_t^{(k)}-r_t^{(m)}\right)$ we get:

i) $\left(hpr_{t+m}^{(k)}-\frac{m}{4}r_t^{(m)}\right)=\tilde{\alpha}_k+\left(\tilde{\beta}_k\frac{k-m}{m}\right)\frac{m}{k-m}\left(r_t^{(k)}-r_t^{(m)}\right)$

ii) $\left(r_t^{(k)}-r_t^{(m)}\right)=0+\left(1\frac{k-m}{m}\right)\times\frac{m}{k-m}\left(r_t^{(k)}-r_t^{(m)}\right)$

iii) $\left(r_{t+m}^{(k-m)}-r_t^{(k)}\right)=\alpha_k+\beta_k\frac{m}{k-m}\left(r_t^{(k)}-r_t^{(m)}\right)$

Thus, for the constants, we simply insert the intercepts from i) to iii) into (17). That is

$$\frac{4}{k-m}\times\text{intercept from i)}=\frac{m}{k-m}\times\text{intercept from ii)}\ -\text{intercept from iii)}$$

$\Updownarrow$

$$\frac{4}{k-m}\times\tilde{\alpha}_k=\frac{m}{k-m}\times 0-\alpha_k$$

$\Updownarrow$

$$\tilde{\alpha}_k=-\alpha_k\frac{k-m}{4}$$

For the loadings, we get

$$\frac{4}{k-m}\times\text{slope from i)}=\frac{m}{k-m}\times\text{slope from ii)}\ -\text{slope from iii)}$$

$$\frac{4}{k-m}\tilde{\beta}_k\frac{k-m}{m}=\frac{m}{k-m}\times\frac{k-m}{m}-\beta_k$$

$\Updownarrow$

$$\frac{4}{m}\tilde{\beta}_k=1-\beta_k$$

Thus the two regressions contain the same information.

For the case of $m=4$, we thus have the simple mappings $\tilde{\alpha}_k=-\alpha_k\frac{k-4}{4}=-\alpha_k\left(\frac{k}{4}-1\right)$ and $\tilde{\beta}_k=1-\beta_k$.

4.5 Asymptotic Distribution

The asymptotic analysis is carried out based on the assumption that any filtering errors caused solely due to the adopted integration approximation in the CDKF are small and none essential for the estimates. The validity of this assumption is explored in the Monte Carlo study presented in this Online Appendix. Given this assumption, the standard QML estimator based on the CDKF is consistent (see Andreasen (2013)). We also have that GMM is consistent, given standard regularity conditions (see Hansen (1982)), and therefore, the proposed estimator with shrinkage is also consistent with respect to $\boldsymbol{\theta}$.

To derive the asymptotic distribution of $\hat{\boldsymbol{\theta}}$ for $T \longrightarrow \infty$, let

$$\mathcal{L} = \frac{1}{T}\sum_{t=1}^{T} \mathcal{L}_t^{CDKF} - \lambda Q,$$

and note that

$$\begin{aligned}\underset{n_\theta \times 1}{\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}}} &= \frac{1}{T}\sum_{t=1}^{T} \underset{n_\theta \times 1}{\frac{\partial \mathcal{L}_t^{CDKF}}{\partial \boldsymbol{\theta}}} - 2\lambda \left(\underset{n_m \times n_\theta}{\frac{\mathbb{E}\left[\mathbf{g}_{1:T}(\boldsymbol{\theta})\right]}{\partial \boldsymbol{\theta}}} \right)' \underset{n_m \times n_m}{\mathbf{W}} \underset{n_m \times 1}{\mathbf{g}_{1:T}(\boldsymbol{\theta})} \\ &= \frac{1}{T}\sum_{t=1}^{T} \left(\mathbf{s}_t(\boldsymbol{\theta}) - 2\lambda \mathbf{G}(\boldsymbol{\theta})' \mathbf{W} \mathbf{g}_t(\boldsymbol{\theta}) \right)\end{aligned}$$

where $\mathbf{s}_t(\boldsymbol{\theta}) \equiv \frac{\partial \mathcal{L}_t^{CDKF}}{\partial \boldsymbol{\theta}}$ is the score function for observation t related to the Central Difference Kalman filter and $\mathbf{G}(\boldsymbol{\theta}) \equiv \frac{\partial \mathbf{g}_{1:T}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} = \frac{1}{T}\sum_{t=1}^{T} \frac{\mathbf{g}_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'}$ is the Jacobian related to the shrinkage moments. The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} = \frac{1}{T}\sum_{t=1}^{T} \left(\mathbf{s}_t\left(\hat{\boldsymbol{\theta}}\right) - 2\lambda \mathbf{G}\left(\hat{\boldsymbol{\theta}}\right)' \mathbf{W} \mathbf{g}_t\left(\hat{\boldsymbol{\theta}}\right) \right) = 0$$

A mean-value expansion of $\mathbf{s}_t$ and $\mathbf{g}_t$ around the true value $\boldsymbol{\theta}_o$ gives

$$\frac{1}{T}\sum_{t=1}^{T} \left\{ \mathbf{s}_t(\boldsymbol{\theta}_o) + \tilde{\mathbf{H}}_t \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o\right) \right\} - 2\lambda \mathbf{G}\left(\hat{\boldsymbol{\theta}}\right)' \mathbf{W} \left\{ \mathbf{g}_{1:T}(\boldsymbol{\theta}_o) + \tilde{\mathbf{G}} \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o\right) \right\} = \mathbf{0}.$$

Here, $\tilde{\mathbf{H}}_t$ is the $n_\theta \times n_\theta$ Hessian matrix of observation t related to the Kalman filtering. The tilde on the Hessian matrix indicates that each row of $\tilde{\mathbf{H}}_t$ is evaluated at a different mean value, which is on the line segment between $\boldsymbol{\theta}_o$ and $\hat{\boldsymbol{\theta}}$. Similarly, $\tilde{\mathbf{G}}$ is the Jacobian related to the shrinkage moments where each row of $\tilde{\mathbf{G}}$ is evaluated at a different mean value, which is on the line segment between $\boldsymbol{\theta}_o$ and $\hat{\boldsymbol{\theta}}$. Thus, we get

$$\frac{1}{T}\sum_{t=1}^{T} \left(\mathbf{s}_t(\boldsymbol{\theta}_o) + \tilde{\mathbf{H}}_t \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o\right) \right) - 2\lambda \mathbf{G}\left(\hat{\boldsymbol{\theta}}\right)' \mathbf{W} \left\{ \mathbf{g}_{1:T}(\boldsymbol{\theta}_o) + \tilde{\mathbf{G}} \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o\right) \right\} = \mathbf{0}$$

$\Updownarrow$

$$\frac{1}{T}\sum_{t=1}^{T} \mathbf{s}_t(\boldsymbol{\theta}_o) - 2\lambda \mathbf{G}\left(\hat{\boldsymbol{\theta}}\right)' \mathbf{W} \left(\frac{1}{T}\sum_{t=1}^{T} \mathbf{g}_t(\boldsymbol{\theta}_o) \right) + \left(\frac{1}{T}\sum_{t=1}^{T} \tilde{\mathbf{H}}_t - 2\lambda \mathbf{G}\left(\hat{\boldsymbol{\theta}}\right)' \mathbf{W} \tilde{\mathbf{G}} \right) \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o\right) = \mathbf{0}$$

because $\mathbf{g}_{1:T}(\boldsymbol{\theta}) \equiv \frac{1}{T}\sum_{t=1}^{T} \mathbf{g}_t(\boldsymbol{\theta})$. Thus

$$\frac{1}{T}\sum_{t=1}^{T} \left(-\mathbf{s}_t(\boldsymbol{\theta}_o) + 2\lambda \mathbf{G}\left(\hat{\boldsymbol{\theta}}\right)' \mathbf{W} \mathbf{g}_t(\boldsymbol{\theta}_o) \right) = \left(\frac{1}{T}\sum_{t=1}^{T} \tilde{\mathbf{H}}_t - 2\lambda \mathbf{G}\left(\hat{\boldsymbol{\theta}}\right)' \mathbf{W} \tilde{\mathbf{G}} \right) \left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o\right)$$

$\Updownarrow$

$$\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o\right) = \left(\frac{1}{T}\sum_{t=1}^{T} \tilde{\mathbf{H}}_t - 2\lambda \mathbf{G}\left(\hat{\boldsymbol{\theta}}\right)' \mathbf{W} \tilde{\mathbf{G}} \right)^{-1} \frac{1}{T}\sum_{t=1}^{T} \left(-\mathbf{s}_t(\boldsymbol{\theta}_o)' + 2\lambda \mathbf{G}\left(\hat{\boldsymbol{\theta}}\right)' \mathbf{W} \mathbf{g}_t(\boldsymbol{\theta}_o) \right)$$

For sufficiently large T, we have

$$\frac{1}{T}\sum_{t=1}^{T} \tilde{\mathbf{H}}_t - 2\lambda \mathbf{G}\left(\hat{\boldsymbol{\theta}}\right)' \mathbf{W} \tilde{\mathbf{G}} \xrightarrow{p} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} \mathbf{H}_t(\boldsymbol{\theta}_o) - 2\lambda \mathbf{G}(\boldsymbol{\theta}_o)' \mathbf{W} \mathbf{G}(\boldsymbol{\theta}_o) \right] \equiv \mathbf{A}_o.$$

Also, let $\mathbf{q}_t(\boldsymbol{\theta}) \equiv -\mathbf{s}_t(\boldsymbol{\theta}) + 2\lambda \mathbf{G}(\boldsymbol{\theta})' \mathbf{W} \mathbf{g_t}(\boldsymbol{\theta})$, then given sufficient regularity conditions, we have that $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \mathbf{q}_t(\boldsymbol{\theta})$ converges to a multivariate normal distribution for $T \longrightarrow \infty$ (see for instance Hansen (1982)). That is,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \mathbf{q}_t\left(\hat{\boldsymbol{\theta}}\right) \xrightarrow{d} \mathcal{N}\left(\mathbb{E}\left[\mathbf{q}_t(\boldsymbol{\theta}_o)\right], Var\left(\mathbf{q}_t(\boldsymbol{\theta}_o)\right)\right)$$

where $\mathbb{E}\left[\mathbf{q}_t(\boldsymbol{\theta}_o)\right] = \mathbf{0}$. To realize that $\mathbb{E}\left[\mathbf{q}_t(\boldsymbol{\theta}_o)\right] = \mathbf{0}$ recall that $\boldsymbol{\theta}_o$ solves the population problem

$$\underset{\boldsymbol{\theta} \in \boldsymbol{\Theta}}{Max} \;\; \mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} \mathcal{L}_t^{CDKF} - \lambda \mathbf{g}_{1:T}(\boldsymbol{\theta})' \mathbf{W} \mathbf{g}_{1:T}(\boldsymbol{\theta})\right],$$

which implies the first-order conditions

$$\frac{\partial}{\partial \boldsymbol{\theta}} \mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} \mathcal{L}_t^{CDKF} - \lambda \mathbf{g}_{1:T}(\boldsymbol{\theta})' \mathbf{W} \mathbf{g}_{1:T}(\boldsymbol{\theta})\right]\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_o} = \mathbf{0}.$$

Given sufficient regularity conditions such that the derivative and the expectation operator can be interchanged, we have

$$\mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} \left(\mathbf{s}_t(\boldsymbol{\theta}) - 2\lambda \mathbf{G}(\boldsymbol{\theta})' \mathbf{W} \mathbf{g}_t(\boldsymbol{\theta})\right)\right]\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_o} = \mathbf{0}.$$

$\Updownarrow$

$$\mathbb{E}\left[\mathbf{q}_t(\boldsymbol{\theta})\right]\big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_o} = \mathbf{0}$$

as desired. Thus, given standard regularity conditions, as stated in Hayashi (2000), we have

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o\right) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \mathbf{A}_o^{-1} Var\left(\mathbf{q}_t(\boldsymbol{\theta}_o)\right) \mathbf{A}_o^{-1}\right)$$

where $\mathbf{q}_t(\boldsymbol{\theta}) \equiv -\mathbf{s}_t(\boldsymbol{\theta}) + 2\lambda \mathbf{G}(\boldsymbol{\theta})' \mathbf{W} \mathbf{g}_t(\boldsymbol{\theta})$. Thus, the asymptotic covariance matric is given by

$$Var\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o\right) = \frac{1}{T} \mathbf{A}_o^{-1} Var\left(\mathbf{q}_t(\boldsymbol{\theta}_o)\right) \mathbf{A}_o^{-1}.$$

We can consistently estimate $\mathbf{A}_o$ by

$$\hat{\mathbf{A}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{H}_t\left(\hat{\boldsymbol{\theta}}\right) - 2\lambda \mathbf{G}\left(\hat{\boldsymbol{\theta}}\right)' \mathbf{W} \mathbf{G}\left(\hat{\boldsymbol{\theta}}\right)$$

As for $Var\left(\mathbf{q}_t(\boldsymbol{\theta}_o)\right)$, we can use standard estimators to account for autocorrelation and heteroskedasticity in a time series, which is generally needed because of autocorrelation and heteroskedasticity in $\mathbf{g}_t(\boldsymbol{\theta}_o)$. We use the estimator of Newey & West (1987), i.e.

$$\widehat{Var}\left(\mathbf{q}_t\left(\hat{\boldsymbol{\theta}}\right)\right) = \hat{\mathbf{\Gamma}}_0 + \sum_{\nu=1}^{k} \left(\hat{\mathbf{\Gamma}}_\nu + \hat{\mathbf{\Gamma}}'_\nu\right)$$

where

$$\hat{\mathbf{\Gamma}}_\nu = \frac{1}{T} \sum_{t=\nu+1}^{T} \left(\mathbf{q}_t\left(\hat{\boldsymbol{\theta}}\right) - \bar{\mathbf{q}}_t\left(\hat{\boldsymbol{\theta}}\right)\right)\left(\mathbf{q}_{t-\nu}\left(\hat{\boldsymbol{\theta}}\right) - \bar{\mathbf{q}}_t\left(\hat{\boldsymbol{\theta}}\right)\right)'$$

and k is a tuning parameter.

In terms of the specific value of λ, we suggest simply to let $\lambda = T$, i.e. the sample size.

4.6 Alternative Interpretation of the Proposed Estimator

One may alternatively consider the proposed estimator as belonging to the class of Laplace type estimators (LTE) or quasi-Bayesian estimators in Chernozhukov & Hong (2003) with the endogenours prior specification of Christiano, Trabandt & Walentin (2011). The use of LTE implies that a potentially misspecified log-likelihood function (as considered in our case) may be used within a Bayesian setting, whereas such a misspecification is generally not accommodated within a standard Bayesian framework.

To realize this, let $L_T(\boldsymbol{\theta}) \equiv \frac{1}{T}\sum_{t=1}^{T}\mathcal{L}_t^{CDKF}$ be the considered extremum statistic and let the priors be denoted by $\pi(\boldsymbol{\theta})$. The quasi-posterior density $p_T(\boldsymbol{\theta})$ for the estimator in Chernozhukov & Hong (2003) is proportional to

$$p_T(\boldsymbol{\theta}) \propto \exp\{L_T(\boldsymbol{\theta})\}\,\pi(\boldsymbol{\theta})$$

$\Updownarrow$

$$\log p_T(\boldsymbol{\theta}) \propto L_T(\boldsymbol{\theta}) + \log\pi(\boldsymbol{\theta}).$$

For the priors we use the endogenours prior specifiction in Christiano et al. (2011) based on sample moments $\mathbf{m}_t$. For a sufficiently large pre-sample size T, it follows that the density of the empirical sample moments $\mathbf{m}_T = \frac{1}{T}\sum_{t=1}^{T}\mathbf{m}_t$ is given by (see Appendix B (page 38) related to Christiano et al. (2011))

$$p(\mathbf{m}_T|\boldsymbol{\theta}) = \left(\frac{T}{2\pi}\right)^{T/2}\left|\hat{\mathbf{S}}\right|^{-1/2}\exp\left\{-T/2\left(\mathbf{m}_T - \mathbb{E}\left[\mathbf{m}(\boldsymbol{\theta})\right]\right)'\hat{\mathbf{S}}^{-1}\left(\mathbf{m}_T - \mathbb{E}\left[\mathbf{m}(\boldsymbol{\theta})\right]\right)\right\},$$

where $\hat{\mathbf{S}}$ is the estimated co-variance matrix of the sample moments. Letting $p(\boldsymbol{\theta})$ denote the primitive priors before observing $\mathbf{m}_T$, then

$$\pi(\boldsymbol{\theta}) = p(\mathbf{m}_T|\boldsymbol{\theta})\,p(\boldsymbol{\theta}),$$

or simply

$$\pi(\boldsymbol{\theta}) = p(\mathbf{m}_T|\boldsymbol{\theta}),$$

when $p(\boldsymbol{\theta})$ is set to flat priors. Accordingly

$$\begin{aligned}\log p_T(\boldsymbol{\theta}) &\propto L_T(\boldsymbol{\theta}) + \log\pi(\boldsymbol{\theta})\\ &\propto L_T(\boldsymbol{\theta}) - T/2\left(\mathbf{m}_T - \mathbb{E}\left[\mathbf{m}(\boldsymbol{\theta})\right]\right)'\hat{\mathbf{S}}^{-1}\left(\mathbf{m}_T - \mathbb{E}\left[\mathbf{m}(\boldsymbol{\theta})\right]\right)\\ &= L_T(\boldsymbol{\theta}) - T/2\times Q\end{aligned}$$

when $\mathbf{W} = \hat{\mathbf{S}}^{-1}$, which is identical to (16) when $\lambda = T/2$.

4.7 Monte Carlo Evidence

This section presents a small simulation study to illustrate the benefit of introducing shrinkage when the New Keynesian model is misspecified. To make the simulation study computational feasible, we consider a reduced version of the New Keynesian model. That is, we omit preference shocks, labor supply shocks, and shocks to the inflation target. Also, we omit consumption habits ($b = 0$) and we do not include wage stickiness ($\kappa_w = 0$), and therefore we also omit wages as an observable in the estimation. The model is solved by third-order perturbation, as the issue related to the accuracy of the solution is not essential in this context. That is, the data generating process is given by a third-order perturbation approximation, which is also used for the estimation. We estimate the parameters listed below, while the remaining paramters take the values $a = -100$, $l_{ss} = 0.33$, $\beta = 0.9925$, $\phi = 0.075$, $\eta = 6$, $(K/Y)_{ss} = 2.5$, $\delta = 0.025$, $\mu_{z,ss} = 1.0055$. We simulate 1,000 samples, each with $T = 250$ observations and estimate the model using the proposed estimation for various values of λ. We study the performance of QML (i.e., $\lambda = 0$) and shrinkage towards the first and second unconditional moments as described in the paper.

Without any misspecification in Panel A in Table 1, we see that QML (i.e., $\lambda = 0$) gives nearly unbiased estimates with low degree of variability as measured by the standard deviation of the sampling distribution. Letting $\lambda = T$ or $\lambda = 10^6$ only worsen the performance of the estimator, as it increases the biases and generates less efficient estimates.

Panel B in Table 1 considers the case, where the data generating model is solved using the true structural parameters (to get the **g**- and the **h**-functions), but when simulating data we use the value $\rho_a = 0.8$ for the persistence in the stationary technology shocks. That is, we introduce a clear misspecification in the model. To ensure that the unconditional variance of the stationary technology shocks are unaffected by this misspecification, we let $\sigma_a = \sqrt{\frac{0.01^2}{1-0.98^2}\left(1-0.80^2\right)}$. Hence, for this case with misspecification, there is no true value of ρ_a and σ_a, and this explains why we do not report their estimates in Panel B. We then estimate the model without accounting for this misspecification, i.e., with the same value of ρ_a being used to solve and simulate the model. Panel B shows that QML with shrinkage ($\lambda = T$) gives parameter estimates that are less biased when compared to QML without shrinkage (i.e., $\lambda = 0$). This is seen clearly from the overall root measure squared biases

$$RMSB^{\theta} = \sqrt{\frac{1}{n_\theta}\sum_{i=1}^{n_\theta}\left(\theta_i - \bar{\theta}_i\right)^2},$$

where θ_i is the true value of the structural parameter and $\bar{\theta}_i$ is the mean estimate of the ith parameter in the Monte Carlo study. Here, n_θ denotes the number of estimated structural parameters. We find $RMSB^{\theta} = 0.116$ with $\lambda = 0$ but only $RMSB^{\theta} = 0.087$ with $\lambda = T$. The cost of robustifying the QML estimator in this way is that we find less efficient estimate with $\lambda = T$ compared to $\lambda = 0$.

We benchmark these results to using an extreme degree of shrinkage with $\lambda = 10^6$, which corresponds to estimating the structural parameters by GMM and obtaining the states afterwards by the CDKF. The Monte Carlo study shows that these GMM estimates display notable biases in finite samples and are clearly less efficient compared to the standard QML estimator ($\lambda = 0$) and the proposed estimator (with $\lambda = T$), both with and without model misspecification. These imprecise GMM estimates of the structural parameters also imply less accurate state estimates when compared to the proposed estimator with $\lambda = T$. This is seen from the following measure related to the estimated states

$$RMSE^{States} = \frac{1}{1,000}\sum_{s=1}^{1,000}\sqrt{\frac{1}{n_x}\sum_{t=1}^{T}\sum_{i=1}^{n_x}\left(x_{i,t}^{(s)} - \hat{x}_{i,t}^{(s)}\right)^2},$$

which is $RMSE^{States} = 0.0223$ with $\lambda = 10^6$ but only $RMSE^{States} = 0.0159$ with $\lambda = T$.

5 Projection Approximation

The considered second order projection approximation reads (unpruned)

$$\mathbf{y}_t = \mathbf{g}_0 + \mathbf{g}_\mathbf{x}\mathbf{x}_t + \mathbf{G}_{\mathbf{xx}}\left(\mathbf{x}_t \otimes \mathbf{x}_t\right)$$

$$\mathbf{x}_{t+1} = \mathbf{h}_0 + \mathbf{h}_\mathbf{x}\mathbf{x}_t + \mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t \otimes \mathbf{x}_t\right) + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

where $\mathbf{G}_{\mathbf{xx}} \equiv reshape\left(\mathbf{g}_{\mathbf{xx}}, n_y, n_x^2\right)$ and $\mathbf{H}_{\mathbf{xx}} \equiv reshape\left(\mathbf{h}_{\mathbf{xx}}, n_x, n_x^2\right)$, whereas the pruned version reads

$$\mathbf{y}_t = \mathbf{g}_0 + \mathbf{g}_\mathbf{x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \mathbf{G}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)$$

$$\mathbf{x}_{t+1}^f = \mathbf{h}_0 + \mathbf{h}_\mathbf{x}\mathbf{x}_t^f + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

$$\mathbf{x}_{t+1}^s = \mathbf{h}_\mathbf{x}\mathbf{x}_t^s + \mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)$$

Thus, the only differences with respect to the pruned state space system for the standard perturbation approximation are i) the presence of $\mathbf{h}_0$ in the law of motion for $\mathbf{x}_t^f$ and ii) the absence of $\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$ in the law of motion for $\mathbf{x}_t^s$. We include $\mathbf{h}_0$ in the law of motion for $\mathbf{x}_t^f$ because it is presence even in a first-order projection solution as it captures a risk-adjustment. Thus, to get the closed-form moments for the pruned projection solution, all we need to do is to extend the results in Andreasen et al. (2018) to account for $\mathbf{h}_0$. This is done below.

For the pruned version, we can show stability and compute the first and second unconditional moments.

Table 1: A Monte Carlo Study

This table reports the biases and standard deviation for the estimated parameters in a simulation study using 1,000 replications and $T = 250$.

		Parameter bias			Standard deviation		
	True value	$\lambda = 0$	$\lambda = T$	$\lambda = 10^6$	$\lambda = 0$	$\lambda = T$	$\lambda = 10^6$
A: No Misspecification							
χ	2.000	0.005	0.076	-0.009	0.102	0.230	0.226
ξ_{Calvo}	0.750	0.010	0.030	0.013	0.040	0.053	0.057
ϕ_π	1.250	0.026	0.092	0.128	0.073	0.147	0.147
$\phi_{\Delta c}$	0.250	-0.001	0.019	0.053	0.045	0.143	0.145
π_{ss}	1.005	0.000	0.001	-0.001	0.002	0.005	0.004
ρ_{μ_z}	0.500	0.051	-0.141	-0.271	0.106	0.191	0.212
σ_{μ_z}	0.001	0.000	0.001	0.000	0.000	0.001	0.001
ρ_a	0.980	-0.001	0.005	0.002	0.004	0.006	0.005
σ_a	0.010	0.000	-0.001	-0.001	0.001	0.001	0.001
RMSB^θ	-	0.041	0.212	0.214	-	-	-
RMSE^{States}	-	0.0066	0.0269	0.0327	-	-	-
B: With misspecification							
χ	2.000	-0.339	-0.084	-0.181	0.347	0.192	0.132
ξ_{Calvo}	0.750	-0.010	-0.079	-0.043	0.079	0.193	0.141
ϕ_π	1.250	0.004	0.175	0.316	0.100	0.217	0.287
$\phi_{\Delta c}$	0.250	-0.015	0.023	0.148	0.091	0.184	0.279
π_{ss}	1.005	0.000	-0.001	-0.002	0.004	0.002	0.002
ρ_{μ_z}	0.500	0.120	0.035	-0.239	0.151	0.213	0.313
σ_{μ_z}	0.001	0.000	0.000	0.000	0.000	0.001	0.001
RMSB^θ	-	0.116	0.087	0.350	-	-	-
RMSE^{States}	-	0.0346	0.0159	0.0223	-	-	-

5.1 Covariance-stationary

Proposition 1:

The pruned second-order approximation for $\mathbf{x}_t^f, \mathbf{x}_t^s$, and $\mathbf{y}_t^s$ is covariance-stationary if

1. If all eigenvalue of $\mathbf{h_x}$ have modulus less than one
2. $\boldsymbol{\epsilon}_{t+1}$ has finite fourth moment

Proof

We now form the extended state vector

$$\mathbf{z}_t \equiv \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}$$

We know the law of motion for $\mathbf{x}_t^f$ and $\mathbf{x}_t^s$, so we only need to find the law of motion for $\mathbf{x}_t^f \otimes \mathbf{x}_t^f$. Hence consider

$$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = \left(\mathbf{h}_0 + \mathbf{h_x}\mathbf{x}_t^f + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \left(\mathbf{h}_0 + \mathbf{h_x}\mathbf{x}_t^f + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)$$

$$= \mathbf{h}_0 \otimes \left(\mathbf{h}_0 + \mathbf{h_x}\mathbf{x}_t^f + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)$$
$$+ \mathbf{h_x}\mathbf{x}_t^f \otimes \left(\mathbf{h}_0 + \mathbf{h_x}\mathbf{x}_t^f + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)$$

$+\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \left(\mathbf{h}_0 + \mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)$

using $(\mathbf{A}+\mathbf{B}) \otimes (\mathbf{C}+\mathbf{D}) = \mathbf{A}\otimes\mathbf{C} + \mathbf{A}\otimes\mathbf{D} + \mathbf{B}\otimes\mathbf{C} + \mathbf{B}\otimes\mathbf{D}$

$$
\begin{aligned}
&= \mathbf{h}_0 \otimes \mathbf{h}_0 + \mathbf{h}_0 \otimes \mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f + \mathbf{h}_0 \otimes \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\
&+\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f \otimes \mathbf{h}_0 + \mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f \otimes \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\
&+\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_0 + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbf{h}_0 \otimes \mathbf{h}_0 + (\mathbf{h}_0 \otimes \mathbf{h}_{\mathbf{x}})\,\mathbf{x}_t^f + (\mathbf{h}_0 \otimes \boldsymbol{\eta})\,\boldsymbol{\epsilon}_{t+1} \\
&+\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f \otimes \mathbf{h}_0 1 + (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h}_{\mathbf{x}} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \\
&+\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_0 1 + (\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\boldsymbol{\eta} \otimes \boldsymbol{\eta})\,(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})
\end{aligned}
$$

using $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = \mathbf{AC}\otimes\mathbf{BD}$

$$
\begin{aligned}
&= \mathbf{h}_0 \otimes \mathbf{h}_0 + (\mathbf{h}_0 \otimes \mathbf{h}_{\mathbf{x}})\,\mathbf{x}_t^f + (\mathbf{h}_0 \otimes \boldsymbol{\eta})\,\boldsymbol{\epsilon}_{t+1} \\
&+ (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_0)\left(\mathbf{x}_t^f \otimes 1\right) + (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h}_{\mathbf{x}} \otimes \boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \\
&+ (\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{0}})\,(\boldsymbol{\epsilon}_{t+1} \otimes 1) + (\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\boldsymbol{\eta} \otimes \boldsymbol{\eta})\,(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})
\end{aligned}
$$

Note that $E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})\right] = vec\left(\mathbf{I}_{n_e}\right)$. Thus, we get

$$
\begin{aligned}
\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f &= \mathbf{h}_0 \otimes \mathbf{h}_0 + (\mathbf{h}_0 \otimes \mathbf{h}_{\mathbf{x}})\,\mathbf{x}_t^f + (\mathbf{h}_0 \otimes \boldsymbol{\eta})\,\boldsymbol{\epsilon}_{t+1} + (\boldsymbol{\eta} \otimes \boldsymbol{\eta})\,vec\left(\mathbf{I}_{n_e}\right) \\
&+ (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_0)\,\mathbf{x}_t^f + (\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h}_{\mathbf{x}} \otimes \boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \\
&+ (\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{0}})\,\boldsymbol{\epsilon}_{t+1} + (\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\boldsymbol{\eta} \otimes \boldsymbol{\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)
\end{aligned}
$$

Accordingly

$$
\begin{bmatrix} \mathbf{x}_{t+1}^f \\ \mathbf{x}_{t+1}^s \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \end{bmatrix}
=
\begin{bmatrix} \mathbf{h}_{\mathbf{x}} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_x^2} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{h}_{\mathbf{x}} & \mathbf{H}_{\mathbf{xx}} \\ \mathbf{h}_0 \otimes \mathbf{h}_{\mathbf{x}} + \mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_0 & \mathbf{0}_{n_x^2 \times n_x} & \mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}} \end{bmatrix}
\begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}
+
\begin{bmatrix} \mathbf{h}_0 \\ \mathbf{0} \\ \mathbf{h}_0 \otimes \mathbf{h}_0 + (\boldsymbol{\eta} \otimes \boldsymbol{\eta})\,vec\left(\mathbf{I}_{n_e}\right) \end{bmatrix}
$$

$$
+
\begin{bmatrix} \boldsymbol{\eta} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{h}_0 \otimes \boldsymbol{\eta} + \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{0}} & \boldsymbol{\eta} \otimes \boldsymbol{\eta} & \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}} & \mathbf{h}_{\mathbf{x}} \otimes \boldsymbol{\eta} \end{bmatrix}
\begin{bmatrix} \boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\ \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \end{bmatrix}
$$

$\Updownarrow$

$$
\mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t + \mathbf{c} + \mathbf{B}\boldsymbol{\xi}_{t+1} \tag{18}
$$

where $Cov\left(\boldsymbol{\xi}_{t+1}, \boldsymbol{\xi}_{t-s}\right) = \mathbf{0}$ for $s = 1, 2, 3, ...$ because $\boldsymbol{\epsilon}_{t+1}$ is independent across time. This follows from the observation that $\boldsymbol{\xi}_{t+1}$ is identical to the value stated for $\boldsymbol{\xi}_{t+1}$ in Andreasen et al. (2018), which show the claimed result..

The absolute value of the eigenvalues in $\mathbf{h}_{\mathbf{x}}$ are all strictly less then one by assumption. Accordingly, all eigenvalues of $\mathbf{A}$ are also strictly less than one. To see this note first that

$p(\lambda) = \left|\mathbf{A} - \lambda\mathbf{I}_{2n_x + n_x^2}\right|$

$$
= \left|\begin{bmatrix} \mathbf{h}_{\mathbf{x}} - \lambda\mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_x^2} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{h}_{\mathbf{x}} - \lambda\mathbf{I}_{n_x} & \mathbf{H}_{\mathbf{xx}} \\ \mathbf{h}_0 \otimes \mathbf{h}_{\mathbf{x}} + \mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_0 & \mathbf{0}_{n_x^2 \times n_x} & \mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}} - \lambda\mathbf{I}_{n_x^2} \end{bmatrix}\right|
$$

$$= \begin{vmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{vmatrix}$$

where we let

$\mathbf{B}_{11} \equiv \begin{bmatrix} \mathbf{h_x} - \lambda \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_x} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{h_x} - \lambda \mathbf{I}_{n_x} \end{bmatrix}$ which is $2n_x \times 2n_x$

$\mathbf{B}_{12} \equiv \begin{bmatrix} \mathbf{0}_{n_x \times n_x^2} \\ \mathbf{H_{xx}} \end{bmatrix}$ which is $2n_x \times n_x^2$

$\mathbf{B}_{21} \equiv \begin{bmatrix} \mathbf{h}_0 \otimes \mathbf{h_x} + \mathbf{h_x} \otimes \mathbf{h}_0 & \mathbf{0}_{n_x^2 \times n_x} \end{bmatrix}$ which is $n_x^2 \times 2n_x$

$\mathbf{B}_{22} \equiv \mathbf{h_x} \otimes \mathbf{h_x} - \lambda \mathbf{I}_{n_x^2}$ which is $n_x^2 \times n_x^2$

$$= |\mathbf{B}_{11}| \, |\mathbf{B}_{22}|$$

using $\begin{vmatrix} \mathbf{U} & \mathbf{C} \\ \mathbf{0} & \mathbf{Y} \end{vmatrix} = |\mathbf{U}| \, |\mathbf{Y}|$ where $\mathbf{U}$ is $m \times m$ and $\mathbf{Y}$ is $n \times n$

$$= \left| \begin{bmatrix} \mathbf{h_x} - \lambda \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_x} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{h_x} - \lambda \mathbf{I}_{n_x} \end{bmatrix} \right| \left| \mathbf{h_x} \otimes \mathbf{h_x} - \lambda \mathbf{I}_{n_x^2} \right|$$

$$= |\mathbf{h_x} - \lambda \mathbf{I}_{n_x}| \, |\mathbf{h_x} - \lambda \mathbf{I}_{n_x}| \left| \mathbf{h_x} \otimes \mathbf{h_x} - \lambda \mathbf{I}_{n_x^2} \right|$$

Hence, the eigenvalue λ solves the problem

$p(\lambda) = 0$

$\Updownarrow$

$|\mathbf{h_x} - \lambda \mathbf{I}_{n_x}| \, |\mathbf{h_x} - \lambda \mathbf{I}_{n_x}| \left| \mathbf{h_x} \otimes \mathbf{h_x} - \lambda \mathbf{I}_{n_x^2} \right| = 0$

$\Updownarrow$

$|\mathbf{h_x} - \lambda \mathbf{I}_{n_x}| = 0$ or $\left| \mathbf{h_x} \otimes \mathbf{h_x} - \lambda \mathbf{I}_{n_x^2} \right| = 0$

The absolute value of all eigenvalues to the first problem are strictly less than one. That is $|\lambda_i| < 1$ $i = 1, 2, ..., n_x$. This is also the case for the second problem because the eigenvalues to $\mathbf{h_x} \otimes \mathbf{h_x}$ are $\lambda_i \lambda_j$ for $i = 1, 2, ..., n_x$ and $j = 1, 2, ..., n_x$

Thus, the system in (18) is covariance stationary if $\boldsymbol{\xi}_{t+1}$ has finite first and second moment. It follows directly that $E\left[\boldsymbol{\xi}_{t+1}\right] = \mathbf{0}$ and $\boldsymbol{\xi}_{t+1}$ has finite second moments if $\boldsymbol{\epsilon}_{t+1}$ has a finite fourth moment. The latter holds by assumption.

For the control variables we have $\mathbf{y}_t = \mathbf{D}\mathbf{z}_t + \mathbf{g}_0$ where $\mathbf{D} \equiv \begin{bmatrix} \mathbf{g_x} & \mathbf{g_x} & \mathbf{G_{xx}} \end{bmatrix}$. That is $\mathbf{y}_t$ is linear function of $\mathbf{z}_t$ and $\mathbf{y}_t$ is therefore also covariance-stationary.

Q.E.D.

5.2 Formulas for the first and second moments

This section computes first and second unconditional moments using the representation of the second-order system stated above. Note first that

$$E\left[\mathbf{x}_{t+1}^f\right] = \mathbf{h}_0 + \mathbf{h_x} E\left[\mathbf{x}_t^f\right]$$

$\Downarrow$

$$E\left[\mathbf{x}_t^f\right] = (\mathbf{I} - \mathbf{h_x})^{-1} \mathbf{h}_0$$

And

$$Var\left[\mathbf{x}_{t+1}^f\right] = \mathbf{h_x} Var\left[\mathbf{x}_t^f\right] \mathbf{h}_\mathbf{x}' + \boldsymbol{\eta}\boldsymbol{\eta}'$$

$\Updownarrow$

$$vec\left(Var\left[\mathbf{x}_{t+1}^f\right]\right) = (\mathbf{h_x} \otimes \mathbf{h_x}) \, vec\left(Var\left[\mathbf{x}_t^f\right]\right) + vec\left(\boldsymbol{\eta}\boldsymbol{\eta}'\right)$$

$\Downarrow$

$$vec\left(Var\left[\mathbf{x}_t^f\right]\right) = \left(\mathbf{I} - \mathbf{h_x} \otimes \mathbf{h_x}\right)^{-1} vec\left(\boldsymbol{\eta}\boldsymbol{\eta}'\right)$$

Hence,

$$\begin{aligned}
Var\left[\mathbf{x}_t^f\right] &= E\left[\left(\mathbf{x}_t^f - E\left[\mathbf{x}_t^f\right]\right)\left(\mathbf{x}_t^f - E\left[\mathbf{x}_t^f\right]\right)'\right] \\
&= E\left[\left(\mathbf{x}_t^f - E\left[\mathbf{x}_t^f\right]\right)\left(\mathbf{x}_t^f\right)'\right] \\
&= E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right] - E\left[\mathbf{x}_t^f\right]E\left[\left(\mathbf{x}_t^f\right)'\right]
\end{aligned}$$

$\Updownarrow$

$$E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right] = Var\left[\mathbf{x}_t^f\right] + E\left[\mathbf{x}_t^f\right]E\left[\left(\mathbf{x}_t^f\right)'\right].$$

The system

$$\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}$$
$$\mathbf{y}_t = \mathbf{D}\mathbf{z}_t + \mathbf{g}_0$$

The mean values are

$$E\left[\mathbf{z}_t\right] = \left(\mathbf{I}_{2n_x+n_x^2} - \mathbf{A}\right)^{-1}\mathbf{c}.$$
$$E\left[\mathbf{y}_t\right] = \mathbf{D}E\left[\mathbf{z}_t\right] + \mathbf{g}_0$$

Then note that $E\left[\mathbf{x}_t^f\right]$ is at the top of $E\left[\mathbf{z}_t\right]$ and $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]$ is at the bottom of $E\left[\mathbf{z}_t\right]$, which is an easy way to get these moments.

For the variances we first have that

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] = E\left[\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)'\right]$$

$$= E\left[\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$

$$\begin{aligned}
&= E\left[\mathbf{c}\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right] \\
&+ E\left[\mathbf{A}\mathbf{z}_t\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right] \\
&+ E\left[\mathbf{B}\boldsymbol{\xi}_{t+1}\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= E\left[\mathbf{c}\mathbf{c}' + \mathbf{c}\mathbf{z}_t'\mathbf{A}' + \mathbf{c}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right] \\
&+ E\left[\mathbf{A}\mathbf{z}_t\mathbf{c}' + \mathbf{A}\mathbf{z}_t\mathbf{z}_t'\mathbf{A}' + \mathbf{A}\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right] \\
&+ E\left[\mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{c}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\mathbf{A}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{c}\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' \\
&+ \mathbf{A}E\left[\mathbf{z}_t\right]\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}' \\
&+ \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'
\end{aligned}$$

We then note that

$$E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right] = E\left[\begin{bmatrix}\mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f\end{bmatrix}\begin{bmatrix}\boldsymbol{\epsilon}_{t+1}' & \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{ne}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\end{bmatrix}\right]$$

$$= E\begin{bmatrix} \mathbf{x}_t^f \boldsymbol{\epsilon}_{t+1}' & \mathbf{x}_t^f \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{ne}\right)\right)' & \mathbf{x}_t^f \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' & \mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' \\ \mathbf{x}_t^s \boldsymbol{\epsilon}_{t+1}' & \mathbf{x}_t^s \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{ne}\right)\right)' & \mathbf{x}_t^s \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' & \mathbf{x}_t^s \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' \\ \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \boldsymbol{\epsilon}_{t+1}' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{ne}\right)\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Thus

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] = \mathbf{c}\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\right]\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

$$= \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

Note also that

$$E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t\right]' = \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)'$$

$$= \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

$$= \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

So

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t\right]' = \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}' - \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' - \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' - \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

$$= \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}' - \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

$$= \mathbf{A}\left(E\left[\mathbf{z}_t\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\right)\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

$$\Updownarrow$$

$$Var\left(\mathbf{z}_{t+1}\right) = \mathbf{A}Var\left(\mathbf{z}_t\right)\mathbf{A}' + \mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'$$

$$\Updownarrow$$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right) = vec\left(\mathbf{A}Var\left(\mathbf{z}_t\right)\mathbf{A}'\right) + vec\left(\mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'\right)$$

$$\Updownarrow$$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right) = \left(\mathbf{A} \otimes \mathbf{A}\right)vec\left(Var\left(\mathbf{z}_t\right)\right) + vec\left(\mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'\right)$$

$$\Updownarrow$$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right)\left(\mathbf{I}_{(2n_x+n_x^2)^2} - \left(\mathbf{A} \otimes \mathbf{A}\right)\right) = vec\left(\mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'\right)$$

$$\Updownarrow$$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right) = \left(\mathbf{I}_{(2n_x+n_x^2)^2} - \left(\mathbf{A} \otimes \mathbf{A}\right)\right)^{-1} vec\left(\mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'\right)$$

Hence we only need to compute $Var\left(\boldsymbol{\xi}_{t+1}\right)$. Given that the expression for $\boldsymbol{\xi}_{t+1}$ is identifical to the one for the pruned standard perturbation approximation, the matrix $Var\left(\boldsymbol{\xi}_{t+1}\right)$ can be computed as outlined in Andreasen et al. (2018). For completeness, we reproduce how this is done below.

$$Var\left(\boldsymbol{\xi}_{t+1}\right) = E\left[\left[\begin{array}{c}\boldsymbol{\epsilon}_{t+1}\\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\end{array}\right]\left[\begin{array}{c}\boldsymbol{\epsilon}_{t+1}\\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\end{array}\right]'\right]$$

$$= E\left[\left[\begin{array}{c}\boldsymbol{\epsilon}_{t+1}\\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\end{array}\right]\begin{array}{cccc}\boldsymbol{\epsilon}'_{t+1} & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\end{array}\right]$$

$$= E\left[\begin{array}{cc}\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1} & \boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)'\\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\boldsymbol{\epsilon}'_{t+1} & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)'\\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}'_{t+1} & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)'\\ \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}'_{t+1} & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)'\end{array}\right.$$

$$\left.\begin{array}{cc}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\\ \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\end{array}\right]$$

We next evaluate each of these terms.
The first row:

$E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\right] = \mathbf{I}_{n_\varepsilon}$

$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)'\right] = E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'-\boldsymbol{\epsilon}_{t+1}vec\left(\mathbf{I}_{n_e}\right)'\right]$

$\quad = E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right] = E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes 1\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\right)'\right)\right]$

$\quad = E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\right)'\right]$

$\quad = E\left[E_t\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\right]\otimes\left(\mathbf{x}_t^f\right)'\right]$

$\quad = E\left[\mathbf{I}_{n_\varepsilon}\otimes\left(\mathbf{x}_t^f\right)'\right]$

$\quad = \mathbf{I}_{n_\varepsilon}\otimes E\left[\left(\mathbf{x}_t^f\right)'\right]$

$E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(1\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}'_{t+1}\right)\right]$

$\quad = E\left[\left(\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\right]$

$\quad = E\left[\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_\varepsilon}$

The second row

$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\boldsymbol{\epsilon}'_{t+1}\right] = E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}'_{t+1}\right]$

$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)'\right] = E\left[\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'-vec\left(\mathbf{I}_{n_e}\right)'\right)\right]$

$\quad = E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'-\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)vec\left(\mathbf{I}_{n_e}\right)'-vec\left(\mathbf{I}_{n_e}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'+vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'\right]$

$$= E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)' - vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)' - vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)' + vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'\right]$$
$$= E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)' - vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'\right]$$
$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right] = E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' - vec\left(\mathbf{I}_{n_e}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$$
$$= E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$$
$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

Third row

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1}'\right] = E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes 1\right)\right]$$
$$= E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f\right]$$
$$= \mathbf{I}_{n_\varepsilon}\otimes E\left[\mathbf{x}_t^f\right]$$
$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)'\right] = E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$
$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right] = E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\right)\right]$$
$$= E\left[\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right)\otimes\left(\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right)\right]$$
$$= \mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]$$

using $\left(\mathbf{A}\otimes\mathbf{B}\right)\left(\mathbf{C}\otimes\mathbf{D}\right) = \mathbf{AC}\otimes\mathbf{BD}$

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

Fourth row

$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}'\right] = E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(1\otimes\boldsymbol{\epsilon}_{t+1}'\right)\right] = E\left[\mathbf{x}_t^f\otimes\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right)\right] = E\left[\mathbf{x}_t^f\right]\otimes\mathbf{I}_{n_e}$$
$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)'\right] = E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$
$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right] = E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$$
$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right)\otimes\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right)\right] = E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}$$

Exploiting that all third moments of $\boldsymbol{\epsilon}_{t+1}$ are zero for the normal distribution, we get

$$Var\left(\boldsymbol{\xi}_{t+1}\right) = \begin{bmatrix} \mathbf{I}_{n_e} & \mathbf{0}_{n_\varepsilon\times n_\varepsilon^2} & \mathbf{I}_{n_\varepsilon}\otimes E\left[\left(\mathbf{x}_t^f\right)'\right] & E\left[\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_\varepsilon} \\ \mathbf{0}_{n_\varepsilon^2\times n_\varepsilon} & E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)' - vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'\right] & \mathbf{0}_{n_\varepsilon^2\times n_\varepsilon n_x} & \mathbf{0}_{n_\varepsilon^2\times n_\varepsilon n_x} \\ \mathbf{I}_{n_\varepsilon}\otimes E\left[\mathbf{x}_t^f\right] & \mathbf{0}_{n_\varepsilon n_x\times n_\varepsilon^2} & \mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right] & E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] \\ E\left[\mathbf{x}_t^f\right]\otimes\mathbf{I}_{n_e} & \mathbf{0}_{n_\varepsilon n_x\times n_\varepsilon^2} & E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right] & E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e} \end{bmatrix}$$

All elements in this matrix can be computed (and coded) directly as shown below. The variance of the control variables is then given by

$$Var\left[\mathbf{y}_t^s\right] = \mathbf{D}Var\left[\mathbf{z}_t\right]\mathbf{D}'$$

5.2.1 Computing the variance of the innovations

1) for $E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Hence the quasi MATLAB codes are :

```
E_eps_eps2 = zeros(ne, (ne)^2)
for phi1 = 1 : ne
      index2 = 0
      for phi2 = 1 : ne
            for phi3 = 1 : ne
                  index2 = index2 + 1
                  if (phi1 = phi2 = phi3)
                        E_eps_eps2(phi1, index2) = m^3 (ϵ_{t+1} (phi1))
                  end
            end
      end
end
```

Note also that $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}'\right]=\left(E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\right)'$

2) $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]-vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'$

Here

$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$=E\left[\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\left(\left\{\epsilon_{t+1}\left(\phi_3,1\right)\left\{\epsilon_{t+1}\left(\phi_4,1\right)\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right)'\right]$$

Hence the quasi MATLAB codes are

```
E_eps2_eps2 = zeros(n_e^2, n_e^2)
index1 = 0
for phi1 = 1 : n_e
      for phi2 = 1 : n_e
            index1 = index1 + 1
            index2 = 0
            for phi3 = 1 : n_e
                  for phi4 = 1 : n_e
                        index2 = index2 + 1
                        % second moments
                        if (phi1 == phi2 && phi3 == phi4 && phi1˜ = phi4)
                              E_eps2_eps2(index1, index2) = 1
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1˜ = phi2)
                              E_eps2_eps2(index1, index2) = 1
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1˜ = phi2)
                              E_eps2_eps2(index1, index2) = 1
                        % fourth moments
                        elseif(phi1 == phi2 && phi1 == phi3 && phi1 == phi4)
                              E_eps2_eps2(index1, index2) = m^4 (ϵ_{t+1} (phi1))
                        end
                  end
            end
      end
```

end

3) $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$

Here

$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$

$$= E\left[\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right]$$

Hence the quasi MATLAB codes are

```
E_epsxf_epsxf = zeros(n_e n_x, n_x n_e)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                index2 = index2 + 1
                if phi1 = phi2
                    E_epsxf_epsxf (index1, index2) = E_xf_xf(gama1, gama2)
                end
            end
        end
    end
end
```

where $E\_xf\_xf = reshape(E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right], nx, nx)$

4) $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

Here

$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E\left[\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

Hence the quasi MATLAB codes are

```
E_epsxf_xfeps = zeros(n_e n_x, n_e n_x)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for phi2 = 1 : ne
                index2 = index2 + 1
                if phi1 = phi2
                    E_epsxf_xfeps (index1, index2) = E_xf_xf(gama1, gama2)
                end
            end
```

end
end
end

5) $E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

Here

$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right] = \left[E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]\right]'$

so $E\_xfeps\_epsxf = E\_epsxf\_xfeps'$

6) $E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

Here

$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}'_{t+1} \otimes \left(\mathbf{x}_t^f\right)'\right)\right]$

$$= E\left[\left\{x_t^f\left(\gamma_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\left\{x_t^f\left(\gamma_2, 1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are

$E\_xfeps\_epsxf = zeros(n_x n_e, n_x n_e)$
$index1 = 0$
$for\ gama1 = 1 : nx$
    $for\ phi1 = 1 : ne$
        $index1 = index1 + 1$
        $index2 = 0$
        $for\ phi2 = 1 : ne$
            $for\ gama2 = 1 : nx$
                $index2 = index2 + 1$
                $if\ phi1 = phi2$
                    $E\_xfeps\_epsxf\left(index1, index2\right) = E\_xf\_xf(gama1, gama2)$
                end
            end
        end
    end
end

where $E\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx)$

5.3 The auto-correlations

This section derives the auto-correlations for the states and the control variables.

5.3.1 The innovations

We start by showing that $\boldsymbol{\xi}_{t+1}$ and $\boldsymbol{\xi}_{t+1+s}$ are uncorrelated for $s = 1, 2, ...$ To see this note that

$E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}'_{t+1+s}\right] =$

$$E\left[\left[\begin{array}{c}\boldsymbol{\epsilon}_{t+1}\\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\end{array}\right]\begin{array}{cccc}\boldsymbol{\epsilon}_{t+1+s}' & \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}-vec\left(\mathbf{I}_{n_e}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' & \left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\end{array}\right]$$

$$=E\left[\begin{array}{cc}\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1+s}' & \boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}-vec\left(\mathbf{I}_{n_e}\right)\right)'\\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\boldsymbol{\epsilon}_{t+1+s}' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}-vec\left(\mathbf{I}_{n_e}\right)\right)'\\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1+s}' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}-vec\left(\mathbf{I}_{n_e}\right)\right)'\\ \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1+s}' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}-vec\left(\mathbf{I}_{n_e}\right)\right)'\end{array}\right.$$

$$\left.\begin{array}{cc}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' & \boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\\ \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\end{array}\right]$$

$$=\left[\begin{array}{cccc}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

5.3.2 The auto-covariances

Recall that we have

$$\mathbf{z}_t=\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\end{array}\right]$$

$$\mathbf{z}_{t+1}=\mathbf{c}+\mathbf{A}\mathbf{z}_t+\mathbf{B}\boldsymbol{\xi}_{t+1}$$

$$\mathbf{y}_t^s=\mathbf{D}\mathbf{z}_t+\mathbf{g}_0$$

To find the one period auto-correlation, i.e. $Cov\left(\mathbf{z}_{t+1},\mathbf{z}_t\right)$, we have

$$Cov\left(\mathbf{z}_{t+1},\mathbf{z}_t\right)=Cov\left(\mathbf{c}+\mathbf{A}\mathbf{z}_t+\mathbf{B}\boldsymbol{\xi}_{t+1},\mathbf{z}_t\right)=\mathbf{A}Cov\left(\mathbf{z}_t,\mathbf{z}_t\right)=\mathbf{A}Var\left(\mathbf{z}_t\right)$$

because $Cov\left(\mathbf{z}_t,\boldsymbol{\xi}_{t+1}\right)=0$ as shown above. And for two periods

$$\begin{aligned}&Cov\left(\mathbf{z}_{t+2},\mathbf{z}_t\right)=Cov\left(\mathbf{c}+\mathbf{A}\mathbf{z}_{t+1}+\mathbf{B}\boldsymbol{\xi}_{t+2},\mathbf{z}_t\right)\\ &\quad=Cov\left(\mathbf{A}\left(\mathbf{c}+\mathbf{A}\mathbf{z}_t+\mathbf{B}\boldsymbol{\xi}_{t+1}\right)+\mathbf{B}\boldsymbol{\xi}_{t+2},\mathbf{z}_t\right)\\ &\quad=Cov\left(\mathbf{A}^2\mathbf{z}_t+\mathbf{B}\boldsymbol{\xi}_{t+2},\mathbf{z}_t\right)\\ &\quad=Cov\left(\mathbf{A}^2\mathbf{z}_t,\mathbf{z}_t\right)\\ &\quad=\mathbf{A}^2Cov\left(\mathbf{z}_t,\mathbf{z}_t\right)\\ &\quad=\mathbf{A}^2Var\left(\mathbf{z}_t\right)\end{aligned}$$

Here, we use the fact that $Cov\left(\boldsymbol{\xi}_{t+2},\mathbf{z}_t\right)=0$. This follows from the same arguements as above, that is consider

$$E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+2}'\right]=E\left[\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\end{array}\right]\left[\begin{array}{cccc}\boldsymbol{\epsilon}_{t+2}' & \left(\boldsymbol{\epsilon}_{t+2}\otimes\boldsymbol{\epsilon}_{t+2}-vec\left(\mathbf{I}_{ne}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\right)' & \left(\mathbf{x}_{t+1}^f\otimes\boldsymbol{\epsilon}_{t+2}\right)'\end{array}\right]\right]$$

$$= E\left[\begin{array}{cccc} \mathbf{x}_t^f\boldsymbol{\epsilon}_{t+2}' & \mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+2}\otimes\boldsymbol{\epsilon}_{t+2}-vec\left(\mathbf{I}_{ne}\right)\right)' & \mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\right)' & \mathbf{x}_t^f\left(\mathbf{x}_{t+1}^f\otimes\boldsymbol{\epsilon}_{t+2}\right)' \\ \mathbf{x}_t^s\boldsymbol{\epsilon}_{t+2}' & \mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+2}\otimes\boldsymbol{\epsilon}_{t+2}-vec\left(\mathbf{I}_{ne}\right)\right)' & \mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\right)' & \mathbf{x}_t^s\left(\mathbf{x}_{t+1}^f\otimes\boldsymbol{\epsilon}_{t+2}\right)' \\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1}' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\boldsymbol{\epsilon}_{t+2}-vec\left(\mathbf{I}_{ne}\right)\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+1}^f\otimes\boldsymbol{\epsilon}_{t+2}\right)' \end{array}\right]$$

$$= 0$$

Hence, in the general case
$Cov\left(\mathbf{z}_{t+l},\mathbf{z}_t\right) = \mathbf{A}^l Var\left(\mathbf{z}_t\right)$

For the control variables:
$Cov\left(\mathbf{y}_{t+l}^s,\mathbf{y}_t^s\right) = Cov\left(\mathbf{D}\mathbf{z}_{t+l}+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2,\mathbf{D}\mathbf{z}_t+\mathbf{g}_0\right)$

$$= Cov\left(\mathbf{D}\mathbf{z}_{t+l},\mathbf{D}\mathbf{z}_t\right)$$

$$= \mathbf{D}Cov\left(\mathbf{z}_{t+l},\mathbf{z}_t\right)\mathbf{D}'$$

$$= \mathbf{D}\mathbf{A}^l Var\left(\mathbf{z}_t\right)\mathbf{D}'$$

5.4 Impulse Response Functions

This section derives closed-form expressions for the generalized impulse response functions in a non-linear DSGE model approximated by the pruned second-order projection solution. The generalized impulse response functions are defined as

$$GIRF_{\mathbf{var}}\left(l,\nu_i,\mathbf{w}_t\right) = E_t\left[\mathbf{var}_{t+l}|\nu_i\right] - E_t\left[\mathbf{var}_{t+l}\right]$$

for a disturbance to innovation i. To reduce the notational burden in the derivations below, we adopt the parsimonious notation

$$IRF_{\mathbf{var}}\left(l,\nu_i,\mathbf{w}_t\right) = E_t\left[\widetilde{\mathbf{var}}_{t+l}\right] - E_t\left[\mathbf{var}_{t+l}\right]$$

in relation to the conditional expectation operators. Note that the formulas we derive below also apply even if we want to explore the joint effects of more than one shock - for instance when simultaneous shocking disturbances i and j, i.e. $GIRF_{\mathbf{var}}\left(l,\nu_i,\nu_j,\mathbf{w}_t\right) = E_t\left[\mathbf{var}_{t+l}|\nu_i,\nu_j\right] - E_t\left[\mathbf{var}_{t+l}\right]$.

5.4.1 The specification for the conditional information

This subsection explains how we will compute conditional expectations by conditioning on ν_i - and possible more disturbances. Let $\mathbf{S}$ be $n_\epsilon \times n_\epsilon$ diagonal selection matrix with either 1 or zeros on the diagonal, and let the shock sizes appear in the vector $\boldsymbol{\nu}$ of dimension $n_\epsilon \times 1$. For shocks which are not hit by a disturbance, we simply put them to zero.

As an example, consider an economy with three shocks and we want to condition our expectations on the first shock. Hence, we need the vector

$$\left[\begin{array}{c} \nu_1 \\ \epsilon_{2,t+1} \\ \epsilon_{3,t+1} \end{array}\right]$$

We can form this vector by letting

$$\mathbf{S} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and

$$\boldsymbol{\nu} = \begin{bmatrix} \nu_1 \\ 0 \\ 0 \end{bmatrix}.$$

Then we have

$$\begin{aligned}
\mathbf{S}\boldsymbol{\nu} + (\mathbf{I} - \mathbf{S})\,\boldsymbol{\epsilon}_{t+1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \nu_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \\ \epsilon_{3,t+1} \end{bmatrix} \\
&= \begin{bmatrix} \nu_1 \\ \epsilon_{2,t+1} \\ \epsilon_{3,t+1} \end{bmatrix}
\end{aligned}$$

Similarly, if we want to condition on the first two shocks, then we let

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\boldsymbol{\nu} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ 0 \end{bmatrix}.$$

meaning that

$$\mathbf{S}\boldsymbol{\nu} + (\mathbf{I} - \mathbf{S})\,\boldsymbol{\epsilon}_{t+1} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \epsilon_{3,t+1} \end{bmatrix}$$

5.4.2 At first order

Recall that we have:

$$\mathbf{x}^f_{t+1} = \mathbf{h}_0 + \mathbf{h}_\mathbf{x}\mathbf{x}^f_t + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

and

$$\begin{aligned}
\mathbf{x}^f_{t+2} &= \mathbf{h}_0 + \mathbf{h}_\mathbf{x}\mathbf{x}^f_{t+1} + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \\
&= \mathbf{h}_0 + \mathbf{h}_\mathbf{x}\left(\mathbf{h}_0 + \mathbf{h}_\mathbf{x}\mathbf{x}^f_t + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \\
&= \mathbf{h}_0 + \mathbf{h}_\mathbf{x}\mathbf{h}_0 + \mathbf{h}^2_\mathbf{x}\mathbf{x}^f_t + \mathbf{h}_\mathbf{x}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{x}^f_{t+3} &= \mathbf{h}_0 + \mathbf{h}_\mathbf{x}\mathbf{x}^f_{t+2} + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\
&= \mathbf{h}_0 + \mathbf{h}_\mathbf{x}\left(\mathbf{h}_0 + \mathbf{h}_\mathbf{x}\mathbf{h}_0 + \mathbf{h}^2_\mathbf{x}\mathbf{x}^f_t + \mathbf{h}_\mathbf{x}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\right) + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\
&= \mathbf{h}_0 + \mathbf{h}_\mathbf{x}\mathbf{h}_0 + \mathbf{h}^2_\mathbf{x}\mathbf{h}_0 + \mathbf{h}^3_\mathbf{x}\mathbf{x}^f_t + \mathbf{h}^2_\mathbf{x}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{h}_\mathbf{x}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\
&= \mathbf{h}^3_\mathbf{x}\mathbf{x}^f_t + \sum_{j=1}^{3}\mathbf{h}^{3-j}_\mathbf{x}\mathbf{h}_0 + \sum_{j=1}^{3}\mathbf{h}^{3-j}_\mathbf{x}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}
\end{aligned}$$

In general

$$\mathbf{x}^f_{t+l} = \mathbf{h}^l_\mathbf{x}\mathbf{x}^f_t + \sum_{j=1}^{l}\mathbf{h}^{l-j}_\mathbf{x}\mathbf{h}_0 + \sum_{j=1}^{l}\mathbf{h}^{l-j}_\mathbf{x}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

With a shock of $\boldsymbol{\nu}$ in period $t+1$, we have

$$\tilde{\mathbf{x}}^f_{t+l} = \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{l} \mathbf{h}^{l-j}_{\mathbf{x}}\mathbf{h}_0 + \sum_{j=1}^{l} \mathbf{h}^{l-j}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

where we define $\boldsymbol{\delta}_t$ such that

$$\boldsymbol{\delta}_{t+j} = \begin{cases} \mathbf{S}\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1} & \text{for } j=1 \\ \boldsymbol{\epsilon}_{t+j} & \text{for } j\neq 1 \end{cases}$$

Agents know the size of the shock $\boldsymbol{\nu}$ at time $t+1$, and it is therefore in agents' information set. I.e. $\boldsymbol{\nu}$ is non-stochastic. So

$$E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right] = E_t\left[\sum_{j=1}^{l} \mathbf{h}^{l-j}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \sum_{j=1}^{l} \mathbf{h}^{l-j}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$= E_t\left[\mathbf{h}^{l-1}_{\mathbf{x}}\boldsymbol{\eta}\left(\mathbf{S}\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\right]$$

$$= \mathbf{h}^{l-1}_{\mathbf{x}}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}$$

and

$$E_t\left[\tilde{\mathbf{y}}^f_{t+l} - \mathbf{y}^f_{t+l}\right] = \mathbf{g}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right]$$

5.4.3 At second order

We need to consider:

$$\mathbf{x}^s_{t+1} = \mathbf{h}_{\mathbf{x}}\mathbf{x}^s_t + \mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right)$$

$$\begin{aligned}\mathbf{x}^s_{t+2} &= \mathbf{h}_{\mathbf{x}}\mathbf{x}^s_{t+1} + \mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right)\\ &= \mathbf{h}_{\mathbf{x}}\left(\mathbf{h}_{\mathbf{x}}\mathbf{x}^s_t + \mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right)\right) + \mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right)\\ &= \mathbf{h}^2_{\mathbf{x}}\mathbf{x}^s_t + \mathbf{h}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right)\end{aligned}$$

$$\begin{aligned}\mathbf{x}^s_{t+3} &= \mathbf{h}_{\mathbf{x}}\mathbf{x}^s_{t+2} + \mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right)\\ &= \mathbf{h}_{\mathbf{x}}\left(\mathbf{h}^2_{\mathbf{x}}\mathbf{x}^s_t + \mathbf{h}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right)\right)\\ &+\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right)\\ &= \mathbf{h}^3_{\mathbf{x}}\mathbf{x}^s_t + \mathbf{h}^2_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \mathbf{h}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right)\\ &+\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right)\\ &= \mathbf{h}^3_{\mathbf{x}}\mathbf{x}^s_t + \mathbf{h}^2_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \mathbf{h}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right) + \mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right)\\ &= \mathbf{h}^3_{\mathbf{x}}\mathbf{x}^s_t + \sum_{j=0}^{2} \mathbf{h}^{2-j}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+j} \otimes \mathbf{x}^f_{t+j}\right)\end{aligned}$$

and in general

$$\mathbf{x}^s_{t+l} = \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^s_t + \sum_{j=0}^{l-1} \mathbf{h}^{(l-1)-j}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+j} \otimes \mathbf{x}^f_{t+j}\right)$$

for $l = 1, 2, 3, \ldots$

Thus, to compute $E_t\left[\tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^s_{t+l}\right]$, we need to find $E_t\left[\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l}\right]$. Hence, consider:

$$\mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l} = \left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{l} \mathbf{h}^{l-j}_{\mathbf{x}}\mathbf{h}_0 + \sum_{j=1}^{l} \mathbf{h}^{l-j}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{l} \mathbf{h}^{l-j}_{\mathbf{x}}\mathbf{h}_0 + \sum_{j=1}^{l} \mathbf{h}^{l-j}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$

$$= \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

and

$$\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f = \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

where we define $\boldsymbol{\delta}_t$ such that:

$$\boldsymbol{\delta}_{t+j} = \begin{cases} \mathbf{S}\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1} & \text{for } j = 1 \\ \boldsymbol{\epsilon}_{t+j} & \text{for } j \neq 1 \end{cases}$$

This means that:

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f\right]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$-\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f - \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 - \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$- \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$- \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$= E_t[\mathbf{0} + \mathbf{0} + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+0 + 0 + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$-\mathbf{0} - \mathbf{0} - \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-0 - 0 - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$- \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0 - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\delta}_{t+j}$$
$$\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \mathbf{h}_0 \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\delta}_{t+j}$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \mathbf{h}_0 + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\delta}_{t+j}$$
$$-0$$
$$-0$$
$$-0 - 0 - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}]$$

as $E_t\left[\boldsymbol{\epsilon}_{t+j}\right] = \mathbf{0}$

$$= E_t[\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu}$$
$$\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \mathbf{h}_0 \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu}$$
$$+ \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \mathbf{h}_0$$
$$+ \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \left(\mathbf{S} \boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1} \right) + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \left(\mathbf{S} \boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1} \right) + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right)$$
$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right)]$$

using that $\boldsymbol{\delta}_{t+j} = \begin{cases} \mathbf{S}\boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1} & \text{for } j = 1 \\ \boldsymbol{\epsilon}_{t+j} & \text{for } j \neq 1 \end{cases}$ and $E_t\left[\boldsymbol{\epsilon}_{t+j}\right] = \mathbf{0}$

$$= E_t[\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu} + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \mathbf{h}_0 \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \mathbf{h}_0$$
$$+ \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \left(\mathbf{S} \boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1} \right) + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \left(\mathbf{S} \boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1} \right) + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right)$$
$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right)]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \mathbf{h}_0 \right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \mathbf{h}_0 \right)$$
$$+ \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \left(\mathbf{S} \boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1} \right) + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \left(\mathbf{S} \boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1} \right) + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right)$$
$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right)]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \mathbf{h}_0 \right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \mathbf{h}_0 \right)$$
$$+ \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1} \right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \left(\mathbf{S} \boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1} \right) + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right)$$
$$+ \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} \mathbf{S} \boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1} \boldsymbol{\eta} (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right)$$

$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$

$$- \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0\right)$$

$$+\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\left(\mathbf{S}\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right) + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$

$$+\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}(\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\left(\mathbf{S}\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right) + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$

$$+ \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}(\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right)$$

$$+ \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

$$- \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

$$- \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0\right)$$

$$+\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{0} + \mathbf{0}$$

$$+\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}(\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}(\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right) + 0$$

$$+0$$

$$+ \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

$$-0$$

$$-0$$

$$- \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

using $E_t[\boldsymbol{\epsilon}_{t+j}] = \mathbf{0}$ and $\boldsymbol{\epsilon}_{t+j}$ are uncorrelated across time

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\mathbf{h}_0\right)$$

$$+\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{0} + \mathbf{0}$$

$$+\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}(\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}(\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right) + 0$$

$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]$$

canceling terms $\sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$

$$= E_t[\left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right) \otimes \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right) + \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}$$
$$+ \left(\mathbf{h}_\mathbf{x}^{l-1} \otimes \mathbf{h}_\mathbf{x}^{l-1}\right)\left(\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)$$
$$- \left(\mathbf{h}_\mathbf{x}^{l-1} \otimes \mathbf{h}_\mathbf{x}^{l-1}\right)\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)]$$

$$= E_t[\left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right) \otimes \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right) + \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}$$
$$+ \left(\mathbf{h}_\mathbf{x}^{l-1} \otimes \mathbf{h}_\mathbf{x}^{l-1}\right)\left(\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} - \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)]$$

$$= \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right) \otimes \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right) + \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}$$
$$+ \left(\mathbf{h}_\mathbf{x}^{l-1} \otimes \mathbf{h}_\mathbf{x}^{l-1}\right)\left(E_t\left[\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right] - E_t\left[\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]\right)$$

Hence, we only need to compute $E_t\left[\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right]$ and $E_t\left[\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$. We know
$E_t\left[\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right] = E_t\left[\left(\boldsymbol{\eta} \otimes \boldsymbol{\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right] = \left(\boldsymbol{\eta} \otimes \boldsymbol{\eta}\right) vec\left(\mathbf{I}\right)$
using $\left(\mathbf{A} \otimes \mathbf{B}\right)\left(\mathbf{C} \otimes \mathbf{D}\right) = \mathbf{AC} \otimes \mathbf{BD}$ if $\mathbf{AC}$ and $\mathbf{BD}$ are defined

and
$E_t\left[\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right] = E_t\left[\left(\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right] = \left(\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right) vec\left(\mathbf{I}\right)$

Thus, we have

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f\right] = \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right) \otimes \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right) + \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}$$
$$+ \left(\mathbf{h}_\mathbf{x}^{l-1} \otimes \mathbf{h}_\mathbf{x}^{l-1}\right)\left(\left(\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right) vec\left(\mathbf{I}\right) - \left(\boldsymbol{\eta} \otimes \boldsymbol{\eta}\right) vec\left(\mathbf{I}\right)\right)$$

$$= \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right) \otimes \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right) + \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}$$
$$+ \left(\mathbf{h}_\mathbf{x}^{l-1} \otimes \mathbf{h}_\mathbf{x}^{l-1}\right)\left(\left(\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right) - \left(\boldsymbol{\eta} \otimes \boldsymbol{\eta}\right)\right) vec\left(\mathbf{I}\right)$$

$$= \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right) \otimes \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right)$$
$$+ \left(\mathbf{h}_\mathbf{x}^{l-1} \otimes \mathbf{h}_\mathbf{x}^{l-1}\right)\left[\left(\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}\right) + \left(\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right) vec\left(\mathbf{I}\right) - \left(\boldsymbol{\eta} \otimes \boldsymbol{\eta}\right) vec\left(\mathbf{I}\right)\right]$$

$$= \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right) \otimes \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} + \mathbf{h}_\mathbf{x}^{l-1}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \left(\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j}\mathbf{h}_0\right)$$
$$+ \left(\mathbf{h}_\mathbf{x}^{l-1} \otimes \mathbf{h}_\mathbf{x}^{l-1}\right)\left[\left(\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu} \otimes \boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}\right) + \boldsymbol{\Lambda}\right]$$

where

$$\boldsymbol{\Lambda} \equiv \left(\left(\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right) - \left(\boldsymbol{\eta} \otimes \boldsymbol{\eta}\right)\right) vec\left(\mathbf{I}\right)$$

Then note that

$$\sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} = \mathbf{h}_\mathbf{x}^{l-1} + \mathbf{h}_\mathbf{x}^{l-2} + ... + \mathbf{h}_\mathbf{x}^{0} = \left(\mathbf{I}-\mathbf{h}_\mathbf{x}\right)^{-1}\left(\mathbf{I}-\mathbf{h}_\mathbf{x}^{l}\right)$$

as we recall the general result that

$$Q = \left(\mathbf{h}_\mathbf{x}\right)^0 + \left(\mathbf{h}_\mathbf{x}\right)^1 + ... + \left(\mathbf{h}_\mathbf{x}\right)^{11} = \left(\mathbf{I}-\mathbf{h}_\mathbf{x}\right)^{-1}\left(\mathbf{I}-\left(\mathbf{h}_\mathbf{x}\right)^{12}\right).$$

Hence, we have

$$E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right]=\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+(\mathbf{I}-\mathbf{h}_{\mathbf{x}})^{-1}\left(\mathbf{I}-\mathbf{h}^l_{\mathbf{x}}\right)\mathbf{h}_0\right)\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}+\mathbf{h}^{l-1}_{\mathbf{x}}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}\otimes\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+(\mathbf{I}-\mathbf{h}_{\mathbf{x}})^{-1}\left(\mathbf{I}-\mathbf{h}^l_{\mathbf{x}}\right)\mathbf{h}_0\right)$$
$$+\left(\mathbf{h}^{l-1}_{\mathbf{x}}\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\right)\left[(\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}\otimes\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu})+\boldsymbol{\Lambda}\right]$$

Or (using another index)

$$E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]=\left(\mathbf{h}^j_{\mathbf{x}}\mathbf{x}^f_t+(\mathbf{I}-\mathbf{h}_{\mathbf{x}})^{-1}\left(\mathbf{I}-\mathbf{h}^j_{\mathbf{x}}\right)\mathbf{h}_0\right)\otimes\mathbf{h}^{j-1}_{\mathbf{x}}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}+\mathbf{h}^{j-1}_{\mathbf{x}}\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}\otimes\left(\mathbf{h}^j_{\mathbf{x}}\mathbf{x}^f_t+(\mathbf{I}-\mathbf{h}_{\mathbf{x}})^{-1}\left(\mathbf{I}-\mathbf{h}^j_{\mathbf{x}}\right)\mathbf{h}_0\right)$$
$$+\left(\mathbf{h}^{j-1}_{\mathbf{x}}\otimes\mathbf{h}^{j-1}_{\mathbf{x}}\right)\left[(\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu}\otimes\boldsymbol{\eta}\mathbf{S}\boldsymbol{\nu})+\boldsymbol{\Lambda}\right]$$

for $j=1,2,3,...$

Thus, we have in general

$$E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]=E_t\left[\sum_{j=0}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}\left(\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}\right)-\sum_{j=0}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right)\right]$$

$$=\sum_{j=1}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$$

the shock hits in period $t+1$, so $\left(\tilde{\mathbf{x}}^f_t\otimes\tilde{\mathbf{x}}^f_t\right)=\mathbf{x}^f_t\otimes\mathbf{x}^f_t$

When implementing the GIRF, it may be useful to have a recursive expression. Here, it is most convenient to use the general expression

$$E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]=\sum_{j=1}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$$

So

$$E_t\left[\tilde{\mathbf{x}}^s_{t+1}-\mathbf{x}^s_{t+1}\right]=0$$

$$E_t\left[\tilde{\mathbf{x}}^s_{t+2}-\mathbf{x}^s_{t+2}\right]=\sum_{j=1}^{1}\mathbf{h}^{1-j}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$$
$$=\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+1}\otimes\tilde{\mathbf{x}}^f_{t+1}-\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right]$$

$$E_t\left[\tilde{\mathbf{x}}^s_{t+3}-\mathbf{x}^s_{t+3}\right]=\sum_{j=1}^{2}\mathbf{h}^{2-j}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$$
$$=\mathbf{h}_{\mathbf{x}}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+1}\otimes\tilde{\mathbf{x}}^f_{t+1}-\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right]$$
$$+\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right]$$
$$=\mathbf{h}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^s_{t+2}-\mathbf{x}^s_{t+2}\right]+\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right]$$

So in general

$$E_t\left[\tilde{\mathbf{x}}^s_{t+k}-\mathbf{x}^s_{t+k}\right]=\mathbf{h}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^s_{t+k-1}-\mathbf{x}^s_{t+k-1}\right]+\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+k-1}\otimes\tilde{\mathbf{x}}^f_{t+k-1}-\mathbf{x}^f_{t+k-1}\otimes\mathbf{x}^f_{t+k-1}\right]$$

For the total state variable:

$$E_t\left[\tilde{\mathbf{x}}_{t+l}-\mathbf{x}_{t+l}\right]=E_t\left[\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\right]+E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]$$

For the control variables:

$$\mathbf{y}^s_{t+l}=\mathbf{g}_0+\mathbf{g}_{\mathbf{x}}\left(\mathbf{x}^f_{t+l}+\mathbf{x}^s_{t+l}\right)+\mathbf{G}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right)$$
$$\tilde{\mathbf{y}}^s_{t+l}=\mathbf{g}_0+\mathbf{g}_{\mathbf{x}}\left(\tilde{\mathbf{x}}^f_{t+l}+\tilde{\mathbf{x}}^s_{t+l}\right)+\mathbf{G}_{\mathbf{xx}}\left(\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}\right)$$

$$E_t\left[\tilde{\mathbf{y}}^s_{t+l}-\mathbf{y}^s_{t+l}\right]=\mathbf{g}_{\mathbf{x}}\left(E_t\left[\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\right]+E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]\right)+\mathbf{G}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right]$$

6 Additional Model Output

This section reports additional output from the model.

6.1 Correlation matrix for the estimated state innovations

The table below shows the correlation matrix for the estimated states in the baseline model, i.e., $\mathcal{M}^{M,CS}$.

	$\epsilon_{\mu_z,t}$	$\epsilon_{d,t}$	$\epsilon_{n,t}$	$\epsilon_{\pi^*,t}$	$\epsilon_{a,t}$
$\epsilon_{\mu_z,t}$	1	-0.12	0.12	0.10	0.09
$\epsilon_{d,t}$		1	-0.21	-0.11	0.08
$\epsilon_{n,t}$			1	0.03	0.26
$\epsilon_{\pi^*,t}$				1	0.07
$\epsilon_{a,t}$					1

6.2 Correlation matrix for the estimated measurement errors

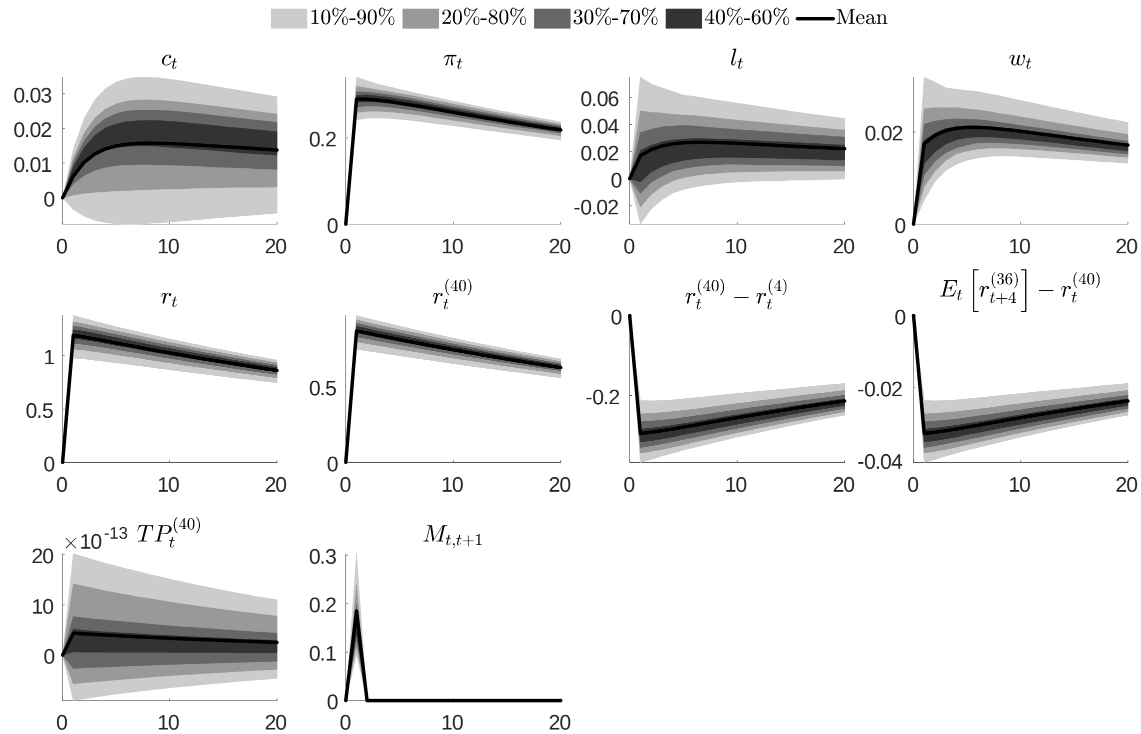
The table below shows the correlation matrix for the estimated measurement errors in by the baseline model, i.e., $\mathcal{M}^{M,CS}$.

	$\hat{l}_t$	$\hat{w}_t$	Δc_t	π_t	r_t	$r_t^{(4)}$	$r_t^{(12)}$	$r_t^{(20)}$	$r_t^{(28)}$	$r_t^{(40)}$
$\hat{l}_t$	1.00	0.23	0.44	0.25	-0.16	-0.26	-0.09	0.07	0.08	-0.11
$\hat{w}_t$	-	1.00	-0.11	-0.44	0.03	0.29	0.25	0.06	-0.21	-0.42
Δc_t	-	-	1.00	0.23	-0.23	-0.50	-0.55	-0.46	-0.35	-0.17
π_t	-	-	-	1.00	-0.09	-0.11	0.03	0.20	0.34	0.32
r_t	-	-	-	-	1.00	0.09	0.05	0.05	0.07	0.15
$r_t^{(4)}$	-	-	-	-	-	1.00	0.72	0.26	-0.08	-0.20
$r_t^{(12)}$	-	-	-	-	-	-	1.00	0.80	0.41	-0.13
$r_t^{(20)}$	-	-	-	-	-	-	-	1.00	0.83	0.12
$r_t^{(28)}$	-	-	-	-	-	-	-	-	1.00	0.58
$r_t^{(40)}$	-	-	-	-	-	-	-	-	-	1.00

6.3 Impulse Reponse Functions to Demand Shocks for a perfect foreslight model solution

Figure 1: A Demand Shock under Perfect Foresight (no Risk)

This figure shows the generalized impulse response functions (IRFs) of a positive one-standard deviation shock to d_t, where the various shadings cover the indicated fraction of the distribution obtained using 1,000 randomly generated initial states. These IRFs are computed in closed-form for the second order projection solution using the approach in Andreasen et al. (2018), except for $M_{t,t+1}$ that is computed using Monte Carlo integration with 1,000 draws. Except for the term premium $TP_t^{(40)}$ and the nominal stochastic discount factor $M_{t,t+1}$, all impulse response functions are expressed in percentage deviations from the steady state (i.e., scaled by 100), with consumption expressed in deviations from the balanced growth path and all bond yields and inflation measured in annualized terms. The term premium $TP_t^{(40)}$ is expressed in annualized basis points.



References

Andreasen, M. M. (2013), 'Non-Linear DSGE Models and the Central Difference Kalman Filter', *Journal of Applied Econometrics* **28**, 929–955.

Andreasen, M. M., Fernandez-Villaverde, J. & Rubio-Ramirez, J. F. (2018), 'The Pruned State Space System for Non-Linear DSGE Models: Theory and Empirical Applications to Estimation', *Review of Economic Studies* **85**(1), 1–49.

Andreasen, M. M. & Jørgensen, K. (2020), 'The Importance of Timing Attitudes in Consumption-Based Asset Pricing Models', *Journal of Monetary Economics* **111**, 95–117.

Blanchard, O. & Gali, J. (2005), 'Real Wage Rigidities and the New Keynesian Model', *Journal of Money, Credit and Banking* **39**(1), 35–65.

Chernozhukov, V. & Hong, H. (2003), 'An MCMC Approach to Classical Estimation', *Journal of Econometrics* **115**(2), 293–346.

Christiano, L. J., Trabandt, M. & Walentin, K. (2011), 'Introducing Financial Frictions and Unemployment into a Small Open Economy Model', *Journal of Economic Dynamic and Control* **35**(12), 1999–2041.

Hamilton, J. D. (2018), 'Why You Should Never Use the Hodrick-Prescott Filter', *The Review of Economics and Statistics* **100**(5), 831–843.

Hansen, L. P. (1982), 'Large Sample Properties of Generalized Method of Moments Estimators', *Econometrica* **50**(4), 1029–1054.

Hayashi, F. (2000), *Econometrics*, Princeton University Press, New Jersey.

Newey, W. K. & West, K. D. (1987), 'A Simple, Positive Semi-definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix', *Econometrica* **55**(3), 703–708.

Norgaard, M., Poulsen, N. K. & Ravn, O. (2000), 'Advances in Derivative-Free State Estimation for Nonlinear Systems', *Automatica* **36**(11), 1627–1638.

Rudebusch, G. D. & Swanson, E. T. (2008), 'Examining the Bond Premium Puzzle with a DSGE Model', *Journal of Monetary Economics* **55**, 111–126.

Rudebusch, G. D. & Swanson, E. T. (2012), 'The Bond Premium in a DSGE Model with Long-Run Real and Nominal Risks', *American Economic Journal: Macroeconomics* **4**(1), 1–43.

Swanson, E. (2012), 'Risk aversion and the labor margin in dynamic equilibrium models', *Working Paper version* .