Internet Appendix for "Incentivizing Investors for a Greener Economy"

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Appendix for Section III.A: Computing the Market Equilibrium

The dynamics of instantaneous returns are given by:

$$dR_b = \frac{Div_b}{V_b}dt + \frac{dV_b}{V_b} = \Lambda_b(\eta)dt + \Gamma_b(\eta)dB_b - \Delta_b(\eta)dB_g - \psi dN;$$

$$dR_g = \frac{Div_g}{V_g}dt + \frac{dV_g}{V_g} = \Lambda_g(\eta)dt + \Gamma_g(\eta)dB_b + \Delta_g(\eta)dB_g - \psi dN;$$

where:

$$\Lambda_b(\eta) = \frac{1}{2p_b(\eta)}\left\{2\alpha_b - \theta_b i_b(\eta)^2 + 2i_b(\eta)\left[-\eta(\eta-1)p_b'(\eta) + p_b(\eta) - 1\right] + 2\eta(\eta-1)i_g(\eta)p_b'(\eta)\right.$$
$$\left. +\eta^2(\eta-1)^2\left({\sigma_b}^2 + {\sigma_g}^2\right)p_b''(\eta) + 2\eta(\eta-1)p_b'(\eta)\left[(\eta-1)\left({\sigma_b}^2 + {\sigma_g}^2\right) - \mu_b + \mu_g\right] + 2\mu_b p_b(\eta)\right\};$$

$$\Lambda_g(\eta) = \frac{1}{2p_g(\eta)}\left\{2\alpha_g - \theta_g i_g(\eta)^2 + 2i_g(\eta)\left[\eta(\eta-1)p_g'(\eta) + p_g(\eta) - 1\right] - 2\eta(\eta-1)i_b(\eta)p_g'(\eta)\right.$$
$$\left. +\eta^2(\eta-1)^2\left({\sigma_b}^2 + {\sigma_g}^2\right)p_g''(\eta) + 2\eta(\eta-1)p_g'(\eta)\left[\eta\left({\sigma_b}^2 + {\sigma_g}^2\right) - \mu_b + \mu_g\right] + 2\mu_g p_g(\eta)\right\};$$

$$\Gamma_b(\eta) = \sigma_b + \sigma_b\eta(1-\eta)\frac{p_b'(\eta)}{p_b(\eta)};$$
$$\Delta_b(\eta) = \sigma_g\eta(1-\eta)\frac{p_b'(\eta)}{p_b(\eta)};$$
$$\Gamma_g(\eta) = \sigma_b\eta(1-\eta)\frac{p_g'(\eta)}{p_g(\eta)};$$
$$\Delta_g(\eta) = \sigma_g - \sigma_g\eta(1-\eta)\frac{p_g'(\eta)}{p_g(\eta)};$$

and

$$Div_n = K_n\left(\alpha_n - i_n(\eta) - \theta_n\frac{i_n^2(\eta)}{2}\right)$$

is the dividend payment made by brown/green firms.

Thus, the wealth process has dynamics:

$$dW = \left[r(\eta)W - c(\eta)W + \pi_b(\eta)W_t\left(\Lambda_b(\eta) - r(\eta)\right) + \pi_g(\eta)W\left(\Lambda_g(\eta) - r(\eta)\right)\right]dt$$
$$+W\left[\pi_b(\eta)\Gamma_b(\eta) + \pi_g(\eta)\Gamma_g(\eta)\right]dB_b + W\left[-\pi_b(\eta)\Delta_b(\eta) + \pi_g(\eta)\Delta_g(\eta)\right]dB_g - W\psi dN,$$

where $c(\eta), \pi_b(\eta), \pi_g(\eta)$ are the fractions of wealth consumed and invested in the brown and green firms, respectively; and $r(\eta)$ is the equilibrium risk-free interest rate.

The value function $M\left(W;\eta\right) = sup_{c,\pi_b,\pi_g}\left\{\mathbb{E}\left[\int_0^\infty e^{-\rho t}u\left(C(t)\right)dt\right]\right\}$ satisfies the HJB equation:

$$\text{(IA.1)}\qquad 0 = \max_{c(\eta),\pi_b(\eta),\pi_g(\eta)}\left\{\begin{array}{l} -\rho M\left(W;\eta\right) + u\left(c(\eta)W\right) \\ +M_W\left(W;\eta\right)W\left[r(\eta) - c(\eta) + \pi_b(\eta)\left(\Lambda_b(\eta) - r(\eta)\right)\right. \\ \left.+\pi_g(\eta)\left(\Lambda_g(\eta) - r(\eta)\right)\right] \\ +\frac{1}{2}M_{WW}\left(W;\eta\right)W^2\left[\left(\pi_b(\eta)\Gamma_b(\eta) + \pi_g(\eta)\Gamma_g(\eta)\right)^2\right. \\ \left.+\left(-\pi_b(\eta)\Delta_b(\eta) + \pi_g(\eta)\Delta_g(\eta)\right)^2\right] \\ +M_\eta\left(W;\eta\right)\Sigma(\eta) + \frac{1}{2}M_{\eta\eta}\left(W;\eta\right)\eta^2\left(1-\eta\right)^2\left(\sigma_b^2 + \sigma_g^2\right) \\ +M_{W\eta}\left(W;\eta\right)W\eta\left(1-\eta\right)\left[\sigma_b\left(\pi_b(\eta)\Gamma_b(\eta) + \pi_g(\eta)\Gamma_g(\eta)\right)\right. \\ \left.+\sigma_g\left(\pi_b(\eta)\Delta_b(\eta) - \pi_g(\eta)\Delta_g(\eta)\right)\right] \\ +\lambda\eta\left[M\left(W - W\psi(\pi_b(\eta) + \pi_g(\eta));\eta\right) - M\left(W;\eta\right)\right] \end{array}\right\}.$$

We guess (and subsequently verify) that:

$$\text{(IA.2)}\qquad M\left(W;\eta\right) = f\left(\eta\right)W^{1-\gamma} + \frac{W^{1-\gamma} - 1}{\rho\left(1-\gamma\right)}.$$

The optimal controls are given by:

$$\text{(IA.3)}\qquad c(\eta) = \left[\left(1-\gamma\right)f\left(\eta\right) + \frac{1}{\rho}\right]^{\frac{-1}{\gamma}};$$

and

$$\text{(IA.4)} \qquad \pi_b(\eta) = \frac{\lambda\eta\psi\left[(1-\gamma)\,f\,(\eta)+\frac{1}{\rho}\right](1-\psi)^{-\gamma} - X - Y\pi_g(\eta) - Z}{T};$$
$$\pi_g(\eta) = \frac{\lambda\eta\psi\left[(1-\gamma)\,f\,(\eta)+\frac{1}{\rho}\right](1-\psi)^{-\gamma} - X' - Y\pi_b(\eta) - Z'}{T'},$$

where

$$X = \left[(1-\gamma)\,f\,(\eta)+\frac{1}{\rho}\right]\left[\Lambda_b(\eta) - r(\eta)\right];$$
$$Y = (-\gamma)\left[(1-\gamma)\,f\,(\eta)+\frac{1}{\rho}\right]\left[\Gamma_g(\eta)\Gamma_b(\eta) - \Delta_g(\eta)\Delta_b(\eta)\right];$$
$$Z = (1-\gamma)\,f'\,(\eta)\,\eta\,(1-\eta)\left[\sigma_b\Gamma_b(\eta) + \sigma_g\Delta_b(\eta)\right];$$
$$T = (-\gamma)\left[(1-\gamma)\,f\,(\eta)+\frac{1}{\rho}\right]\left[\Gamma_b^2(\eta) + \Delta_b^2(\eta)\right];$$
$$X' = \left[(1-\gamma)\,f\,(\eta)+\frac{1}{\rho}\right]\left[\Lambda_g(\eta) - r(\eta)\right];$$
$$Z' = (1-\gamma)\,f'\,(\eta)\,\eta\,(1-\eta)\left[\sigma_b\Gamma_g(\eta) - \sigma_g\Delta_g(\eta)\right];$$
$$T' = (-\gamma)\left[(1-\gamma)\,f\,(\eta)+\frac{1}{\rho}\right]\left[\Gamma_g^2(\eta) + \Delta_g^2(\eta)\right].$$

Substituting equations (IA.2), (IA.3), and (IA.4) into equation (IA.1) we obtain:

$$\text{(IA.5)} \qquad 0 = -\rho\frac{\left[(1-\gamma)\,f\,(\eta)+\frac{1}{\rho}\right]}{(1-\gamma)} + \frac{\gamma\left[(1-\gamma)\,f\,(\eta)+\frac{1}{\rho}\right]^{\frac{-(1-\gamma)}{\gamma}}}{1-\gamma}$$
$$+\left[(1-\gamma)\,f\,(\eta)+\frac{1}{\rho}\right]\left[r(\eta) + \pi_b(\eta)\left(\Lambda_b(\eta) - r(\eta)\right) + \pi_g(\eta)\left(\Lambda_g(\eta) - r(\eta)\right)\right]$$
$$-\frac{\gamma}{2}\left[(1-\gamma)\,f\,(\eta)+\frac{1}{\rho}\right]\left[\left(\pi_b(\eta)\Gamma_b(\eta) + \pi_g(\eta)\Gamma_g(\eta)\right)^2 + \left(-\pi_b(\eta)\Delta_b(\eta) + \pi_g\Delta_g(\eta)\right)^2\right]$$
$$+f'\,(\eta)\,\Sigma(\eta) + \frac{1}{2}f''\,(\eta)\,\eta^2\,(1-\eta)^2\left(\sigma_b^2 + \sigma_g^2\right)$$
$$+\,(1-\gamma)\,f'\,(\eta)\,\eta\,(1-\eta)\left[\sigma_b\left(\pi_b\Gamma_b(\eta) + \pi_g\Gamma_g(\eta)\right) + \sigma_g\left(\pi_b\Delta_b(\eta) - \pi_g\Delta_g(\eta)\right)\right]$$
$$+\lambda\eta\frac{\left[(1-\gamma)\,f\,(\eta)+\frac{1}{\rho}\right]}{(1-\gamma)}\left[(1-\psi)^{1-\gamma} - 1\right].$$

In equilibrium, the following market clearing conditions must hold:

(A). Goods market equilibrium condition: $c_t(\eta)W = Div_b + Div_g$.

(B). The agent holds both trees in equilibrium and none of the bond: $\pi_b(\eta) = \frac{K_b p_b(\eta)}{W}$, $\pi_g(\eta) = \frac{K_g p_g(\eta)}{W}$, and $\pi_b(\eta) + \pi_g(\eta) = 1$.

Equating the optimal allocations in equation (IA.4) with condition (B), we have that in equilibrium the price-to-capital ratios and risk-free interest rate should satisfy:

$$\text{(IA.6)} \qquad \begin{aligned} \frac{\eta p_b(\eta)}{\eta p_b(\eta) + (1-\eta)p_g(\eta)} = \frac{-1}{\gamma\left[\Gamma_b^2(\eta) + \Delta_b^2(\eta)\right]} \{ & \lambda\eta\psi\,(1-\psi)^{-\gamma} - [\Lambda_b(\eta) - r(\eta)] \\ & + [\Gamma_g(\eta)\Gamma_b(\eta) - \Delta_g(\eta)\Delta_b(\eta)]\frac{\gamma(1-\eta)p_g(\eta)}{\eta p_b(\eta) + (1-\eta)p_g(\eta)} \\ & - \frac{(1-\gamma)\,f'(\eta)}{(1-\gamma)\,f(\eta) + \frac{1}{\rho}}\eta\,(1-\eta)\,[\sigma_b\Gamma_b(\eta) + \sigma_g\Delta_b(\eta)]\,\}, \end{aligned}$$

and

$$\text{(IA.7)} \qquad \begin{aligned} \frac{(1-\eta)p_g(\eta)}{\eta p_b(\eta) + (1-\eta)p_g(\eta)} = \frac{-1}{\gamma\left[\Gamma_g^2(\eta) + \Delta_g^2(\eta)\right]} \{ & \lambda\eta\psi\,(1-\psi)^{-\gamma} - [\Lambda_g(\eta) - r(\eta)] \\ & + [\Gamma_g(\eta)\Gamma_b(\eta) - \Delta_g(\eta)\Delta_b(\eta)]\frac{\gamma\eta p_b(\eta)}{\eta p_b(\eta) + (1-\eta)p_g(\eta)} \\ & - \frac{(1-\gamma)\,f'(\eta)}{(1-\gamma)\,f(\eta) + \frac{1}{\rho}}\eta\,(1-\eta)\,[\sigma_b\Gamma_g(\eta) - \sigma_g\Delta_g(\eta)]\,\}. \end{aligned}$$

In addition, using the optimal consumption in equation (IA.3) together with the firms' investment decisions in equation (23), we can rewrite the market clearing condition (A) as:

$$\text{(IA.8)} \qquad \begin{aligned} & \left[(1-\gamma)\,f(\eta) + \frac{1}{\rho}\right]^{\frac{-1}{\gamma}} [\eta p_b(\eta) + (1-\eta)p_g(\eta)] \\ & = \eta\left[\alpha_b - \left(\frac{1}{\theta_b}(p_b(\eta) - 1)\right) - \frac{\theta_b}{2}\left(\frac{1}{\theta_b}(p_b(\eta) - 1)\right)^2\right] \\ & + (1-\eta)\left[\alpha_g - \left(\frac{1}{\theta_g}(p_g(\eta) - 1)\right) - \frac{\theta_g}{2}\left(\frac{1}{\theta_g}(p_g(\eta) - 1)\right)^2\right]. \end{aligned}$$

We solve for $p_b(\eta), p_g(\eta), r(\eta), f(\eta)$ that satisfy the system of equations (IA.5)-(IA.6)-

(IA.7)-(IA.8) using the following procedure: First, we obtain $f(\eta)$ in terms of $p_b(\eta), p_g(\eta)$ from equation (IA.8), and $r(\eta)$ in terms of $p_b(\eta), p_g(\eta)$ from equation (IA.6). Substituting them into equations (IA.5) and (IA.7), we have a system of ODEs for $p_b(\eta), p_g(\eta)$, of which the boundary conditions come from the single-sector economy when we replace $\eta = 0$ or $\eta = 1$ into the system. Finally, we numerically solve the system of ODEs for $p_b(\eta), p_g(\eta)$, and subsequently retrieve $f(\eta)$ and $r(\eta)$.

Appendix for Section III.B: Computing the Social Planner Solution

We provide a brief description of the social planner's solution. The planner's value function $G(K_b; K_g)$ satisfies the following HJB equation:

(IA.9)

$$0 = \max_{i_b(\eta), i_g(\eta)} \left\{ \begin{array}{l} -\rho G(K_b; K_g) + u\left(K_b\left(\alpha_b - i_b(\eta) - \frac{\theta_b}{2} i_b^2(\eta)\right) + K_g\left(\alpha_g - i_g(\eta) - \frac{\theta_g}{2} i_g^2(\eta)\right)\right) \\ +G_{K_b}(K_b; K_g) K_b (i_b(\eta) + \mu_b) + \frac{1}{2} G_{K_b K_b}(K_b; K_g) K_b^2 \sigma_b^2 \\ +G_{K_g}(K_b; K_g) K_g (i_g(\eta) + \mu_g) + \frac{1}{2} G_{K_g K_g}(K_b; K_g) K_g^2 \sigma_g^2 \\ +\lambda \frac{K_b}{K_b + K_g} \left[G(K_b - K_b\psi; K_g - K_g\psi) - G(K_b; K_g)\right] \end{array} \right\}.$$

We guess (and verify) the value function is given by:

$$G(K_b; K_g) = g\left(\frac{K_b}{K_b + K_g}\right)(K_b + K_g)^{1-\gamma} + \frac{(K_b + K_g)^{1-\gamma} - 1}{\rho(1-\gamma)}. \quad \text{(IA.10)}$$

Replacing equation (IA.10) into equation (IA.9) we obtain:

(IA.11)

$$0=\max_{i_b(\eta),i_g(\eta)}\left\{\begin{array}{l} g(\eta)\left\{-\rho-\frac{(1-\gamma)\gamma}{2}\left[\sigma_b^2\eta^2+\sigma_g^2(1-\eta)^2\right]\right. \\ \left.+g'(\eta)(1-\gamma)\left[(i_b(\eta)+\mu_b)\eta+(i_g(\eta)+\mu_g)(1-\eta)\right]+\lambda\eta\left[(1-\psi)^{1-\gamma}-1\right]\right\} \\ +\left[\eta(1-\eta)(i_b(\eta)+\mu_b-i_g(\eta)-\mu_g)-\gamma\sigma_b^2\eta^2(1-\eta)+\gamma\sigma_g^2\eta(1-\eta)^2\right] \\ +\frac{1}{2}g''(\eta)\eta^2(1-\eta)^2\left(\sigma_b^2+\sigma_g^2\right) \\ +\left[\eta\left(\alpha_b-i_b(\eta)-\frac{\theta_b}{2}i_b^2(\eta)\right)+(1-\eta)\left(\alpha_g-i_g(\eta)-\frac{\theta_g}{2}i_g^2(\eta)\right)\right]^{1-\gamma} \end{array}\right\}.$$

The optimal investment rates satisfy:

(IA.12)

$$\begin{aligned}0=(\theta_b i_b(\eta)+1)&\left[-i_b(\eta)\eta+\alpha_b\eta-\frac{\eta\theta_b i_b^2(\eta)}{2}-\frac{(1-\eta)\left(\theta_g i_g^2(\eta)+2i_g(\eta)-2\alpha_g\right)}{2}\right]^{-\gamma}\\ &+\frac{-1-\rho(1-\gamma)g(\eta)-\rho(1-\eta)g'(\eta)}{\rho},\end{aligned}$$

and

(IA.13)

$$\begin{aligned}0=(\theta_g i_g(\eta)+1)&\left[-i_b(\eta)\eta+\alpha_b\eta-\frac{\eta\theta_g i_b^2(\eta)}{2}-\frac{(1-\eta)\left(\theta_g i_g^2(\eta)+2i_g(\eta)-2\alpha_g\right)}{2}\right]^{-\gamma}\\ &+\frac{-1-\rho(1-\gamma)g(\eta)+\rho\eta g'(\eta)}{\rho}.\end{aligned}$$

For arbitrary values of γ, the system of DAEs (Differential-Algebraic Equations) (IA.11)-(IA.12)-(IA.13) cannot be reduced to a system of ODEs. The numerical method to solve the system is discussed in the Appendix for Section IV.A.

Appendix for Section IV.A

For arbitrary values of γ, we use the following iterative algorithm to solve the social planner problem and the competitive equilibrium under carbon/investment income taxes.

We begin by setting $\gamma = 1$, in which case the system of equations (IA.11)-(IA.12)-(IA.13) reduces to:

(IA.14)

$$0 = \max_{i_b(\eta), i_g(\eta)} \left\{ \begin{array}{l} -\rho g(\eta) + \eta (1-\eta) g'(\eta) \left[(i_b(\eta) + \mu_b - i_g(\eta) - \mu_g) - \sigma_b^2 \eta + \sigma_g^2 (1-\eta) \right] \\ +\frac{1}{2} g''(\eta) \eta^2 (1-\eta)^2 \left(\sigma_b^2 + \sigma_g^2 \right) \\ + \log \left[\eta \left(\alpha_b - i_b(\eta) - \frac{\theta_b}{2} i_b^2(\eta) \right) + (1-\eta) \left(\alpha_g - i_g(\eta) - \frac{\theta_g}{2} i_g^2(\eta) \right) \right] \\ +\frac{1}{\rho} \left\{ -\frac{1}{2} \left[\sigma_b^2 \eta^2 + \sigma_g^2 (1-\eta)^2 \right] + \lambda \eta \log(1-\psi) \right. \\ \left. + \left[(i_b(\eta) + \mu_b) \eta + (i_g(\eta) + \mu_g)(1-\eta) \right] \right\} \end{array} \right\},$$

(IA.15)

$$(1-\eta) g'(\eta) + \frac{1}{\rho} = \left[\eta \left(\alpha_b - i_b(\eta) - \frac{\theta_b}{2} i_b^2(\eta) \right) + (1-\eta) \left(\alpha_g - i_g(\eta) - \frac{\theta_g}{2} i_g^2(\eta) \right) \right]^{-1} (1 + \theta_b i_b(\eta)),$$

and

(IA.16)

$$(1 + \theta_g i_g(\eta)) = \frac{\left[-\eta g'(\eta) + \frac{1}{\rho} \right]}{\left[(1-\eta) g'(\eta) + \frac{1}{\rho} \right]} (1 + \theta_b i_b(\eta)).$$

We solve the system of equations (IA.15)-(IA.16) to obtain $i_b(\eta)$ and $i_g(\eta)$ in terms of $g(\eta)$, and replace them into equation (IA.14) to get an ordinary differential equation for $g(\eta)$, which we solve numerically together with the following boundary conditions from the single-sector economy:

$$g(0) = \frac{1}{2\rho^2\theta_g}\left\{-2\left(1+\sqrt{1+2\theta_g\alpha_g+\rho^2\theta_g^2}\right)+\theta_g\left(2\mu_g-\sigma_g^2-2\rho\right)\right.$$
$$\left.+2\rho\theta_g\log\left[-\rho\left(\rho\theta_g+\sqrt{1+2\theta_g\alpha_g+\rho^2\theta_g^2}\right)\right]\right\};$$

$$g(1) = \frac{1}{2\rho^2\theta_b}\left\{-2\left(1+\sqrt{1+2\theta_b\alpha_b+\rho^2\theta_b^2}\right)+\theta_b\left(2\mu_b-\sigma_b^2-2\rho\right)\right.$$
$$\left.+2\rho\theta_b\log\left[-\rho\left(\rho\theta_b+\sqrt{1+2\theta_b\alpha_b+\rho^2\theta_b^2}\right)\right]-2\theta_b\lambda\log(1-\psi)\right\}.$$

Next, we increase gamma by an increment of 0.1. We use the solution $g(\eta)$ obtained above as the initial guess to plug in and solve the system of algebraic equations (IA.12)-(IA.13). The numerical solutions $i_b(\eta), i_g(\eta)$ are then replaced back into ODE (IA.11), which we solve numerically and obtain a new function $g(\eta)$. We repeat the process, and stop when the second norm of the difference between two consecutive iterations is less than the tolerance level, which we choose to be 10^{-6}. We continue increasing γ by an increment of 0.1, and carry out the previous steps until we reach the desired level of γ.

The output of the iteration are denoted as $i_b^{FB}(\eta), i_g^{FB}(\eta)$, where the superscript FB stands for First-Best. Since under carbon/investment income taxes, the competitive equilibrium investment rates and valuation ratios are identical to the First-Best ones attained by the social planner, we have that the competitive equilibrium valuation ratios under taxation are:

$$p_b^{FB}(\eta) = \theta_b i_b^{FB}(\eta) + 1,$$

$$p_g^{FB}(\eta) = \theta_g i_g^{FB}(\eta) + 1.$$

The interest rate can be retrieved using equation (47), and the equity premia are calculated accordingly.

Appendix for Section V

(a) First, we establish the competitive market equilibrium without taxes where the two sectors have symmetric technologies, productivity, and adjustment costs.

The stock investor's problem is as follows:

$$\max_{C_{S,b},\, C_{S,g}} u_S(C_{S,b}, C_{S,g}) = \log\left(C_{S,b}^{\epsilon_S}\, C_{S,g}^{1-\epsilon_S}\right) \tag{IA.17}$$

$$s.t.\ C_{S,b}\,\xi(\eta) + C_{S,g} = C_S,$$

where $C_{S,b}$ and $C_{S,g}$ are the stock investor's consumption of brown and green output respectively, and C_S is the total consumption expenditure that represents the budget constraint.

The solutions are:

$$C_{S,b} = \frac{\epsilon_S\, C_S}{\xi(\eta)},\ C_{S,g} = (1-\epsilon_S)\, C_S. \tag{IA.18}$$

Therefore,

$$u_S\big(C_{S,b}, C_{S,g}\big) = \log\big(C_S\big) + \log\left[(1-\epsilon_S)\left(\frac{1-\epsilon_S}{\epsilon_S}\,\xi(\eta)\right)^{-\epsilon_S}\right].$$

Similarly, for the non-stock investor, his consumption of brown and green outputs are:

$$C_{NS,b} = \frac{\epsilon_{NS}\, C_{NS}}{\xi(\eta)},\ C_{NS,g} = (1-\epsilon_{NS})\, C_{NS}, \tag{IA.19}$$

and

$$u_{NS}\big(C_{NS,b}, C_{NS,g}\big) = \log\big(C_{NS}\big) + \log\left[(1-\epsilon_{NS})\left(\frac{1-\epsilon_{NS}}{\epsilon_{NS}}\,\xi(\eta)\right)^{-\epsilon_{NS}}\right].$$

When an agent decides how to choose his/her consumption in a given state of the world, he/she ignores the second part because it is taken as given. Thus, for a given budget set, the preferences will be the same as with $\log(C)$ preferences.

Therefore, at the partial equilibrium level, both agents will consume a constant fraction of her wealth:

$$C_S = \rho\, \kappa\, W,\; C_{NS} = \rho\,(1-\kappa)\, W. \tag{IA.20}$$

The aggregate consumption of green and brown goods respectively are:

$$C_b = C_{S,b} + C_{NS,b} = \frac{\rho\big(1-\widetilde{\epsilon}\big)\, W_t}{\xi(\eta)}, \qquad C_g = C_{S,g} + C_{NS,g} = \rho\, \widetilde{\epsilon}\, W, \tag{IA.21}$$

where $\widetilde{\epsilon} = (\kappa-1)\epsilon_{NS} - \kappa\, \epsilon_S + 1$.

The clearing conditions for the markets of green and brown goods require the above aggregate consumptions to equate the supplies of brown output S_b and green output S_g. Let us denote by $z_b(\eta) = \frac{Z_b(\eta)}{K_b}$ and $z_g(\eta) = \frac{Z_g(\eta)}{K_g}$ the fractions of brown and green output that are not sold for consumption but instead used to produce investment goods. Then:

$$S_b(\eta) = \big(\alpha - z_b(\eta)\big)\, K_b, \qquad S_g(\eta) = \big(\alpha - z_g(\eta)\big)\, K_g. \tag{IA.22}$$

From equations (IA.22) and (IA.21), by equating the supply and demand of each type of output, we obtain:

$$\begin{aligned} \frac{\rho\, W\Big((1-\kappa)\epsilon_{NS} + \kappa\epsilon_S\Big)}{\xi(\eta)} &= \big(\alpha - z_b(\eta)\big)\, K_b \\ \iff \; \xi(\eta) &= \frac{\rho\,(1-\widetilde{\epsilon})\, \widetilde{p}(\eta)}{\eta\,\Big(z_b(\eta) - \alpha\Big)}, \end{aligned} \tag{IA.23}$$

and

$$\begin{aligned} \rho\, W\Big(1 - (1-\kappa)\epsilon_{NS} - \kappa\epsilon_S\Big) &= \big(\alpha - z_g(\eta)\big) K_g \\ \iff \; z_g(\eta) &= \alpha + \frac{\rho\, \widetilde{\epsilon}\, \widetilde{p}(\eta)}{1-\eta}, \end{aligned} \tag{IA.24}$$

where $\widetilde{p}(\eta) = (\eta - 1)\, p_g(\eta) - \eta\, p_b(\eta)$, and $\widetilde{\epsilon} = (\kappa - 1)\,\epsilon_{NS} - \kappa\,\epsilon_S + 1$.

By equation (31), it follows that:

$$Z_g(\eta) = Z_b(\eta) \iff z_b(\eta) = z_g(\eta)\left(\frac{1-\eta}{\eta}\right), \tag{IA.25}$$

and thus the price of a unit of investment goods is:

$$\chi(\eta) = \frac{\xi(\eta) + 1}{2}. \tag{IA.26}$$

The market clearing condition for the investment goods market requires:

$$2\, z_g(\eta)\, K_g = 2\, z_b(\eta)\, K_b = g\big(i_b(\eta)\big)\, K_b + g\big(i_g(\eta)\big)\, K_g \tag{IA.27}$$

$$\iff z_g(\eta) = \frac{\left[\big(\xi(\eta)+1\big)^2 - 4\,\eta\, p_b^2(\eta) + 4\,(\eta-1)\, p_g^2(\eta)\right]}{4\,\theta\,(\eta-1)\,\big(\xi(\eta)+1\big)^2}, \tag{IA.28}$$

where $g(i_n) = i_n + \theta\,\frac{i_n^2}{2}$.

The value function of the stock investors, $G^S\left(K_b; K_g\right) = sup_{i_b, i_g}\left\{\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log\left(C_{S,b}^{\epsilon_S}\, C_{S,g}^{1-\epsilon_S}\right)dt\right]\right\}$, satisfies the HJB equation:

$$0 = \max_{i_b(\eta), i_g(\eta)}\begin{Bmatrix} -\rho\, G^S\left(K_b; K_g\right) + \log\left(C_{S,b}^{\epsilon_S}\, C_{S,g}^{1-\epsilon_S}\right) \\ +G^S_{K_b}\left(K_b; K_g\right) K_b\left[i_b\left(\eta\right) + \mu\right] + G^S_{K_g}\left(K_b; K_g\right) K_g\left[i_g\left(\eta\right) + \mu\right] \\ +\frac{1}{2} G^S_{K_b K_b}\left(K_b; K_g\right) K_b^2 \sigma_b^2 + \frac{1}{2} G^S_{K_g K_g}\left(K_b; K_g\right) K_g^2 \sigma_g^2 \\ +\lambda\,\eta\left[G^S\big(K_b - K_b\,\psi; K_g - K_g\,\psi\big) - G^S\big(K_b; K_g\big)\right]\end{Bmatrix}. \tag{IA.29}$$

Following equations (IA.18), (IA.19), (IA.22) and (IA.27), the consumption must satisfy:

$$C_{S,b} = \frac{\kappa\,\epsilon_S\,\left(\alpha - z_b(\eta)\right) K_b}{\kappa\,\epsilon_S + (1-\kappa)\epsilon_{NS}} = \frac{\kappa\,\epsilon_S\, K_b\left[\alpha - \frac{1}{2}\left(g\big(i_b(\eta)\big) + \frac{1-\eta}{\eta} g\big(i_g(\eta)\big)\right)\right]}{\kappa\,\epsilon_S + (1-\kappa)\epsilon_{NS}}, \tag{IA.30}$$

and

(IA.31)

$$C_{S,g} = \frac{\kappa\ (1-\epsilon_S)\ (\alpha - z_g(\eta))\, K_g}{\kappa\ (1-\epsilon_S) + (1-\kappa)\,(1-\epsilon_{NS})} = \frac{\kappa\ (1-\epsilon_S)\ K_g \left[\alpha - \frac{1}{2}\left(\frac{\eta}{1-\eta} g\big(i_b(\eta)\big) + g\big(i_g(\eta)\big)\right)\right]}{\kappa\ (1-\epsilon_S) + (1-\kappa)\,(1-\epsilon_{NS})}.$$

We guess (and subsequently verify) that:

$$G^S\left(K_b; K_g\right) = f^S\left(\frac{K_b}{K_b+K_g}\right) + \frac{\log(K_b+K_g)}{\rho}. \tag{IA.32}$$

Substitute equations (IA.30), (IA.31), and (IA.32) into equation (IA.29) we obtain:

(IA.33)

$$0 = \max_{i_b(\eta), i_g(\eta)} \left\{ \begin{array}{l} \frac{1}{2}\eta^2\,(\eta-1)^2\left(\sigma_b^2+\sigma_g^2\right) f^{S''}(\eta) - \eta\,(\eta-1)\left(i_b - i_g - \eta\sigma_b^2 + (1-\eta)\sigma_g^2\right) f^{S'}(\eta) \\ -\rho f^S(\eta) + \epsilon_S\ \log\left[\epsilon_S \times \frac{-\eta\left(-4\alpha+\theta {i_b}^2+2i_b\right)+\theta(\eta-1){i_g}^2+2(\eta-1)i_g}{\theta(1-\widetilde{\epsilon})}\right] \\ +\,(1-\epsilon_S)\ \log\left[(1-\epsilon_S) \times \frac{4\alpha-\eta\left(4\alpha+\theta i_b^2+2i_b\right)+\theta(\eta-1){i_g}^2+2(\eta-1)i_g}{\theta\widetilde{\epsilon}}\right] \\ +\frac{\eta\, i_b+(1-\eta)\, i_g}{\rho} - \frac{\eta^2\sigma_b^2+(\eta-1)^2\sigma_g^2}{2\rho} + \lambda\eta\frac{\log(1-\psi)}{\rho} + \log\left(\frac{\kappa\,\theta}{4}\right) + \frac{\mu}{\rho} \end{array} \right\}.$$

The FOCs for i_b and i_g result in:

$$\begin{aligned} 0 = \eta\,(1-\eta) f^{S'}(\eta) &- \frac{2\,\eta\,(\epsilon_S-1)\,(1+\theta\, i_b)}{(\eta-1)\left(4\alpha-\theta\, {i_g}^2-2i_g\right)+\eta\left(\theta\, {i_b}^2+2i_b\right)} \\ &+ \frac{2\,\eta\,\epsilon_S\,(1+\theta\, i_b)}{\eta\left(-4\alpha+\theta\, {i_b}^2+2i_b\right)-(\eta-1)\left(\theta\, {i_g}^2+2i_g\right)} + \frac{\eta}{\rho}, \end{aligned} \tag{IA.34}$$

and

$$\begin{aligned} 0 = &-\frac{2(\eta-1)\,(\epsilon_S-1)\,(1+\theta\, i_g) - 2\eta\,(\epsilon_S-1)\,(1+\theta\, i_b)}{(\eta-1)\left(4\alpha-\theta\, {i_g}^2-2i_g\right)+\eta\left(\theta\, {i_b}^2+2i_b\right)} \\ &+ \frac{2(\eta-1)\,\epsilon_S\,(1+\theta\, i_g) - 2\eta\,\epsilon_S\,(1+\theta\, i_b)}{\eta\left(-4\alpha+\theta\, {i_b}^2+2i_b\right)-(\eta-1)\left(\theta\, {i_g}^2+2i_g\right)} - \frac{1}{\rho}. \end{aligned} \tag{IA.35}$$

The system of differential-algebraic equations (IA.33)-(IA.34)-(IA.35) thus characterizes the market equilibrium.

Under condition (1), we provide a full analytical characterization of the steady-state market equilibrium. In this case, the law of motion of η is:

$$d\eta = \eta\,(1-\eta)\,\big[i_b(\eta) - i_g(\eta)\big]\,dt. \tag{IA.36}$$

At the steady state $\eta = \eta_{SS}$, it must be that $d\eta = 0$. Thus, firms' investment rates satisfy:

$$i_b(\eta_{SS}) = i_g(\eta_{SS}) \iff p_b(\eta_{SS}) = p_g(\eta_{SS}), \tag{IA.37}$$

following equation (33).

In addition, since both sectors share the same exposure to the climate risk, at the steady state, the (after-tax) dividend yields for both types of firms have to be identical:

$$\begin{aligned} &\frac{\alpha\,\xi(\eta_{SS}) - \chi(\eta_{SS})\,g\big(i_b(\eta_{SS})\big)}{p_b(\eta_{SS})} = \frac{\alpha - \chi(\eta_{SS})\,g\big(i_g(\eta_{SS})\big)}{p_g(\eta_{SS})} \\ &\iff z_g(\eta_{SS}) = -\frac{\alpha\,(\eta_{SS} - 1 + \widetilde{\epsilon})}{(\eta_{SS} - 1)\,(2\,\widetilde{\epsilon} - 1)}. \end{aligned} \tag{IA.38}$$

Evaluating equations (IA.23), (IA.24), (IA.25), (IA.26), (IA.28) at η_{SS}, and combining them with equations (IA.37), (IA.38), we obtain:

$$\begin{aligned} p_b(\eta_{SS}) = p_g(\eta_{SS}) = p_{SS} &= -\rho\,\theta + \sqrt{2\alpha\,\theta + \rho^2\,\theta^2 + 1}, \\ \eta_{SS} &= \frac{1}{2} - \frac{\rho\,\big(2\,\widetilde{\epsilon} - 1\big)\,p_{SS}}{2\alpha}, \\ z_g(\eta_{SS}) &= \frac{\alpha^2 - \alpha\,\rho\,p_{SS}}{\alpha + \rho\,\big(2\,\widetilde{\epsilon} - 1\big)\,p_{SS}}, \\ z_b(\eta_{SS}) &= \frac{\alpha^2 - \alpha\,\rho\,p_{SS}}{\alpha - \rho\,\big(2\,\widetilde{\epsilon} - 1\big)\,p_{SS}}, \\ \chi(\eta_{SS}) &= \xi(\eta_{SS}) = 1, \end{aligned} \tag{IA.39}$$

where $\widetilde{\epsilon} = (\kappa - 1)\,\epsilon_{NS} - \kappa\,\epsilon_S + 1$.

The equilibrium interest rate is:

$$r_{SS} = \frac{\alpha\big[2\,(\kappa-\psi)\,(p_{SS}+\rho\,\theta+\mu\,\theta-1)-\kappa\,\theta\,\lambda\,\psi\big]+\kappa\,\theta\,\lambda\,\psi\,\rho\,\big(2\,\widetilde{\epsilon}-1\big)\,p_{SS}}{2\,\alpha\,\theta\,(\kappa-\psi)}.$$

We verify that under conditions (1), $\eta_{SS}\in(0,1)$, and $z_g(\eta_{SS})$, $z_b(\eta_{SS})\in(0,\alpha)$. The steps are as follows: First,

$$\frac{z_b(\eta_{SS})}{z_g(\eta_{SS})} = \frac{\alpha+\rho\,\big(2\,\widetilde{\epsilon}-1\big)\,p_{SS}}{\alpha-\rho\,\big(2\,\widetilde{\epsilon}-1\big)\,p_{SS}}.$$

Thus,

$$\begin{aligned}\frac{z_b(\eta_{SS})}{z_g(\eta_{SS})}+1 &= \frac{2\alpha}{\alpha-\rho\,\big(2\,\widetilde{\epsilon}-1\big)\,p_{SS}} = \frac{1}{\eta_{SS}}\\ \iff \frac{z_b(\eta_{SS})}{z_g(\eta_{SS})} &= \frac{1}{\eta_{SS}}-1.\end{aligned}$$

If $0<\eta_{SS}<1$ holds, then $z_b(\eta_{SS})$ has the same sign as $z_g(\eta_{SS})$. Therefore, if one of them is non-negative, so is the other.

Next, note that as $2\,\widetilde{\epsilon}-1<1$,

$$\eta_{SS} = \frac{\alpha-\rho\left(2\,\widetilde{\epsilon}-1\right)\,\left(-\rho\,\theta+\sqrt{2\alpha\,\theta+\rho^2\,\theta^2+1}\right)}{2\alpha} > 0,$$

thanks to conditions (1).

We also have $\eta_{SS}<1$ if and only if:

$$\alpha+\rho\left(2\,\widetilde{\epsilon}-1\right)\,\left(-\rho\,\theta+\sqrt{2\alpha\,\theta+\rho^2\,\theta^2+1}\right) > 0. \tag{IA.40}$$

In addition, $z_g(\eta_{SS})>0$ if and only if:

$$\frac{\alpha^2-\alpha\,\rho\,p_{SS}}{\alpha+\rho\left(2\,\widetilde{\epsilon}-1\right)\,p_{SS}} > 0. \tag{IA.41}$$

If inequality (IA.40) holds, then inequality (IA.41) also holds if and only if:

$$\begin{aligned} & \alpha - \rho\, p_{SS} > 0 \\ \iff\ & \frac{\alpha}{\rho} > -\rho\,\theta + \sqrt{2\alpha\,\theta + \rho^2\,\theta^2 + 1}, \end{aligned} \tag{IA.42}$$

which is satisfied under conditions (1). Because of inequality (IA.42), for inequality (IA.40) to hold, it is sufficient to have:

$$\begin{aligned} & 1 - 2\,\widetilde{\epsilon} \leq 1 \\ \iff\ & \widetilde{\epsilon} = (\kappa - 1)\,\epsilon_{NS} - \kappa\,\epsilon_S + 1 \geq 0, \end{aligned}$$

which is indeed true.

It is trivial that $z_g(\eta_{SS}) < \alpha$, and $z_b(\eta_{SS}) < \alpha$. Hence, under conditions (1), $\eta_{SS} \in (0,1)$, and $z_g(\eta_{SS})$, $z_b(\eta_{SS}) \in (0, \alpha)$.

(b) Second, after characterizing the market equilibrium without taxes, we move onto the equilibrium under the budget-neutral carbon tax scheme $\delta = (\delta_g, \delta_b)$ that satisfies condition (24). In this case, equation (IA.38) changes to:

$$\begin{aligned} & \frac{\alpha\,\xi(\eta_{SS}) - \chi(\eta_{SS})\,g\big(i_b(\eta_{SS})\big) - \alpha\,\delta_b}{p_b(\eta_{SS})} = \frac{\alpha - \chi(\eta_{SS})\,g\big(i_g(\eta_{SS})\big) - \alpha\,\delta_g}{p_g(\eta_{SS})} \\ \iff\ & z_g(\eta^\delta_{SS}) = \frac{\alpha\big[\widetilde{\epsilon}\,(\delta_b\,\eta^\delta_{SS} - \eta^\delta_{SS} + 1) - (1 - \eta^\delta_{SS})^2\big]}{(\eta^\delta_{SS} - 1)\big[(\eta^\delta_{SS} - 1)\,(2\,\widetilde{\epsilon} - 1) - \delta_b\widetilde{\epsilon}\big]}. \end{aligned} \tag{IA.43}$$

The steady-state equilibrium η^δ_{SS} arising from the system of equations (IA.23), (IA.24), (IA.25), (IA.26), (IA.28), (IA.37), (IA.43) is the solution to the fixed-point problem:

$$\eta^\delta_{SS} = \frac{4\,\alpha\,(\eta^\delta_{SS} - 1)^2(\delta_b - \eta^\delta_{SS} + 1)^2 + \rho\varOmega\,(\delta_b - 2\,\eta^\delta_{SS} + 2)\,(\varTheta - \varPsi)}{8\,\alpha\,(\eta^\delta_{SS} - 1)^4\left(\frac{\delta_b}{1 - \eta^\delta_{SS}} + 1\right)^2},$$

where:

$$\Omega = (\delta_b - 2\,\eta_{SS}^{\delta} + 2)\,(\widetilde{\epsilon} - 1) + \delta_b - \eta_{SS}^{\delta} + 1,$$
$$\Psi = \sqrt{4\,(\eta_{SS}^{\delta} - 1)^2 (2\,\alpha\,\theta + 1)\,(\delta_b - \eta_{SS}^{\delta} + 1)^2 + \Theta^2},$$
$$\Theta = \theta\rho\,(\delta_b - 2\,\eta_{SS}^{\delta} + 2)\left[\delta_b\,(\widetilde{\epsilon} - 1) + \delta_b - \eta_{SS}^{\delta} + 1\right].$$

In addition, other steady-state quantities are:

$$\xi(\eta_{SS}^{\delta}) = 1 + \frac{\delta_b}{1 - \eta_{SS}^{\delta}},$$
$$\chi(\eta_{SS}^{\delta}) = 1 + \frac{\delta_b}{2\left(1 - \eta_{SS}^{\delta}\right)},$$
$$z_b(\eta_{SS}^{\delta}) = \left(\frac{1}{\eta_{SS}^{\delta}} - 1\right) z_g(\eta_{SS}^{\delta}),$$
$$p_b(\eta_{SS}^{\delta}) = p_g(\eta_{SS}^{\delta}) = p_{SS}^{\delta} = \frac{(\eta_{SS}^{\delta} - 1)\Big(z_g(\eta_{SS}^{\delta}) - \alpha\Big)}{\rho\,\widetilde{\epsilon}}.$$

(c) To show Proposition (5), we first note that under conditions (1), the value function of the non-stock investor is:

(IA.44)
$$f^{NS}\left(\eta_{SS}^{\delta}\right) = \frac{\mu + i_{SS}^{\delta} + \lambda\,\eta_{SS}^{\delta}\,\log(1-\psi) + \rho\,\log\left(\widehat{p_{SS}^{\delta}}\right) - \rho\,\epsilon_{NS}\,\log\left[\xi(\eta_{SS}^{\delta})\left(\frac{1}{\epsilon_{NS}} - 1\right)\right]}{\rho^2},$$

where $i_{SS}^{\delta} = \frac{p_{SS}^{\delta} - \chi(\eta_{SS}^{\delta})}{\theta\,\chi(\eta_{SS}^{\delta})}$, and $\widehat{p_{SS}^{\delta}} = \rho\,(\kappa - 1)\,(\epsilon_{NS} - 1)\,p_{SS}^{\delta}$.

Introducing an (infinitesimally) small carbon tax would make the non-stock investor strictly worse off if and only if:

$$\frac{\partial f^{NS}}{\partial \delta_b}\,|_{\delta_b = 0} \leq 0.$$

Taking the derivative of f^{NS} with respect to δ_b, and evaluating at 0, we rewrite the above inequality as:

$$0 \geq \frac{1}{4\rho\,\theta\,\alpha^2\,\Phi\,(\Phi-\rho)}\Bigg\{\lambda\log(1-\psi)\Big[-4\,\rho^2\,\theta^2\,(\Phi-\rho)\,\Big(2(\widetilde{\epsilon}-1)\widetilde{\epsilon}+1\Big)+2\alpha\,\theta\Big(4\rho\,(\widetilde{\epsilon}-1)\,\widetilde{\epsilon}+3\rho-\Phi\Big)$$
$$+\,4\rho\,(\widetilde{\epsilon}-1)\,\widetilde{\epsilon}+3\rho-\Phi\Big]-\frac{4\,\alpha\,\kappa\,(\epsilon_{NS}-\epsilon_S)\,\Big(\rho\,\theta^2\,(\rho-\Phi)+2\alpha\,\theta+1\Big)}{\theta}\Bigg\}$$

(IA.45)

$$\iff\quad 0\geq\lambda\log(1-\psi)\,\Big[4\rho^2\theta^2\,(\rho-\Phi)\,\Big(2\,(\widetilde{\epsilon}-1)\,\widetilde{\epsilon}+1\Big)+(2\alpha\,\theta+1)\Big(4\rho\,(\widetilde{\epsilon}-1)\,\widetilde{\epsilon}+3\rho-\Phi\Big)\Big]$$
$$-\frac{4\,\alpha\,\kappa\,(\epsilon_{NS}-\epsilon_S)\,\Big[\rho\,\theta^2\,(\rho-\Phi)+2\alpha\,\theta+1\Big]}{\theta},$$

where $\widetilde{\epsilon}=(\kappa-1)\,\epsilon_{NS}-\kappa\,\epsilon_S+1$, and $\Phi=\sqrt{\frac{2\alpha}{\theta}+\frac{1}{\theta^2}+\rho^2}$.

Note that $2\alpha\,\theta+1=\theta^2\,(\Phi^2-\rho^2)$. Inequality (IA.45) is equivalent to:

$$\frac{4\,\alpha\,\kappa\,(\epsilon_{NS}-\epsilon_S)}{\theta\,\lambda\log(1-\psi)}\leq\frac{4\rho^2\,(\rho-\Phi)\,\Big(2\,(\widetilde{\epsilon}-1)\,\widetilde{\epsilon}+1\Big)+(\Phi^2-\rho^2)\,\Big(4\rho\,(\widetilde{\epsilon}-1)\,\widetilde{\epsilon}+3\rho-\Phi\Big)}{\rho\,\,(\rho-\Phi)+(\Phi^2-\rho^2)}$$

(IA.46)
$$\iff\quad\frac{4\,\alpha\,\kappa\,(\epsilon_{NS}-\epsilon_S)}{\theta\,\lambda\log(1-\psi)}\leq\rho\,(2\,\widetilde{\epsilon}-1)^2\,\Big(1-\frac{\rho}{\Phi}\Big)+\rho-\Phi.$$

For inequality (IA.46) to hold, it is sufficient to have the following equivalent of (39):

$$\frac{4\,\alpha\,\kappa\,(\epsilon_{NS}-\epsilon_S)}{\theta\,\lambda\log(1-\psi)}\leq\rho-\Phi.$$

Alternative production functions in the investment goods market

With Cobb-Douglas production function for the investment good:

$$I=Z_b^{\zeta}\,Z_g^{1-\zeta},$$

equation (IA.25) changes to:

$$\text{(IA.47)} \qquad \frac{Z_b\,\widetilde{\xi(\eta)}}{Z_g} = \frac{\zeta}{1-\zeta} \iff z_b(\eta) = \frac{z_g(\eta)}{\widetilde{\xi(\eta)}} \left(\frac{1-\eta}{\eta}\right) \left(\frac{\zeta}{1-\zeta}\right),$$

where $\widetilde{\xi(\eta)}$ denotes the price of brown output. Hence, the price of a unit of investment goods is:

$$\text{(IA.48)} \qquad \widetilde{\chi(\eta)} = \frac{\left[\left(\frac{1-\zeta}{\zeta}\right)\widetilde{\xi(\eta)}\right]^{\zeta}}{1-\zeta}.$$

The market clearing condition for the investment goods implies:

$$\big(z_b(\eta)\,K_b\big)^{\zeta}\big(z(\eta)\,K_g\big)^{1-\zeta} = g\big(i_b(\eta)\big)\,K_b + g\big(i_g(\eta)\big)\,K_g$$

(IA.49)

$$\iff z_g(\eta) = \frac{\left[\left(\frac{1-\zeta}{\zeta}\right)\widetilde{\xi(\eta)}\right]^{-\zeta}\left[(1-\zeta)^2\,(\eta-1)\,p_g^2(\eta) + (1-\zeta)^2\,\eta\,p_b^2(\eta) - \left(\left(\frac{1-\zeta}{\zeta}\right)\widetilde{\xi(\eta)}\right)^{2\zeta}\right]}{2\theta\,\big(\eta-1\big)}.$$

At the steady state $\eta_{SS}^{\delta *}$ under the budget-neutral carbon tax scheme $\delta^* = (\delta_g^*, \delta_b^*)$, the (after-tax) dividend yields for both types of firms have to be identical, which requires:

$$\frac{\alpha\,\widetilde{\xi(\eta_{SS}^{\delta *})} - \widetilde{\chi(\eta_{SS}^{\delta *})}\,g\big(i_b(\eta_{SS}^{\delta *})\big) - \alpha\,\delta_b^*}{p_b(\eta_{SS}^{\delta *})} = \frac{\alpha - \widetilde{\chi(\eta_{SS}^{\delta *})}\,g\big(i_g(\eta_{SS}^{\delta *})\big) - \alpha\,\delta_g^*}{p_g(\eta_{SS}^{\delta *})}$$

$$\text{(IA.50)} \qquad \iff z_g(\eta_{SS}^{\delta *}) = \frac{\alpha\left[\widetilde{\epsilon}\,(\delta_b^*\,\eta_{SS}^{\delta *} - \eta_{SS}^{\delta *} + 1) - (1-\eta_{SS}^{\delta *})^2\right]}{(\eta_{SS}^{\delta *} - 1)\left[(\eta_{SS}^{\delta *} - 1)\,(2\,\widetilde{\epsilon} - 1) - \delta_b^*\,\widetilde{\epsilon}\right]}.$$

Evaluating equations (IA.23), (IA.24), (IA.47), (IA.48), (IA.49) at $\eta_{SS}^{\delta *}$, and combining them with equations (IA.37), (IA.50), we obtain:

$$\widetilde{\xi(\eta_{SS}^{\delta*})} = 1 + \frac{\delta_b^*}{1-\eta_{SS}^{\delta*}},$$

$$\widetilde{\chi(\eta_{SS}^{\delta*})} = \frac{\left[\left(\frac{1-\zeta}{\zeta}\right)\left(1+\frac{\delta_b^*}{1-\eta_{SS}^{\delta*}}\right)\right]^{\zeta}}{1-\zeta},$$

$$z_b(\eta_{SS}^{\delta*}) = \left(\frac{1}{\eta_{SS}^{\delta*}} - 1\right)\left(\frac{\zeta}{1-\zeta}\right)\frac{z_g(\eta_{SS}^{\delta*})}{1+\frac{\delta_b^*}{1-\eta_{SS}^{\delta*}}},$$

$$p_b(\eta_{SS}^{\delta*}) = p_g(\eta_{SS}^{\delta*}) = p_{SS}^{\delta*} = \frac{(\eta_{SS}^{\delta*}-1)\Big(z_g(\eta_{SS}^{\delta*})-\alpha\Big)}{\rho\,\widetilde{\epsilon}}.$$

In this case, $\eta_{SS}^{\delta*}$ is the solution to the fixed-point problem:

$$\eta_{SS}^{\delta*} = \frac{(\eta_{SS}^{\delta*}-1)\left[\Theta'\sqrt{\Xi} - \alpha(\zeta-1)\,\theta\rho^2\left(\frac{\Omega'}{\zeta(\eta_{SS}^{\delta*}-1)}\right)^{\zeta}(\zeta+\widetilde{\epsilon}-1)(-\delta_b^*\,\widetilde{\epsilon}+\eta_{SS}^{\delta*}-1) + \alpha^2(\zeta-1)^2\zeta\,\Psi'\right]}{\alpha^2(\zeta-1)^2\,\Psi'^2},$$

where:

$$\Xi = \frac{\alpha^2(\zeta-1)^2\left(\frac{\alpha\,\zeta^2(\eta_{SS}^{\delta*}-1)}{\Omega'}\right)^{-\zeta}\left[2\,\alpha\,\theta\,(\Upsilon-\Pi) + \left(\frac{\alpha\,\Omega'}{\eta_{SS}^{\delta*}-1}\right)^{\zeta}\left(\Psi'^2+\theta^2\rho^2(-\delta_b^*\,\widetilde{\epsilon}+\eta_{SS}^{\delta*}-1)^2\right)\right]}{\theta(\eta_{SS}^{\delta*}-1)\Theta'},$$

$$\Omega' = (\zeta-1)(\delta_b^*-\eta_{SS}^{\delta*}+1),$$

$$\Psi' = \delta_b^*(\zeta-1)+\eta_{SS}^{\delta*}-1,$$

$$\Theta' = \theta\rho^2(\eta_{SS}^{\delta*}-1)(\zeta+\widetilde{\epsilon}-1)^2,$$

$$\Upsilon = (\alpha\,\zeta)^{\zeta}\left[\delta_b^{*2}\,(\zeta-1)^2 + 3\,\delta_b^*\,\zeta\,(\eta_{SS}^{\delta*}-1) - \zeta\,(\eta_{SS}^{\delta*}-1)^2\right],$$

$$\Pi = (\alpha\,\zeta)^{\zeta}\,(\eta_{SS}^{\delta*}-1)\left[\delta_b^*\left(\zeta^2+2\right) - \eta_{SS}^{\delta*}+1\right].$$

Carbon taxes would fail to gain a political majority under the following condition:

$$\frac{\partial\widetilde{f^{NS}}}{\partial\delta_b^*}\,|_{\delta_b^*=0} \leq 0, \tag{IA.51}$$

where $\widetilde{f^{NS}}$ denotes the standardized value function of the non-stock investor.

The form of $\widetilde{f^{NS}}$ is identical to the one in equation (IA.44), which is:

$$\text{(IA.52)}\quad \widetilde{f^{NS}}\left(\eta_{SS}^{\delta *}\right)=\frac{\mu+i_{SS}^{\delta}+\lambda\,\eta_{SS}^{\delta *}\,\log(1-\psi)+\rho\,\log\left(\widehat{p_{SS}^{\delta *}}\right)-\rho\,\epsilon_{NS}\,\log\left[\widetilde{\xi(\eta_{SS}^{\delta *})}\left(\frac{1}{\epsilon_{NS}}-1\right)\right]}{\rho^{2}},$$

where $i_{SS}^{\delta}=\frac{p_{SS}^{\delta *}-\widetilde{\chi(\eta_{SS}^{\delta *})}}{\theta\,\widetilde{\chi(\eta_{SS}^{\delta *})}}$, and $\widehat{p_{SS}^{\delta *}}=\rho\,(\kappa-1)\,(\epsilon_{NS}-1)\,p_{SS}^{\delta *}$.

Taking the derivative of $\widetilde{f^{NS}}\left(\eta_{SS}^{\delta *}\right)$ and evaluate at $\delta_b^*=0$, we can rewrite inequality (IA.51) as:

$$\text{(IA.53)}\quad \begin{aligned}0\geq M=&-p\,(\alpha,\zeta,\kappa,\lambda,\psi,\theta,\rho,\lambda)\,(\epsilon_{NS}-\epsilon_{S})^{2}-q\,(\alpha,\zeta,\kappa,\lambda,\psi,\theta,\rho,\lambda)\,(\epsilon_{NS}-\epsilon_{S})\\&-s\,(\alpha,\zeta,\kappa,\lambda,\psi,\theta,\rho,\lambda,\epsilon_{NS},\widetilde{\epsilon})\,,\end{aligned}$$

where:

$$p\,(\alpha,\zeta,\kappa,\lambda,\psi,\theta,\rho,\lambda)=\frac{2\,\theta^{2}\,\kappa^{2}\,\lambda\,\log(1-\psi)(\alpha-\alpha\,\zeta)^{\zeta}}{2\,\theta\,\alpha^{2}(\zeta-1)(\alpha\,\zeta)^{\zeta}-(\alpha-\alpha\,\zeta)^{\zeta}\,(1+\rho^{2}\,\theta^{2})},$$

$$q\,(\alpha,\zeta,\kappa,\lambda,\psi,\theta,\rho,\lambda)=\frac{\kappa}{\alpha\,\rho}+\frac{2\,\kappa\,\theta^{2}\lambda\,\log(1-\psi)(1-\zeta)^{\zeta-1}\,(-2\,\zeta^{2}+(2\,\epsilon_{NS}+1)(\zeta-1))}{2\,\alpha^{2}\,\theta(\zeta-1)\zeta^{\zeta}-(1-\zeta)^{\zeta}\,(1+\rho^{2}\,\theta^{2})},$$

and

$$\begin{aligned}&s\,(\alpha,\zeta,\kappa,\lambda,\psi,\theta,\rho,\lambda,\epsilon_{NS},\widetilde{\epsilon})=\\&\frac{-\lambda\,\log(1-\psi)\left(\frac{\alpha}{1-\zeta}\right)^{1-\zeta/2}}{\alpha^{3}\,\rho\big[2\,\alpha\,\theta(\zeta-1)\zeta^{\zeta}-(1-\zeta)^{\zeta}\,(1+\rho^{2}\theta^{2})\,\big]}\,\big[2\,\theta\,\zeta^{1+\zeta}\,(1-\zeta)^{3-\zeta/2}\,\alpha^{2+\zeta/2}\\&+\zeta\,(1-\zeta)^{2+\zeta/2}\,\alpha^{1+\zeta/2}+\rho\,\theta\,\vartheta\,\zeta^{2}(1-\zeta)^{\zeta}\,\big(2+3\,\rho^{2}\,\theta^{2}\big)\\&-\rho^{2}\,\theta\,\zeta^{-\zeta}\,\alpha^{\zeta/2}\,(1-\zeta)^{3\zeta/2}\,\big(1+\rho^{2}\,\theta^{2}\big)\,(3\,\zeta+\widetilde{\epsilon}-2)\widetilde{\zeta}\\&+\rho^{2}\,\theta^{2}\,\zeta(1-\zeta)^{\zeta/2}\,\alpha^{\zeta/2+1}\,\big(7\,\zeta^{2}-2\,\zeta-4\,\zeta\,\epsilon_{NS}-1\big)\\&+2\,\rho^{2}\,\theta^{2}\,(\zeta-1)\,(1-\zeta)^{\zeta/2}\,(\epsilon_{NS}^{2}+\epsilon_{NS})\,\alpha^{\zeta/2+1}\\&+\rho\,\theta\,\vartheta\,(1-\zeta)^{\zeta}\big[\big((2\zeta-1)(\widetilde{\epsilon}-1)-\zeta\big)+\rho^{2}\,\theta^{2}\big((\widetilde{\epsilon}-1)(4\zeta+\widetilde{\epsilon}-2)-\zeta\big)\big]\\&+2\,\alpha\,\rho\,\theta^{2}\,\Big[2\,\zeta\,\rho\,(\alpha-\alpha\zeta)^{\zeta/2}\big(1-2\zeta+(\zeta-2)(\widetilde{\epsilon}-1)\big)-\vartheta\,\zeta^{\zeta}\,(\zeta-1)(2\,\zeta-1)\,\widetilde{\zeta}\Big]\Big]\,,\end{aligned}$$

$\vartheta = \frac{1}{\theta}\sqrt{\zeta^{-2\zeta}\left[(\alpha-\alpha\,\zeta)^{\zeta}\,(1+\rho^2\,\theta^2) - 2\,\alpha\,\theta\,(\zeta-1)\,(\alpha\zeta)^{\zeta}\right]}$, $\widetilde{\zeta} = \zeta + \widetilde{\epsilon} - 1$, and $\widetilde{\epsilon} = (\kappa - 1)\,\epsilon_{NS} - \kappa\,\epsilon_S + 1$.

Suppose that $0 \leq \zeta \leq 1 - \varepsilon$, where ε is arbitrarily small, so that all the above functions are well-behaved. We observe that $p\,(\alpha, \zeta, \kappa, \lambda, \psi, \theta, \rho, \lambda) > 0$. There are three possibilities with regards to the polynomial M on the right-hand side of inequality (IA.53), which we consider separately below:

- If $\Delta^* = q\,(\alpha, \zeta, \kappa, \lambda, \psi, \theta, \rho, \lambda)^2 - 4p\,(\alpha, \zeta, \kappa, \lambda, \psi, \theta, \rho, \lambda) \times s\,(\alpha, \zeta, \kappa, \lambda, \psi, \theta, \rho, \lambda, \epsilon_{NS}, \widetilde{\epsilon}) < 0$, then:

$$\frac{\partial \widetilde{f^{NS}}}{\partial \delta_b^*}\,|_{\delta_b^*=0} = M < 0.$$

- If $\Delta^* = 0$,then:

$$\frac{\partial \widetilde{f^{NS}}}{\partial \delta_b^*}\,|_{\delta_b^*=0} = M \leq 0.$$

- If $\Delta^* > 0$,for $M \leq 0$, for inequality (IA.53) to hold, it is sufficient to have:

$$\epsilon_{NS} - \epsilon_S \geq \frac{-q\,(\alpha, \zeta, \kappa, \lambda, \psi, \theta, \rho, \lambda) + \sqrt{\Delta^*}}{2p\,(\alpha, \zeta, \kappa, \lambda, \psi, \theta, \rho, \lambda)} = u\,(\alpha, \zeta, \kappa, \lambda, \psi, \theta, \rho, \lambda, \epsilon_{NS}, \widetilde{\epsilon})\,. \tag{IA.54}$$

The function on the right-hand side is continuous on its compact domain. By Weierstrass Theorem, it attains the maximum u^*,which is the upper bound of u. The above sufficient condition then becomes:

$$\epsilon_{NS} - \epsilon_S \geq u^*. \tag{IA.55}$$

Note that even though u depends on ϵ_{NS} and ϵ_S, u^* is independent of those two parameters. As long as inequality (IA.55) is satisfied, inequality (IA.54), and in turn inequality (IA.51), is going to hold. Therefore, under all the above three possible circumstances, if $\epsilon_{NS} - \epsilon_S \geq u^*$, carbon taxes would fail to gain a political majority.

Under specific values for ζ, we can obtain an estimate for u^*. Let us illustrate by considering the case of $\zeta = 1/2$, under which inequality (IA.51) simplifies to:

$$-\frac{\lambda \log(1-\psi)}{4\alpha\kappa\rho}\left(\frac{2\theta\rho^2\left(\theta\rho-\sqrt{\theta\left(\alpha+\theta\rho^2\right)+1}\right)(2\widetilde{\epsilon}-1)^2}{\sqrt{\theta\left(\alpha+\theta\rho^2\right)+1}}+\alpha\right) \leq \epsilon_{NS}-\epsilon_S.$$

Since $0 \leq \widetilde{\epsilon} \leq 1$, an estimate for the upper bound of the left-hand side is:

$$LHS \leq -\frac{\lambda \log(1-\psi)}{4\kappa\rho}.$$

Hence, the sufficient condition for inequality (IA.51) to hold is :

$$-\frac{\lambda \log(1-\psi)}{4\kappa\rho} \leq \epsilon_{NS}-\epsilon_S,$$

which follows the same line of economic intuitions under the Leontief production function (39) because:

- As the intensity (λ) or severity of climate disaster (ψ) increases, there need to be a bigger gap between the preference for the carbon tax not to be supported by the non-stock investors. As they are affected more by climate disasters, the benefits delivered to them by carbon taxes grow. As a result, they only oppose to such a tax if its impact on the price of brow output is high enough to outweigh the benefits, or when they have a stronger preference for the brown goods.

- As investors are more patient, they care more about the climate change that may destroy their future consumption. Thus, they are more likely to support carbon taxes, which is reflected by a higher chance of the above inequality being satisfied.

- When the population of the non-stock investors grows, their total consumption of the brown output rises, and the brown sector inflates. This contributes to more frequent climate disasters, which renders carbon taxes more valuable in curbing those extreme events that destroy future consumption. Hence, non-stock investors are more likely to support carbon pricing.