Dynamic Adverse Selection and Belief Update in Credit Markets

Online Appendices

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Online Appendix A: Proof

Proof of Lemma 1. The proof is done by showing that $0 < \frac{b(\underline{\theta})-\sqrt{b(\underline{\theta})^2-4b(\underline{\theta})r}}{\underline{\theta}} < 1$. First, we observe that $\frac{b(\underline{\theta})-\sqrt{b(\underline{\theta})^2-4b(\underline{\theta})r}}{\underline{\theta}} > 0$ is well-defined because $b(\underline{\theta}) = \left[\frac{1}{\bar{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{\bar{\theta}}\frac{1}{\theta}d\theta\right]^{-1} > \left[\frac{1}{\bar{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{\bar{\theta}}\frac{1}{\underline{\theta}}d\theta\right]^{-1} = \underline{\theta} \geq 4r$. Next, note that $\frac{b(\underline{\theta})}{\underline{\theta}} - \sqrt{\left(\frac{b(\underline{\theta})}{\underline{\theta}}\right)^2 - \frac{4r}{\underline{\theta}}\frac{b(\underline{\theta})}{\underline{\theta}}}$ decreases with $\frac{b(\underline{\theta})}{\underline{\theta}}$ and $\frac{b(\underline{\theta})}{\underline{\theta}} > 1$. Thus, we obtain $\frac{b(\underline{\theta})}{\underline{\theta}} - \sqrt{\left(\frac{b(\underline{\theta})}{\underline{\theta}}\right)^2 - \frac{4r}{\underline{\theta}}\frac{b(\underline{\theta})}{\underline{\theta}}} < 1 - \sqrt{1-\frac{4r}{\underline{\theta}}} \leq 1$, which completes the proof. ■

Proof of Lemma 2. First, consider an entrepreneur in period s who has not started a business. If the entrepreneur decides to establish a company and be matched with a lender, this entrepreneur faces the lender's belief $\mu(\cdot,(s,A^{s-1}))$. On the other hand, if the entrepreneur chooses not to establish a company in period s but does so in period $s+1$, then this entrepreneur faces the lender's belief $\mu(\cdot,(s+1,A^{s}))$. Note that $\mu(\cdot,(s,A^{s-1})) = \mu(\cdot,(s+1,A^{s}))$ due to the restriction on μ. Therefore, if the entrepreneur has an incentive not to establish in period s, then he/she also has the same incentive in period $s+1$. Then, by induction, if the entrepreneur has no incentive to establish a company in period s, this entrepreneur also does not have such incentive in the future, which results in zero continuation value. However, offering an incentive-compatible contract provides a positive continuation value. Thus, every entrepreneur has the incentive to establish a company immediately when he/she is born.

For the rest of the proof, we show that no entrepreneur will ever temporarily stop running his/her business. Consider any history h_{t-1} and an entrepreneur with h_{t-1} and entrepreneurial productivity θ who is currently running his/her business. Notice that the entrepreneur's period-t default decision D_t after offering a contract x_t must satisfy $\left[0, \frac{x_t}{\theta}\right) \subseteq D_t$. The lender's expected payoff from accepting contract x_t satisfies

$$\begin{aligned}
&\int_{\Theta} (1 - |D_t|) x_t d\mu(x_t, h_{t-1}) \\
\leq &\int_{\Theta} \left(1 - \left|\left[0, \frac{x_t}{\theta}\right)\right|\right) x_t d\mu(x_t, h_{t-1}) \\
\leq &\left(1 - \left|\left[0, \frac{x_t}{\bar{\theta}}\right)\right|\right) x_t = \left(1 - \frac{x_t}{\bar{\theta}}\right) x_t
\end{aligned}$$

Since $x_{min} \equiv \frac{\bar{\theta} - \sqrt{\bar{\theta}^2 - 4\bar{\theta} r}}{2}$ is the smallest x_t that satisfies $\left(1 - \frac{x_t}{\bar{\theta}}\right) x_t = r$, it is necessary for incentive compatibility that $x_t \geq x_{min}$, regardless of μ. The above inequality indicates that offering x_{min} is incentive-compatible only if the lender believes that the entrepreneur's productivity is $\bar{\theta}$ for sure and the corresponding default decision satisfies $D_t = \left[0, \frac{x_{min}}{\bar{\theta}}\right)$.

We now show that an entrepreneur has the incentive to offer x_{min} with $D_t = \left[0, \frac{x_{min}}{\bar{\theta}}\right)$ if the lender believes that the entrepreneur's productivity is $\bar{\theta}$ with certainty. The entrepreneur can always choose to offer a contract and default on it, which gives $\frac{\theta}{2}$ units of expected payoff to the type (θ, s) entrepreneur. This implies that $V_{t+1}(\theta, h_t) \geq \frac{\theta}{2}$, where $h_t = (s, \{A^{t-1}, A_t\})$. By assumption 1, we have:

$$\beta V_{t+1}(\theta, h_t) \geq \frac{\beta\theta}{2} \geq \frac{\beta\underline{\theta}}{2} > \frac{b(\underline{\theta}) - \sqrt{b(\underline{\theta})^2 - 4b(\underline{\theta})r}}{2} > \frac{\bar{\theta} - \sqrt{\bar{\theta}^2 - 4\bar{\theta}r}}{2} = x_{min}.$$

Thus, it is optimal to set $D_t = \left[0, \frac{x_{min}}{\bar{\theta}}\right)$.

Note that $V_t(\theta, h_{t-1})$ increases with θ because a more productive entrepreneur is capable of mimicking a less productive entrepreneur. Furthermore, an entrepreneur cannot offer a contract lower than x_{min} as explained above. Therefore, the highest feasible continuation value that an entrepreneur can achieve in the economy is when an entrepreneur with productivity $\bar{\theta}$ consistently faces the lender's belief that his/her productivity is $\bar{\theta}$ with the certainty at every period. In this case, the entrepreneur offers x_{min} and defaults only if $A_t < \frac{x_{min}}{\bar{\theta}}$. It has been proven in the previous paragraph that this arrangement is incentive-compatible in every period. Let V^* be such the highest continuation value. Then,

$$V^* = \mathbb{E}_{A_t}\left[A_t\bar{\theta}\right] + \left(1 - \left|\left[0, \frac{x_{min}}{\bar{\theta}}\right)\right|\right)\mathbb{E}_{A_t}\left[-x_{min} + \beta V^*\right].$$

This gives $V^* = \frac{\bar{\theta} - 2r}{2 - \beta - \beta\sqrt{1 - \frac{4r}{\bar{\theta}}}}$. Note that the entrepreneur's expected future continuation value cannot exceed V^* in any period.

Now, suppose that the entrepreneur decides not to run their business at some period after the establishment of the company. Because the cost κ to restart the business is higher than V^*, the entrepreneur will never restart the business again, resulting in a zero continuation value for the entrepreneur. Therefore, the entrepreneur would never stop running the business in any period t. ■

Proof of proposition 1. Consider an entrepreneur with history $h_{t-1} = (s, A^{t-1})$ and let θ be the entrepreneurial productivity of this entrepreneur. Based on lemma 2 and the incentive compatibility condition, the entrepreneur offers a contract in $S \equiv \{\hat{x} : \omega_\mu(\hat{x}, D_t, h_{t-1}) \geq r\}$, which is nonempty under the restriction on μ. If S is a singleton, the proof is done. Thus, for the rest of the proof, we assume that S is not a singleton. Let $x_{t,1} = \min S$ and $x_{t,2} \in S \backslash \{x_{t,1}\}$. It follows that $x_{t,2} > x_{t,1}$. Let $D_{t,i}$ denote the default set associated with $x_{t,i}$ for $i = 1, 2$. By (5), $A_t \in D_{t,i}$ if and only if either $x_{t,i} > \beta V_{t+1}(\theta, h_t)$ or $x_{t,i} > A_t\theta$ for each $i = 1, 2$, where $h_t = (s, \{A^{t-1}, A_t\})$. Thus, $D_{t,1} \subseteq D_{t,2}$. Further, note, from (5), that $-x_{t,1} + \beta V_{t+1}(\theta, h_t) \geq 0$ whenever $A_t \notin D_{t,1}$, which implies that

$$\mathbb{E}_{A_t}\left[-x_{t,1} + \beta V_{t+1}(\theta, h_t) | A_t \notin D_{t,1}\right] - \mathbb{E}_{A_t}\left[-x_{t,1} + \beta V_{t+1}(\theta, h_t) | A_t \notin D_{t,2}\right]$$
$$= \mathbb{E}_{A_t}\left[-x_{t,1} + \beta V_{t+1}(\theta, h_t) | A_t \in D_{t,2} \backslash D_{t,1}\right] \geq 0.$$

Finally, it is necessary that $[0, 1] \backslash D_{t,2}$ has a positive measure because the lender's expected payoff from accepting $x_{t,2}$ is no less than r.

Given the above observations, we obtain

$$(1 - |D_{t,1}|)\mathbb{E}_{A_t}\left[-x_{t,1} + \beta V_{t+1}(\theta, h_t) \mid A_t \notin D_{t,1}\right]$$
$$\geq (1 - |D_{t,2}|)\mathbb{E}_{A_t}\left[-x_{t,1} + \beta V_{t+1}(\theta, h_t) \mid A_t \notin D_{t,2}\right]$$
$$> (1 - |D_{t,2}|)\mathbb{E}_{A_t}\left[-x_{t,2} + \beta V_{t+1}(\theta, h_t) \mid A_t \notin D_{t,2}\right].$$

Thus, the expected payoff from offering $x_{t,1}$ is strictly higher than that from offering $x_{t,2}$ as shown (1). As $x_{t,2}$ is chosen arbitrarily, it follows that in equilibrium, the entrepreneur chooses $\min S$ regardless of the entrepreneurial productivity level. ■

Proof of Lemma 3. Take any history $h_{t-1} = (s, A^{t-1})$ such that $\operatorname{supp} \hat{\Omega}_{h_{t-1}} \neq \varnothing$ in equilibrium. For each $\tau = s, \ldots, t$, let $A^{\tau-1}$ be the truncated subsequence of A^{t-1} such that $A^{\tau-1} = \{\varnothing, \ldots, A_{\tau-1}\}$. If $t = s$, then $\hat{\Omega}_{h_{t-1}} = U_{[\underline{\theta},\bar{\theta}]}$ because all entrepreneurs with this history established their company in period s. Now suppose that $s < t$ and let $h_{k-1} = (s, A^{k-1})$ for each $k \in \{s, \ldots, t-1\}$. Suppose that for some $k \in \{s, \ldots, t-1\}$, there exists $\hat{\theta}_k \in \Theta$ such that $\hat{\Omega}_{h_{k-1}} = U_{[\hat{\theta}_k,\bar{\theta}]}$. Then, the proof is done by induction if we show that there exists $\hat{\theta}_{k+1} \in \Theta$ such that $\hat{\Omega}_{h_k} = U_{[\hat{\theta}_{k+1},\bar{\theta}]}$.

By applying lemma 2 and proposition 1, all entrepreneurs with h_{k-1} offer the same contract in period k, denoted as x_k. Since $\operatorname{supp} \hat{\Omega}_{h_{t-1}} \neq \varnothing$, some entrepreneurs did not default under the realization of A_k in period k. Now suppose that an entrepreneur with entrepreneurial productivity $\theta' \in [\hat{\theta}_k, \bar{\theta}]$ did not default on contract x_k in period k, which implies that $\min\{A_k\theta', \beta V_{k+1}(\theta', h_k)\} \geq x_k$, as stated in (5). Note that for any $\theta'' \geq \theta'$, $V_{k+1}(\theta'', h_k) \geq V_{k+1}(\theta', h_k)$ because an entrepreneur with θ'' is capable of mimicking entrepreneur with θ', achieving a larger payoff due to higher productivity. Thus, $\min\{A_k\theta'', \beta V_{k+1}(\theta'', h_k)\} \geq x_k$ holds for all $\theta'' \geq \theta'$, indicating that entrepreneur with entrepreneurial productivity larger than θ' also did not default. This implies that there exists $\hat{\theta}_{k+1} \in [\hat{\theta}_k, \bar{\theta}]$ such that entrepreneurs with h_{k-1} did not default on x_k in period $t = k$ if and only if their entrepreneurial productivity is larger than or equal to $\hat{\theta}_{k+1}$. Furthermore, $\hat{\Omega}_{h_{k-1}}$ is uniformly distributed, so the entrepreneurial productivity of the survivors is also uniformly distributed. That is, $\hat{\Omega}_{h_k} = U_{[\hat{\theta}_{k+1},\bar{\theta}]}$ for some $\hat{\theta}_{k+1} \in \Theta$, which completes the proof. ■

Proof of Lemma 4. Take any history $h_{t-1} = (s, A^{t-1})$ such that $\hat{\Omega}_{h_{t-1}} = U_{[\hat{\theta}_t,\bar{\theta}]}$ for some $\hat{\theta}_t \in \Theta$, i.e., $\hat{\theta}_t = \min \operatorname{supp} \hat{\Omega}_{h_{t-1}}$. According to lemma 2 and proposition 1, all entrepreneurs with h_{t-1} offer the same contract x_t in period t. Thus, the lender's expected payoff from accepting contract x_t is given by (6), which decreases with the measure of default sets D_t. This implies that

$$\begin{aligned}\omega_\mu(x_t, D_t(\cdot, h_{t-1}), h_{t-1}) &= \int_\Theta \int_{[0,1]\setminus D(\theta,h_{t-1})} x_t \mathbf{m}_{[0,1]}(dA_t)\mathbf{m}_{[\hat{\theta}_t,\bar{\theta}]}(d\theta) \\ &\le \int_\Theta \int_{[\frac{x_t}{\theta},1]} x_t \mathbf{m}_{[0,1]}(dA_t)\mathbf{m}_{[\hat{\theta}_t,\bar{\theta}]}(d\theta) = x_t - \frac{x_t^2}{b(\hat{\theta}_t)}.\end{aligned}$$

Using the definition of $x^*(\cdot)$ in (7), we have $x^*(\hat{\theta}_t) = \min\{x : x - \frac{x^2}{b(\hat{\theta}_t)} \ge r\}$. Thus, the lender will never accept x_t if $x_t < x^*(\hat{\theta}_t)$. Therefore, any contract x_t must satisfy $x_t \ge x^*(\hat{\theta}_t)$.

We now show that $x^*(\cdot)$ is a decreasing convex function. From assumption 1, we obtain $\frac{\partial b(\theta)}{\partial\theta} = \frac{\frac{\bar{\theta}}{\theta}-1-\log(\frac{\bar{\theta}}{\theta})}{(\log\bar{\theta}-\log\theta)^2} > 0$ for all $\theta < \bar{\theta}$ and $\frac{\partial b(\theta)}{\partial\theta}\Big|_{\theta=\bar{\theta}} = \lim_{\theta\to\bar{\theta}} \frac{b(\bar{\theta})-b(\theta)}{\bar{\theta}-\theta} = \lim_{\theta\to\bar{\theta}} \frac{\partial b(\theta)}{\partial\theta} = \frac{1}{2} > 0$. Thus,

$$\frac{\partial x^*(\theta)}{\partial\theta} = \frac{\partial x^*(\theta)}{\partial b(\theta)}\frac{\partial b(\theta)}{\partial\theta} = \frac{1}{2}\left\{1 - \frac{b(\theta)-2r}{\sqrt{b(\theta)^2-4b(\theta)r}}\right\}\frac{\partial b(\theta)}{\partial\theta} < 0. \tag{16}$$

Next, by letting $u(\theta) = \frac{\bar{\theta}}{\theta} \ge 1$ for each $\theta \in \Theta$, we obtain $\frac{\partial^2 b(\theta)}{\partial\theta^2} = -\frac{(u(\theta)+1)\log u(\theta) - 2(u(\theta)-1)}{\theta(\log u(\theta))^3}$. The term $(u(\theta)+1)\log u(\theta) - 2(u(\theta)-1)$ increases with $u(\theta) \ge 1$, and it is zero when $u(\theta) = 1$, so $\frac{\partial^2 b(\theta)}{\partial\theta^2} < 0$ for all $\theta < \bar{\theta}$. Additionally, $\frac{\partial^2 b(\theta)}{\partial\theta^2}\Big|_{\theta=\bar{\theta}} = -\frac{1}{6\bar{\theta}} < 0$. Then, from (16), we obtain

$$\frac{\partial^2 x^*(\theta)}{\partial\theta^2} = \frac{\sqrt{b(\theta)^2-4b(\theta)r}-(b(\theta)-2r)}{2\sqrt{b(\theta)^2-4b(\theta)r}} \times \frac{\partial^2 b(\theta)}{\partial\theta^2} + 2r^2\left(b(\theta)^2-4b(\theta)r\right)^{-\frac{3}{2}}\left(\frac{\partial b(\theta)}{\partial\theta}\right)^2 > 0,$$

which completes the proof. ■

Proof of Propositions 2 and 3. Here, we prove propositions 2 and 3 together. Consider the entrepreneur's strategy (x, D) that satisfies the following conditions: For any history h_{t-1}, if $\hat{\Omega}_{h_{t-1}} = U_{[\hat{\theta},\bar{\theta}]}$ for some $\hat{\theta} \in \Theta$, then for all $\theta \in [\hat{\theta}, \bar{\theta}]$,
$(x(\theta, h_{t-1}), D(\theta, h_{t-1})) = \left(x^*(\hat{\theta}), \left[0, \frac{x^*(\hat{\theta})}{\theta}\right)\right)$, where $x^*(\cdot)$ is defined in (7). We call the entrepreneur's strategy the "S_e^*-strategy" if it satisfies the above conditions.[1]

We first introduce and prove the following claim, which provides a useful intermediate step.

Claim 1 *Suppose that entrepreneurs adopt the S_e^*-strategy and take any $h_{t-1} = (s, A^{t-1}) \in \mathbb{H}$. For each $\tau = s, \ldots, t$, let $A^{\tau-1}$ denote the truncated subsequence of A^{t-1} such that $A^{\tau-1} = \{\varnothing, \ldots, A_{\tau-1}\}$, and $h_{\tau-1} = (s, A^{\tau-1})$. If $\operatorname{supp} \hat{\Omega}_{h_{t-1}} \neq \varnothing$, then $\hat{\Omega}_{h_{\tau-1}} = U_{[\hat{\theta}_\tau,\bar{\theta}]}$ for each $\tau = s, \ldots, t$, where $\hat{\theta}_\tau$ is given by (8) in proposition 3.*

Proof of claim 1. The statement holds if $\tau = s$ because the initial distribution of the entrepreneurs' productivity at the establishment period is $U_{[\underline{\theta},\bar{\theta}]}$. To prove the claim by induction, assume that the statement holds for $\tau = k \in \{s, \ldots, t-1\}$, namely, $\hat{\Omega}_{h_{k-1}} = U_{[\hat{\theta}_k,\bar{\theta}]}$, where $\hat{\theta}_k$ is derived by the rule in (8). Then, according to the S_e^*-strategy, all entrepreneurs with h_{k-1} offer $x^*(\hat{\theta}_k)$ and default if and only if $A_k\theta < x^*(\hat{\theta}_k)$. Considering the fact that $\operatorname{supp} \hat{\Omega}_{h_{t-1}} \neq \varnothing$, $\frac{x^*(\hat{\theta}_k)}{A_k} \leq \bar{\theta}$ holds; otherwise, all entrepreneurs with h_{k-1} would had defaulted in period k, resulting in $\operatorname{supp} \hat{\Omega}_{h_{t-1}} = \varnothing$. Thus, $\hat{\Omega}_{h_k} = U_{\left[\max\left\{\hat{\theta}_k, \frac{x^*(\hat{\theta}_k)}{A_k}\right\},\bar{\theta}\right]} = U_{[\hat{\theta}_{k+1},\bar{\theta}]}$. Therefore, the statement also holds for $\tau = k+1$, which completes the proof of claim 1. ■

[1]The S_e^*-strategy does not specify any rules for h_{t-1} if $\hat{\Omega}_{h_{t-1}}$ is not the form of $U_{[\hat{\theta},\bar{\theta}]}$ for some $\hat{\theta} \in \Theta$. Further, without a specification of the lender's belief system, it is not guaranteed at all that S_e^*-strategy solves for (3).

Claim 1 asserts that if an equilibrium exists in which entrepreneurs adopt the S_e^*-strategy, then such an equilibrium satisfies the statements of propositions 2 and 3. Moreover, if an equilibrium where entrepreneurs adopt the S_e^*-strategy exists, it must be the $\boldsymbol{e}^*$ equilibrium, since entrepreneurs offer the lower bound for the set of equilibrium offers, as described in lemma 4.[2]

We complete the proof by showing the existence of an equilibrium in which entrepreneurs adopt the S_e^*-strategy. Suppose that entrepreneurs adopt the S_e^*-strategy, and the lender's belief system μ satisfies that for any $h_{t-1} \in \mathbb{H}$ in any period t, $\mu(x^*(\hat{\theta}_t), h) = U_{[\hat{\theta}_t, \bar{\theta}]}$, where $\hat{\theta}_t$ is defined by (8) in proposition 3. Then, μ is consistent, according to claim 1. Also, the lender's expected payoff from accepting contract $x^*(\hat{\theta}_t)$ offered by an entrepreneur with h_{t-1} is

$$\int_\Theta \left(1 - \left|\left[0, \frac{x^*(\hat{\theta}_t)}{\theta}\right)\right|\right) x^*(\hat{\theta}_t) dU_{[\hat{\theta}_t, \bar{\theta}]} = x^*(\hat{\theta}_t) - \frac{x^*(\hat{\theta}_t)^2}{b(\hat{\theta}_t)} = r.$$

Thus, the entrepreneur's strategy is incentive-compatible under μ.

Finally, we show that the S_e^*-strategy is optimal. Consider any $h_{t-1} = (s, A^{t-1}) \in \mathbb{H}$. First, by lemma 2, all entrepreneurs with h_{t-1} offer a contract. Furthermore, according to proposition 1 and the lender's belief system μ constructed in the aforementioned way, it is optimal for all entrepreneurs with h_{t-1} to offer $x^*(\hat{\theta}_t)$ in period t. We finish by showing that $\left[0, \frac{x^*(\hat{\theta}(h_{t-1}))}{\theta}\right)$ is the optimal default decision associated with contract $x^*(\hat{\theta}_t)$. By (5), it suffices to show that $x^*(\hat{\theta}_t) \leq \beta V_{t+1}(\theta, h_t)$, where $h_t = (s, \{A^{t-1}, A_t\})$. By the results of claim 1 and the

[2]Specifically, consider any $h_{t-1} = (s, A^{t-1}) \in \mathbb{H}$ such that $\operatorname{supp} \hat{\Omega}_{h_{t-1}} \neq \varnothing$. By claim 1, there exists $\hat{\theta} \in \Theta$ such that $\hat{\Omega}_{h_{t-1}} = U_{[\hat{\theta}, \bar{\theta}]}$ if entrepreneurs adopt the S_e^*-strategy, and all entrepreneurs with h_{t-1} offer $x^*(\hat{\theta})$. Now, consider another equilibrium in which $\hat{\Omega}_{h'_{t'-1}} = U_{[\hat{\theta}, \bar{\theta}]}$ for some $h'_{t'-1} = (s', {A'}^{t'-1}) \in \mathbb{H}$. According to lemma 4, the contract that entrepreneurs with $h'_{t'-1}$ offer must be no less than $x^*(\hat{\theta})$ in this equilibrium.

way of constructing μ above, entrepreneurs with any history in any equilibrium are capable of offering an incentive-compatible contract. Thus, $V_{t+1}(\theta, h_t) \geq \mathbb{E}_{A_{t+1}}[A_{t+1}\theta] = \frac{\theta}{2}$, because an entrepreneur can always choose to offer an incentive-compatible contract in period $t+1$ and default on it, even if it may not be an optimal behavior. Next, given assumption 1, we have

$$\frac{\beta\underline{\theta}}{2} > \frac{b(\underline{\theta}) - \sqrt{b(\underline{\theta})^2 - 4b(\underline{\theta})r}}{\underline{\theta}} \times \frac{\underline{\theta}}{2} = x^*(\underline{\theta}).$$

Further, $x^*(\hat{\theta}_t) \leq x^*(\underline{\theta})$ by lemma 4. As a result, for any $\theta \in \Theta$, we have

$$x^*(\hat{\theta}_t) < \frac{\beta\underline{\theta}}{2} \leq \frac{\beta\theta}{2} \leq \beta V_{t+1}(\theta, h_t),$$

which completes the proof. ■

Proof of lemma 5. If suffices to show that $\int_\Theta \frac{x^*(\hat{\theta})}{\theta} dU_{[\hat{\theta},\bar{\theta}]}$ decreases with $\hat{\theta}$. Take any $\hat{\theta}^1, \hat{\theta}^2 \in \Theta$ such that $\hat{\theta}^1 < \hat{\theta}^2$. Then, because $x^*(\cdot)$ is a decreasing function, we obtain

$$\begin{aligned}\int_\Theta \frac{x^*(\hat{\theta}^1)}{\theta} dU_{[\hat{\theta}^1,\bar{\theta}]} =& x^*(\hat{\theta}^1)\left(\log(\bar{\theta}) - \log(\hat{\theta}^1)\right) \\ >& x^*(\hat{\theta}^2)\left(\log(\bar{\theta}) - \log(\hat{\theta}^2)\right) = \int_\Theta \frac{x^*(\hat{\theta}^2)}{\theta} dU_{[\hat{\theta}^2,\bar{\theta}]},\end{aligned}$$

which completes the proof. ■

Proof of proposition 5. Consider any $A^{t-1} \in \mathbb{A}^{t-1}$ and $s^o, s^y \in \{0, \ldots, t\}$ in the $\boldsymbol{e}^*$ equilibrium such that $\operatorname{supp} \hat{\Omega}_{h^o_{t-1}} \neq \varnothing$ and $\operatorname{supp} \hat{\Omega}_{h^y_{t-1}} \neq \varnothing$, where $h^o_{t-1} = (s^o, A^{t-1})$ and $h^y_{t-1} = (s^y, A^{t-1})$. For each $i = \{o, y\}$, let $\hat{\theta}^i_t = \min \operatorname{supp} \hat{\Omega}_{h^i_{t-1}}$ and $\hat{\theta}^i_{t+1} = \min \operatorname{supp} \hat{\Omega}_{h^i_t}$ whenever $\operatorname{supp} \hat{\Omega}_{h^i_t} \neq \varnothing$, where $h^i_t = (s, \{A^{t-1}, A_t\}))$. Suppose that $\hat{\theta}^y_t < \hat{\theta}^o_t$, which implies $x^*(\hat{\theta}^y_t) > x^*(\hat{\theta}^o_t)$ by lemma 4.

Note that all entrepreneurs with h^y_{t-1} leave the economy after defaulting in period t if $A_t \in \left[0, \frac{x^*(\hat\theta^y_t)}{\bar\theta}\right)$. Thus, in what follows, we focus on the case with $A_t \in \left[\frac{x^*(\hat\theta^y_t)}{\bar\theta}, 1\right]$, which implies $\operatorname{supp}\hat\Omega_{h^i_t} \neq \varnothing$ for both $i = o, y$. From proposition 3, we obtain:

$$\hat\theta^o_{t+1} = \max\left\{\frac{x^*(\hat\theta^o_t)}{A_t}, \hat\theta^o_t\right\} \text{ and } \hat\theta^y_{t+1} = \max\left\{\frac{x^*(\hat\theta^y_t)}{A_t}, \hat\theta^y_t\right\}. \tag{17}$$

We now consider three relevant cases.

First, if $A_t \in \left[\frac{x^*(\hat\theta^y_t)}{\hat\theta^y_t}, 1\right]$, then $A_t > \frac{x^*(\hat\theta^o_t)}{\hat\theta^o_t}$ given that $\hat\theta^y_t < \hat\theta^o_t$ and $x^*(\hat\theta^y_t) > x^*(\hat\theta^o_t)$. Thus, we have $\hat\theta^o_{t+1} = \hat\theta^o_t$ and $\hat\theta^y_{t+1} = \hat\theta^y_t$ from (17), resulting in $\hat\theta^y_{t+1} < \hat\theta^o_{t+1}$. Second, if $A_t \in \left[\frac{x^*(\hat\theta^o_t)}{\hat\theta^o_t}, \frac{x^*(\hat\theta^y_t)}{\hat\theta^y_t}\right)$, then we obtain $\hat\theta^o_{t+1} = \hat\theta^o_t$ and $\hat\theta^y_{t+1} = \frac{x^*(\hat\theta^y_t)}{A_t}$ from (17). In this case, we have $\hat\theta^y_{t+1} \leq \hat\theta^o_{t+1}$ if and only if $A_t \geq \frac{x^*(\hat\theta^y_t)}{\hat\theta^o_t}$. Third, if $A_t \in \left[\frac{x^*(\hat\theta^y_t)}{\bar\theta}, \frac{x^*(\hat\theta^o_t)}{\hat\theta^o_t}\right)$, then $A_t \in \left[\frac{x^*(\hat\theta^y_t)}{\bar\theta}, \frac{x^*(\hat\theta^y_t)}{\hat\theta^y_t}\right)$ is also implied, which leads to $\hat\theta^o_{t+1} = \frac{x^*(\hat\theta^o_t)}{A_t}$ and $\hat\theta^y_{t+1} = \frac{x^*(\hat\theta^y_t)}{A_t}$ from (17). In this case, we have $\hat\theta^y_{t+1} > \hat\theta^o_{t+1}$ because $x^*(\hat\theta^o_t) < x^*(\hat\theta^y_t)$.

By summarizing the above three cases, we conclude that $\hat\theta^y_{t+1} \leq \hat\theta^o_{t+1}$ for all $A_t \in \left[\frac{x^*(\hat\theta^y_t)}{\hat\theta^o_t}, 1\right]$ and $\hat\theta^y_{t+1} > \hat\theta^o_{t+1}$ for all $A_t \in \left[\frac{x^*(\hat\theta^y_t)}{\bar\theta}, \frac{x^*(\hat\theta^y_t)}{\hat\theta^o_t}\right)$. Then, using the fact that $\lambda(h^o_t) \lesseqgtr \lambda(h^y_t)$ if and only if $\hat\theta^y_{t+1} \lesseqgtr \hat\theta^o_{t+1}$ by lemma 5 and letting $A_L = \frac{x^*(\hat\theta^y_t)}{\bar\theta}$ and $A_H = \frac{x^*(\hat\theta^y_t)}{\hat\theta^o_t}$, we obtain the results of proposition 5. ■

Proof of proposition 6. Consider any $A^{t-1} \in \mathbb{A}^{t-1}$ and $s^o, s^y \in \{0, \ldots, t\}$ in the $\boldsymbol{e}^*$ equilibrium such that $\operatorname{supp}\hat\Omega_{h^o_{t-1}} \neq \varnothing$ and $\operatorname{supp}\hat\Omega_{h^y_{t-1}} \neq \varnothing$, where $h^o_{t-1} = (s^o, A^{t-1})$ and $h^y_{t-1} = (s^y, A^{t-1})$. According to lemma 3, there exist $\theta_o, \theta_y \in \Theta$ such that $\hat\Omega_{h^i_{t-1}} = U_{[\theta_i, \bar\theta]}$ for $i = \{o, y\}$. Assume that $\lambda(h^o_{t-1}) < \lambda(h^y_{t-1})$, which implies $\theta_o > \theta_y$ by lemma 5. Then, it suffices to show that

$$\mathbb{E}_{A_t}\left[\theta'_o - \theta'_y \mid \operatorname{supp}\hat\Omega_{h^o_t} \neq \varnothing \text{ and } \operatorname{supp}\hat\Omega_{h^y_t} \neq \varnothing\right] > 0,$$

where $h_t^i = (s^i, \{A^{t-1}, A_t\})$ and $\theta_i' = \min \operatorname{supp} \hat{\Omega}_{h_t^i}$ for each $i = \{o, y\}$ whenever $\operatorname{supp} \hat{\Omega}_{h_t^i} \neq \varnothing$.

By proposition 2, for each $i \in \{o, y\}$, an entrepreneur with h_{t-1}^i and $\theta \in \operatorname{supp} \hat{\Omega}_{h_{t-1}^i}$ plays $\left(x^*(\theta_i), \left[0, \frac{x^*(\theta_i)}{\theta}\right)\right)$, so $\theta_i' = \max\left\{\frac{x^*(\theta_i)}{A_t}, \theta_o\right\}$ if $\operatorname{supp} \hat{\Omega}_{h_t^i} \neq \varnothing$. Consequently, $\operatorname{supp} \hat{\Omega}_{h_t^i} \neq \varnothing$ for both $i \in \{o, y\}$ if and only if $A_t \geq \max\left\{\frac{x^*(\theta_o)}{\bar{\theta}}, \frac{x^*(\theta_y)}{\bar{\theta}}\right\} = \frac{x^*(\theta_y)}{\bar{\theta}}$, given the assumption that $\theta_o > \theta_y$. Therefore, the proof is completed by showing that

$$\Xi \equiv \left(1 - \frac{x^*(\theta_y)}{\bar{\theta}}\right) E_{A_t}\left[\theta_o' - \theta_y' \mid A_t \geq \frac{x^*(\theta_y)}{\bar{\theta}}\right] > 0. \tag{18}$$

Let θ^* be such that $\frac{x^*(\theta_y)}{\bar{\theta}} = \frac{x^*(\theta^*)}{\theta^*}$, that is, $\frac{x^*(\theta^*)}{\theta^*}\frac{\bar{\theta}}{x^*(\theta_y)} = 1$. Here, $\theta^* \in (\theta_y, \bar{\theta})$ is uniquely determined because $\frac{x^*(\bar{\theta})}{\bar{\theta}}\frac{\bar{\theta}}{x^*(\theta_y)} = \frac{x^*(\bar{\theta})}{x^*(\theta_y)} < 1$, $\frac{x^*(\theta_y)}{\theta_y}\frac{\bar{\theta}}{x^*(\theta_y)} = \frac{\bar{\theta}}{\theta_y} > 1$, and $\frac{x^*(\theta)}{\theta}\frac{\bar{\theta}}{x^*(\theta_y)}$ decreases with θ. Consequently, $\frac{x^*(\theta_y)}{\bar{\theta}} \leq \frac{x^*(\theta_o)}{\theta_o}$ if and only if $\theta_o \leq \theta^*$.

First, consider the case where $\frac{x^*(\theta_y)}{\bar{\theta}} \leq \frac{x^*(\theta_o)}{\theta_o}$, i.e., $\theta_o \leq \theta^*$. From (18), we obtain

$$\begin{aligned}\Xi &= \int_{\frac{x^*(\theta_y)}{\bar{\theta}}}^{\frac{x^*(\theta_o)}{\theta_o}} \frac{x^*(\theta_o)}{A_t} dA_t + \theta_o\left(1 - \frac{x^*(\theta_o)}{\theta_o}\right) - \int_{\frac{x^*(\theta_y)}{\bar{\theta}}}^{\frac{x^*(\theta_y)}{\theta_y}} \frac{x^*(\theta_y)}{A_t} dA_t - \theta_y\left(1 - \frac{x^*(\theta_y)}{\theta_y}\right) \\ &= \theta_o - \theta_y + x^*(\theta_y) - x^*(\theta_o) + x^*(\theta_o)\log\left(\frac{x^*(\theta_o)\bar{\theta}}{\theta_o x^*(\theta_y)}\right) - x^*(\theta_y)\log\frac{\bar{\theta}}{\theta_y}. \qquad (19)\end{aligned}$$

Now, define a function $F(\theta)$ for each $\theta \in [\theta_y, \theta^*]$ as follows:

$$F(\theta) = (\theta - \theta_y) + (x^*(\theta_y) - x^*(\theta)) + x^*(\theta)\log\left(\frac{x^*(\theta)}{\theta}\frac{\bar{\theta}}{x^*(\theta_y)}\right) - x^*(\theta_y)\log\frac{\bar{\theta}}{\theta_y}. \tag{20}$$

Note, from (19) and (20), that $F(\theta_o) = \Xi$, so it suffices to show $F(\theta_o) > 0$. Taking the first and second derivatives of $F(\theta)$ with respect to θ, we have:

$$F'(\theta) = 1 + \frac{\partial x^*(\theta)}{\partial \theta} \log\left(\frac{x^*(\theta)}{\theta}\frac{\bar{\theta}}{x^*(\theta_y)}\right) - \frac{x^*(\theta)}{\theta} \tag{21}$$

$$F''(\theta) = \frac{\partial^2 x^*(\theta)}{\partial \theta^2} \log\left(\frac{x^*(\theta)}{\theta}\frac{\bar{\theta}}{x^*(\theta_y)}\right) + \left(\frac{\partial x^*(\theta)}{\partial \theta}\right)^2 \times \frac{1}{x^*(\theta)} - \frac{\partial x^*(\theta)}{\partial \theta} \times \frac{2}{\theta} + \frac{x^*(\theta)}{\theta^2}. \tag{22}$$

From lemma 4, we know that $\frac{\partial x^*(\theta)}{\partial \theta} < 0$ and $\frac{\partial^2 x^*(\theta)}{\partial \theta^2} > 0$. Moreover, since $\frac{x^*(\theta)}{\theta}\frac{\bar{\theta}}{x^*(\theta_y)} \geq 1$ for all $\theta \in [\theta_y, \theta^*]$, we can conclude, from (22), that $F''(\theta) > 0$ for all $\theta \in [\theta_y, \theta^*]$. Consequently, $F'(\theta_o) > F'(\theta_y)$. Since we have $F(\theta_y) = 0$ according to equation (20), if $F'(\theta_y) > 0$, it follows that $\Xi = F(\theta_o) > 0$. Substituting $\theta = \theta_y$ into equation (21), we obtain

$$F'(\theta_y) = 1 + \left.\frac{\partial x^*(\theta)}{\partial \theta}\right|_{\theta=\theta_y} \log\left(\frac{\bar{\theta}}{\theta_y}\right) - \frac{x^*(\theta_y)}{\theta_y}.$$

Using the facts that $\frac{\partial}{\partial \theta_y}\left[\frac{x^*(\theta_y)}{\theta_y}\right] < 0$, $b(\underline{\theta}) = \frac{\bar{\theta}-\underline{\theta}}{\log(\frac{\bar{\theta}}{\underline{\theta}})}$, and

$$\frac{\partial}{\partial \theta_y}\left[\left.\frac{\partial x^*(\theta)}{\partial \theta}\right|_{\theta=\theta_y} \log\left(\frac{\bar{\theta}}{\theta_y}\right)\right] = \left.\frac{\partial^2 x^*(\theta)}{\partial \theta^2}\right|_{\theta=\theta_y} \log\left(\frac{\bar{\theta}}{\theta_y}\right) - \left.\frac{\partial x^*(\theta)}{\partial \theta}\right|_{\theta=\theta_y} \frac{1}{\theta_y} > 0,$$

we obtain

$$F'(\theta_y) \geq 1 + \left.\frac{\partial x^*(\theta)}{\partial \theta}\right|_{\theta=\underline{\theta}} \log\left(\frac{\bar{\theta}}{\underline{\theta}}\right) - \frac{x^*(\underline{\theta})}{\underline{\theta}} = 1 - \frac{1}{2\underline{\theta}}G(b(\underline{\theta})),$$

where $G : (4r, \infty) \to \mathbb{R}$ is a function defined as:

$$G(b) = \left(\frac{b-2r}{\sqrt{b^2-4rb}} - 1\right)(b-\underline{\theta}) + b - \sqrt{b^2-4rb}.$$

Note that $G'(b) < 0$ for all $b > 4r$. Therefore, we can deduce that

$$F'(\theta_y) \geq 1 - \frac{1}{2\underline{\theta}} G(b(\underline{\theta})) > 1 - \frac{1}{2\underline{\theta}} G(\underline{\theta}) = 1 - \frac{1}{2\underline{\theta}} \left(\underline{\theta} - \sqrt{\underline{\theta}^2 - 4r\underline{\theta}} \right) > 0,$$

which implies $F(\theta_o) > 0$. This completes the proof for the case when $\theta_o \in (\theta_y, \theta^*]$.

Second, let us suppose that $\frac{x^*(\theta_y)}{\bar{\theta}} \geq \frac{x^*(\theta_o)}{\theta_o}$, i.e., $\theta_o \geq \theta^*$. In this case, we have:

$$\begin{aligned}
\Xi &= \theta_o \left(1 - \frac{x^*(\theta_y)}{\bar{\theta}}\right) - \int_{\frac{x^*(\theta_y)}{\bar{\theta}}}^{\frac{x^*(\theta_y)}{\theta_y}} \frac{x^*(\theta_y)}{A_t} dA_t - \theta_y \left(1 - \frac{x^*(\theta_y)}{\theta_y}\right) \\
&= \theta_o - \theta_y + x^*(\theta_y) \left[1 - \frac{\theta_o}{\bar{\theta}} - \log \frac{\bar{\theta}}{\theta_y}\right].
\end{aligned}$$

Since Ξ increases with θ_o, and we know that $\Xi > 0$ when $\theta_o = \theta^*$ (as shown in the first case), it follows that $\Xi > 0$ when $\theta_o > \theta^*$. ■

Proof of proposition 7. First, consider the case where $\tilde{A} \in \left(0, \frac{x^*(\theta)}{\bar{\theta}}\right] \cup \left[\frac{x^*(\theta)}{\underline{\theta}}, 1\right]$. Suppose that $\Omega_t = U_{[\underline{\theta},\bar{\theta}]}$ in a given period $t \geq 0$. Notice that Ω_t is the average of $\hat{\Omega}_{h_{t-1}}$ weighted by the mass of entrepreneurs with each history $h_{t-1} \in \mathbb{H}_{t-1}$. Furthermore, according to lemma 3, for all $h_{t-1} \in \mathbb{H}_{t-1}$ such that $\operatorname{supp} \hat{\Omega}_{h_{t-1}} \neq \varnothing$, there must exist $\theta' \in \Theta$ such that $\hat{\Omega}_{h_{t-1}} = U_{[\theta',\bar{\theta}]}$. Therefore, $\Omega_t = U_{[\underline{\theta},\bar{\theta}]}$ implies $\hat{\Omega}_{h_{t-1}} = U_{[\underline{\theta},\bar{\theta}]}$ for all such h_{t-1}, and thus, all entrepreneurs in period t play $\left(x^*(\underline{\theta}), \left[0, \frac{x^*(\theta)}{\theta}\right)\right)$.

Given that $\Omega_t = U_{[\underline{\theta},\bar{\theta}]}$, if $\tilde{A} \in \left(0, \frac{x^*(\theta)}{\bar{\theta}}\right)$, all entrepreneurs default in period t. On the other hand, if $\tilde{A} \in \left[\frac{x^*(\theta)}{\underline{\theta}}, 1\right]$, every entrepreneur survives. In either case, $\Omega_{t+1} = U_{[\underline{\theta},\bar{\theta}]}$. If $\tilde{A} = \frac{x^*(\theta)}{\bar{\theta}}$, then an entrepreneur survives if and only if $\theta = \bar{\theta}$. Consequently, the mass of defaulted entrepreneurs is 1, and thus, $\Omega_{t+1} = U_{[\underline{\theta},\bar{\theta}]}$. Therefore, for any $\tilde{A} \in \left(0, \frac{x^*(\theta)}{\bar{\theta}}\right] \cup \left[\frac{x^*(\theta)}{\underline{\theta}}, 1\right]$,

$\Omega_t = U_{[\underline{\theta},\bar{\theta}]}$ implies $\Omega_{t+1} = U_{[\underline{\theta},\bar{\theta}]}$. Finally, since $\Omega_0 = U_{[\underline{\theta},\bar{\theta}]}$, $\Omega_t = U_{[\underline{\theta},\bar{\theta}]}$ for all $t \geq 0$ by induction. Therefore, the aggregate production in each period t is given as $\hat{Y}(\tilde{A}^t) = \frac{1}{2}\tilde{A}(\underline{\theta} + \bar{\theta})$.

Now suppose that $\tilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$. Consider any $h_{t-1} = (s, \tilde{A}^{t-1}) \in \mathbb{H}$ such that $\hat{\Omega}_{h_{t-1}} = U_{[\underline{\theta},\bar{\theta}]}$. Let $M \in (0,1]$ be the mass of entrepreneurs with h_{t-1}. According to proposition 2, all entrepreneurs with h_{t-1} offer $x^*(\underline{\theta})$, and those with entrepreneurial productivity smaller than $\frac{x^*(\underline{\theta})}{\tilde{A}}$ default. Therefore, the mass of survivors with h_{t-1} is $\frac{\bar{\theta} - \frac{x^*(\underline{\theta})}{\tilde{A}}}{\bar{\theta} - \underline{\theta}} M$. Their entrepreneurial productivity is uniformly distributed over $\left[\frac{x^*(\underline{\theta})}{\tilde{A}}, \bar{\theta}\right]$, and they offer $x^*\left(\frac{x^*(\underline{\theta})}{\tilde{A}}\right)$ in the next period. By lemma 4, we know that $x^*\left(\frac{x^*(\underline{\theta})}{\tilde{A}}\right) < x^*(\underline{\theta})$, which implies $\tilde{A}\theta > x^*\left(\frac{x^*(\underline{\theta})}{\tilde{A}}\right)$ for all $\theta \in \left[\frac{x^*(\underline{\theta})}{\tilde{A}}, \bar{\theta}\right]$. Therefore, all the survivors with h_{t-1} continue to survive in the next period and remain in the economy for all succeeding periods without defaulting by offering $x^*\left(\frac{x^*(\underline{\theta})}{\tilde{A}}\right)$. The mass of defaulters with h_{t-1} is $\frac{\frac{x^*(\underline{\theta})}{\tilde{A}} - \underline{\theta}}{\bar{\theta} - \underline{\theta}} M$, and they are replaced with new entrepreneurs in the next period. Let $\Delta \equiv \frac{\frac{x^*(\underline{\theta})}{\tilde{A}} - \underline{\theta}}{\bar{\theta} - \underline{\theta}}$. Note that $\Delta \in (0,1)$, since $\frac{x^*(\underline{\theta})}{\tilde{A}} \in (\underline{\theta}, \bar{\theta})$. Additionally, the economy starts with a unit mass of entrepreneurs in period 0 and $\Omega_0 = U_{[\underline{\theta},\bar{\theta}]}$. Then, by induction, in period $t > 0$, the economy consists of Δ^t mass of entrepreneurs whose entrepreneurial productivities are uniformly distributed over $\left[\underline{\theta}, \bar{\theta}\right]$ and $1 - \Delta^t$ mass of entrepreneurs whose entrepreneurial productivities are uniformly distributed over $\left[\frac{x^*(\underline{\theta})}{\tilde{A}}, \bar{\theta}\right]$. Thus, the cdf $\hat{\Omega}_{\tilde{A}^{t-1}}$ is given by:

$$\hat{\Omega}_{\tilde{A}^{t-1}} = \begin{cases} \Delta^t \frac{\theta - \underline{\theta}}{\bar{\theta} - \underline{\theta}} & \text{if } \theta \in \left[\underline{\theta}, \frac{x^*(\underline{\theta})}{\tilde{A}}\right) \\ \Delta^t \frac{\theta - \underline{\theta}}{\bar{\theta} - \underline{\theta}} + (1 - \Delta^t)\frac{\theta\tilde{A} - x^*(\underline{\theta})}{\bar{\theta}\tilde{A} - x^*(\underline{\theta})} & \text{if } \theta \in \left[\frac{x^*(\underline{\theta})}{\tilde{A}}, \bar{\theta}\right]. \end{cases} \tag{23}$$

Substituting (23) into (10), we obtain the aggregate production as $\hat{Y}(\tilde{A}^t) = \frac{1}{2}\Delta^t\tilde{A}(\underline{\theta} + \bar{\theta}) + \frac{1}{2}(1 - \Delta^t)\left(x^*(\underline{\theta}) + \tilde{A}\bar{\theta}\right)$, which completes the proof. ∎

Proof of proposition 8. First, suppose that $\tilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$. According to proposition 7-1, we have

$\hat{Y}(\hat{A}^{\eta-1}) = \frac{\tilde{A}(\underline{\theta}+\bar{\theta})}{2}$, $\Omega_\eta = U_{[\underline{\theta},\bar{\theta}]}$, and every entrepreneur offers $x^*(\underline{\theta})$ in period η. Since $A'\underline{\theta} > \tilde{A}\underline{\theta} \geq x^*(\underline{\theta})$, all entrepreneurs in period η make the repayment. Thus, we have $\Omega_{\eta+1} = U_{[\underline{\theta},\bar{\theta}]}$. Therefore, for all $t \geq \eta + 1$, we have $\Omega_t = U_{[\underline{\theta},\bar{\theta}]}$ and $\hat{Y}(\hat{A}^t) = \frac{\tilde{A}(\underline{\theta}+\bar{\theta})}{2} = \hat{Y}(\hat{A}^{\eta-1})$.

Second, consider the case where $\tilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$. According to proposition 7-2, we have $\hat{Y}(\hat{A}^{\eta-1}) = \frac{x^*(\underline{\theta})+\tilde{A}\bar{\theta}}{2}$, $\Omega_\eta = U_{\left[\frac{x^*(\underline{\theta})}{\tilde{A}},\bar{\theta}\right]}$, and every entrepreneur offers $x^*\left(\frac{x^*(\underline{\theta})}{\tilde{A}}\right)$ in period η. Note that $\frac{x^*(\underline{\theta})}{\tilde{A}} > \underline{\theta}$ because $\tilde{A} < \frac{x^*(\underline{\theta})}{\underline{\theta}}$. Thus, we have $A'\frac{x^*(\underline{\theta})}{\tilde{A}} > x^*(\underline{\theta}) > x^*\left(\frac{x^*(\underline{\theta})}{\tilde{A}}\right)$, so all entrepreneurs in period η make the repayment. Therefore, $\Omega_{\eta+1} = U_{\left[\frac{x^*(\underline{\theta})}{\tilde{A}},\bar{\theta}\right]}$. As a result, for all $t \geq \eta + 1$, we have $\Omega_t = U_{\left[\frac{x^*(\underline{\theta})}{\tilde{A}},\bar{\theta}\right]}$ and $\hat{Y}(\hat{A}^t) = \frac{x^*(\underline{\theta})+\tilde{A}\bar{\theta}}{2} = \hat{Y}(\hat{A}^{\eta-1})$.

Third, consider the case where $\tilde{A} \in \left(0, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right]$ and $A' \in \left(\tilde{A}, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right] \cup \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$. According to proposition 7-1, we have $\hat{Y}(\hat{A}^{\eta-1}) = \frac{\tilde{A}(\bar{\theta}+\underline{\theta})}{2}$, $\Omega_\eta = U_{[\underline{\theta},\bar{\theta}]}$, and all entrepreneurs offer $x^*(\underline{\theta})$ in period η. If $A' \in \left(\tilde{A}, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right]$, then all entrepreneurs whose entrepreneurial productivity below $\bar{\theta}$ default because $A'\theta < \frac{x^*(\underline{\theta})}{\bar{\theta}}\bar{\theta} = x^*(\underline{\theta})$ for all $\theta < \bar{\theta}$, which implies $\Omega_{\eta+1} = U_{[\underline{\theta},\bar{\theta}]}$. On the other hand, if $A' \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$, then all entrepreneurs make the repayment in period η because $A'\underline{\theta} \geq \frac{x^*(\underline{\theta})}{\underline{\theta}}\underline{\theta} = x^*(\underline{\theta})$, and thus, $\Omega_{\eta+1} = U_{[\underline{\theta},\bar{\theta}]}$. In both cases, the economy returns to the pre-shock level from period $\eta + 1$. Therefore, for all $t \geq \eta + 1$, we have $\Omega_t = U_{[\underline{\theta},\bar{\theta}]}$ and $\hat{Y}(\hat{A}^t) = \frac{\tilde{A}(\underline{\theta}+\bar{\theta})}{2} = \hat{Y}(\hat{A}^{\eta-1})$.

From now on, consider the case where $\tilde{A} \in \left(0, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right]$ and $A' \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$. Define $\theta'(A') = \frac{x^*(\underline{\theta})}{A'} \in (\underline{\theta},\bar{\theta})$. Given that $\tilde{A} \in \left(0, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right]$, all entrepreneurs with entrepreneurial productivity below $\bar{\theta}$ who establish their companies in period $\eta + 1$ or later will offer $x^*(\underline{\theta})$ and eventually default. Consequently, in periods $t \geq \eta + 1$, the economy consists of at most two groups of entrepreneurs: 1) Those who have survived since period η (existing entrepreneurs), and 2) those who established their companies in period t (new entrepreneurs).[3]

In period η, all entrepreneurs offer $x^*(\underline{\theta})$. Among them, those with entrepreneurial productivity $\theta \geq \theta'(A')$ repay the debt, while the rest default. Consequently, in period $\eta+1$, there are $\frac{\bar{\theta}-\theta'(A')}{\bar{\theta}-\underline{\theta}}$ mass of the existing entrepreneurs and $\frac{\theta'(A')-\underline{\theta}}{\bar{\theta}-\underline{\theta}}$ mass of new entrepreneurs. The cdf of the entrepreneurial productivity for the existing entrepreneurs is $U_{[\theta'(A'),\bar{\theta}]}$, and that of the newly established entrepreneurs is $U_{[\underline{\theta},\bar{\theta}]}$, respectively.

Note from proposition 7 that $\hat{Y}(\hat{A}^{\eta-1}) = \frac{\tilde{A}(\underline{\theta}+\bar{\theta})}{2}$. Then, given the common productivity in period $\eta+1$ as $\tilde{A}$, we obtain

$$\begin{aligned}
\hat{Y}(\hat{A}^{\eta+1}) &= \frac{\bar{\theta}-\theta'(A')}{\bar{\theta}-\underline{\theta}} \times \frac{\tilde{A}(\theta'(A')+\bar{\theta})}{2} + \frac{\theta'(A')-\underline{\theta}}{\bar{\theta}-\underline{\theta}} \times \hat{Y}(\hat{A}^{\eta-1}) \\
&= \hat{Y}(\hat{A}^{\eta-1}) + \frac{\bar{\theta}-\theta'(A')}{\bar{\theta}-\underline{\theta}} \times \frac{\tilde{A}(\theta'(A')-\underline{\theta})}{2}.
\end{aligned}$$

In period $\eta+1$, the existing entrepreneurs offer $x^*(\theta'(A'))$, and those with entrepreneurial productivity $\theta \geq \frac{x^*(\theta'(A'))}{\tilde{A}}$ repay the debt while the others default. This leads to three relevant cases.

First, if $\tilde{A} \geq \frac{x^*(\theta'(A'))}{\theta'(A')}$, all the existing entrepreneurs make the repayment in period $\eta+1$ and remain in the economy for all succeeding periods by offering $x^*(\theta'(A'))$. Thus, $\hat{Y}(\hat{A}^t) = \hat{Y}(\hat{A}^{\eta+1})$ for all $t \geq \eta+2$.

Second, if $\frac{x^*(\theta'(A'))}{\bar{\theta}} > \tilde{A}$, then all the existing entrepreneurs default in period $\eta+1$. As a

[3]We can neglect entrepreneurs who establish their companies in period $\eta+1$ or later whose entrepreneurial productivity is $\overline{\theta}$, as they constitute a negligible portion of the overall representation.

result, the economy starts with all new entrepreneurs in the morning in period $\eta+2$, and thus $\hat{Y}(\hat{A}^t) = \frac{\tilde{A}(\bar{\theta}+\underline{\theta})}{2} = \hat{Y}(\hat{A}^{\eta-1})$ for all $t \geq \eta+2$.

Finally, if $\frac{x^*(\theta'(A'))}{\bar{\theta}} \leq \tilde{A} < \frac{x^*(\theta'(A'))}{\theta'(A')}$, then the existing entrepreneurs with entrepreneurial productivity above $\frac{x^*(\theta'(A'))}{\tilde{A}}$ make the repayment in period $\eta+1$, while those with entrepreneurial productivity below $\frac{x^*(\theta'(A'))}{\tilde{A}}$ default. Consequently, in period $\eta+2$, there are $\frac{\bar{\theta}-\frac{x^*(\theta'(A'))}{\tilde{A}}}{\bar{\theta}-\underline{\theta}}$ mass of the existing entrepreneurs and $\frac{\frac{x^*(\theta'(A'))}{\tilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}$ mass of newly established entrepreneurs in period $\eta+2$, and the cdf of the entrepreneurial productivities for these two groups are given by $U_{\left[\frac{x^*(\theta'(A'))}{\tilde{A}},\bar{\theta}\right]}$ and $U_{[\underline{\theta},\bar{\theta}]}$, respectively. Thus, we obtain

$$
\begin{aligned}
\hat{Y}(\hat{A}^{\eta+2}) &= \frac{\bar{\theta}-\frac{x^*(\theta'(A'))}{\tilde{A}}}{\bar{\theta}-\underline{\theta}} \times \frac{\tilde{A}\left(\frac{x^*(\theta'(A'))}{\tilde{A}}+\bar{\theta}\right)}{2} + \frac{\frac{x^*(\theta'(A'))}{\tilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}} \times \frac{\tilde{A}(\underline{\theta}+\bar{\theta})}{2} \\
&= \frac{\tilde{A}(\underline{\theta}+\bar{\theta})}{2} + \frac{\tilde{A}\bar{\theta}-x^*(\theta'(A'))}{\tilde{A}(\bar{\theta}-\underline{\theta})} \times \frac{x^*(\theta'(A'))-\tilde{A}\underline{\theta}}{2}.
\end{aligned}
$$

Furthermore, note that the existing entrepreneurs will offer $x^*\left(\frac{x^*(\theta'(A'))}{\tilde{A}}\right) < x^*(\theta'(A'))$ and repay the debt without defaults for all periods $t \geq \eta+2$. Therefore, $\hat{Y}(\bar{A}^t) = \hat{Y}(\hat{A}^{\eta+2})$ for all $t > \eta+2$.

Note that $\frac{x^*(\theta'(A'))}{\theta'(A')}$ increases in A' because $\theta'(A')$ decreases in A' and $x^*(\cdot)$ is a decreasing function. Moreover, $\lim_{A'\to 0}\frac{x^*(\theta'(A'))}{\theta'(A')} = 0$, $\lim_{A'\to\frac{x^*(\underline{\theta})}{\bar{\theta}}}\frac{x^*(\theta'(A'))}{\theta'(A')} = \frac{x^*(\bar{\theta})}{\bar{\theta}}$, and $\lim_{A'\to\frac{x^*(\underline{\theta})}{\underline{\theta}}}\frac{x^*(\theta'(A'))}{\theta'(A')} = \frac{x^*(\underline{\theta})}{\underline{\theta}} > \frac{x^*(\underline{\theta})}{\bar{\theta}}$. Thus, there exists B^* such that $\tilde{A} = \frac{x^*(\theta'(B^*))}{\theta'(B^*)}$. Here, if $\tilde{A} \in \left(\frac{x^*(\bar{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right]$, then $B^* \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, while B^* is weakly below $\frac{x^*(\underline{\theta})}{\bar{\theta}}$ when $\tilde{A} \in \left(0, \frac{x^*(\bar{\theta})}{\bar{\theta}}\right]$. By defining $A^* = \max\left\{\frac{x^*(\bar{\theta})}{\bar{\theta}}, B^*\right\} \in \left[\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, we obtain that $\tilde{A} \geq \frac{x^*(\theta'(A'))}{\theta'(A')}$ if and only if $A' \leq A^*$.

By combining the above cases using the definition of A^*, we obtain the results of proposition 8. ■

Proof of proposition 9. First, consider the case where $\tilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$ and $A' \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$. According to the proof of proposition 7, the population of entrepreneurs consists of two parts: $\frac{\bar{\theta}-\frac{x^*(\underline{\theta})}{A'}}{\bar{\theta}-\underline{\theta}}$ mass of survivors whose entrepreneurial productivity is uniformly distributed over $\left[\frac{x^*(\underline{\theta})}{A'}, \bar{\theta}\right]$, and $\frac{\frac{x^*(\underline{\theta})}{A'}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}$ mass of new entrepreneurs in period $\eta + 1$. Since $\tilde{A} \geq \frac{x^*(\underline{\theta})}{\underline{\theta}}$, all entrepreneurs make repayments and remain in the economy for all periods $t \geq \eta + 1$.[4] Thus,

$$\hat{Y}(\hat{A}^t) = \frac{\frac{x^*(\underline{\theta})}{A'} - \underline{\theta}}{\bar{\theta} - \underline{\theta}} \times \frac{\tilde{A}(\underline{\theta} + \bar{\theta})}{2} + \frac{\frac{x^*(\underline{\theta})}{A'} - \underline{\theta}}{\bar{\theta} - \underline{\theta}} \times \frac{\tilde{A}\left(\frac{x^*(\underline{\theta})}{A'} + \bar{\theta}\right)}{2}$$

for all $t \geq \eta + 1$. By letting $\Delta' = \frac{\frac{x^*(\underline{\theta})}{A'}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}$ and rearranging the above analysis, we obtain the first part of proposition 9.

Next, consider the case where $\tilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$ and $A' \in \left(0, \frac{x^*(\tilde{\theta})}{\bar{\theta}}\right]$. By proposition 7-2, $\Omega_\eta = U_{[\tilde{\theta},\bar{\theta}]}$, where $\tilde{\theta} \equiv \frac{x^*(\underline{\theta})}{\tilde{A}}$, and every entrepreneur offers $x^*(\tilde{\theta})$ in period η. Since $A'\bar{\theta} \leq x^*(\tilde{\theta})$, all entrepreneurs with entrepreneurial productivity below$\bar{\theta}$ default in period η, so $\Omega_{\eta+1} = U_{[\underline{\theta},\bar{\theta}]}$. Then, by proposition 7-2, $\hat{Y}(\tilde{A}^t) = \Delta^{t-\eta-1}\frac{\tilde{A}(\underline{\theta}+\bar{\theta})}{2} + [1 - \Delta^{t-\eta-1}]\frac{x^*(\underline{\theta})+\tilde{A}\bar{\theta}}{2}$ for $t \geq \eta + 1$, where $\Delta = \frac{\frac{x^*(\underline{\theta})}{\tilde{A}}-\underline{\theta}}{\bar{\theta}-\underline{\theta}}$. Now, consider the case where $\tilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$ and $A' \in \left(\frac{x^*(\tilde{\theta})}{\bar{\theta}}, \frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right)$. In this case, entrepreneurs with entrepreneurial productivity in $\left[\tilde{\theta}, \frac{x^*(\tilde{\theta})}{A'}\right)$ default and are replaced with new entrepreneurs in period $\eta + 1$, and the other entrepreneurs with entrepreneurial productivity in $\left[\frac{x^*(\tilde{\theta})}{A'}, \bar{\theta}\right]$ survive. The mass of defaulted and surviving entrepreneurs are given as $\frac{\frac{x^*(\tilde{\theta})}{A'}-\tilde{\theta}}{\bar{\theta}-\tilde{\theta}}$ and $\frac{\bar{\theta}-\frac{x^*(\tilde{\theta})}{A'}}{\bar{\theta}-\tilde{\theta}}$, respectively. Thus,

$$\hat{Y}(\hat{A}^t) = \frac{\frac{x^*(\tilde{\theta})}{A'} - \tilde{\theta}}{\bar{\theta} - \tilde{\theta}} \left[\Delta^{t-\eta-1}\frac{\tilde{A}(\underline{\theta} + \bar{\theta})}{2} + [1 - \Delta^{t-\eta-1}]\frac{x^*(\underline{\theta}) + \tilde{A}\bar{\theta}}{2}\right] + \frac{\bar{\theta} - \frac{x^*(\tilde{\theta})}{A'}}{\bar{\theta} - \tilde{\theta}}\frac{\tilde{A}}{2}\left(\frac{x^*(\tilde{\theta})}{A'} + \bar{\theta}\right).$$

[4] Surviving entrepreneurs offer $x^*\left(\frac{x^*(\underline{\theta})}{A'}\right) < \tilde{A}\underline{\theta}$ and new entrepreneurs offer $x^*(\underline{\theta}) \leq \tilde{A}\underline{\theta}$.

By letting $\tilde{\Delta}' = \min\left\{1, \frac{\frac{x^*(\tilde{\theta})}{A'} - \tilde{\theta}}{\bar{\theta} - \tilde{\theta}}\right\}$ and combining the two cases aforementioned with $\tilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, we obtain the second part of proposition 9.

We continue with the remaining parts of the proof. First, suppose that $\tilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$ and $A' \in \left(0, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right] \cup \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, \tilde{A}\right)$. In this case, we have $\Omega_t = U_{[\underline{\theta}, \bar{\theta}]}$ and $\hat{Y}(\hat{A}^t) = \frac{\tilde{A}(\underline{\theta} + \bar{\theta})}{2} = \hat{Y}(\hat{A}^{\eta-1})$ for all $t \geq \eta + 1$. Next, suppose that $\tilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$ and $A' \in \left[\frac{x^*(\tilde{\theta})}{\bar{\theta}}, \tilde{A}\right)$. By proposition 7-2, $\Omega_\eta = U_{[\tilde{\theta}, \bar{\theta}]}$, where $\tilde{\theta} \equiv \frac{x^*(\underline{\theta})}{\tilde{A}}$, and every entrepreneur offers $x^*(\tilde{\theta})$ in period η. Moreover, all entrepreneurs survive for all $A' \in \left[\frac{x^*(\tilde{\theta})}{\bar{\theta}}, \tilde{A}\right)$. Thus, $\hat{Y}(\hat{A}^t) = \frac{\tilde{A}(\tilde{\theta} + \bar{\theta})}{2} = \frac{x^*(\underline{\theta}) + \tilde{A}\bar{\theta}}{2} = \hat{Y}(\hat{A}^{\eta-1})$ for all $t \geq \eta + 1$. Finally, suppose that $\tilde{A} \in \left(0, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right]$, which implies that $A' \in \left(0, \frac{x^*(\underline{\theta})}{\bar{\theta}}\right)$. In this case, by proposition 7-1, we have $\Omega_t = U_{[\underline{\theta}, \bar{\theta}]}$ and $\hat{Y}(\hat{A}^t) = \frac{\tilde{A}(\underline{\theta} + \bar{\theta})}{2} = \hat{Y}(\hat{A}^{\eta-1})$ for all $t \geq \eta + 1$. ■

Proof of proposition 10. Note that $\sum_{t=0}^{\eta-1} \beta^t \hat{Y}(\hat{A}^t) = \sum_{t=0}^{\eta-1} \beta^t \hat{Y}(\tilde{A}^t)$ and $\hat{Y}(\hat{A}^\eta) < \hat{Y}(\tilde{A}^\eta)$. Thus, if $\hat{Y}(\hat{A}^t) \leq \hat{Y}(\tilde{A}^t)$ for all $t \geq \eta + 1$, then we have $\sum_{t=0}^{\infty} \beta^t [\hat{Y}(\hat{A}^t) - \hat{Y}(\tilde{A}^t)] < 0$. Therefore, for a negative shock to be constructive, there must exist a time period $\tau > \eta$ such that $\hat{Y}(\hat{A}^\tau) > \hat{Y}(\tilde{A}^\tau)$. Based on proposition 9, it suffices to focus on the following two cases: 1) $\tilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$ with a shock $A' \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$ and 2) $\tilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$ with a shock $A' \in \left(\frac{x^*(\tilde{\theta})}{\bar{\theta}}, \frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right)$.

First, consider the case where $\tilde{A} \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$ and $A' \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$. From proposition 9, we obtain the expression:

$$\beta^{-\eta} \sum_{t=0}^{\infty} \beta^t [\hat{Y}(\hat{A}^t) - \hat{Y}(\tilde{A}^t)] = (A' - \tilde{A}) \frac{\underline{\theta} + \bar{\theta}}{2} + \frac{\beta}{1 - \beta} \times \frac{\bar{\theta} - \frac{x^*(\underline{\theta})}{A'}}{\bar{\theta} - \underline{\theta}} \times \frac{\tilde{A}}{2} \left(\frac{x^*(\underline{\theta})}{A'} - \underline{\theta}\right).$$

Then, we have $\sum_{t=0}^{\infty} \beta^t [\hat{Y}(\hat{A}^t) - \hat{Y}(\tilde{A}^t)] > 0$ if and only if $\beta > \hat{\beta}(\tilde{A}, A')$, where

$$\hat{\beta}(\tilde{A}, A') \equiv \frac{\bar{\theta}^2 - \underline{\theta}^2}{\bar{\theta}^2 - \underline{\theta}^2 + \frac{\tilde{A}}{\tilde{A} - A'} \left(\bar{\theta} - \frac{x^*(\underline{\theta})}{A'}\right) \left(\frac{x^*(\underline{\theta})}{A'} - \underline{\theta}\right)}. \tag{24}$$

Note that $\hat{\beta}(\tilde{A}, A') \in (0,1)$ because $\underline{\theta} < \frac{x^*(\underline{\theta})}{A'} < \bar{\theta}$ and $\tilde{A} - A' > 0$. Therefore, for sufficiently high values of β, the set $I(\tilde{A}, \beta)$ is nonempty. Furthermore, note that $A' \in I(\tilde{A}, \beta)$ if and only if

$$
\begin{aligned}
F_1(A') &\equiv 2A'^2(\bar{\theta} - \underline{\theta})\beta^{-\eta} \sum_{t=0}^{\infty} \beta^t [\hat{Y}(\hat{A}^t) - \hat{Y}(\tilde{A}^t)] \\
&= A'^2(A' - \tilde{A})(\bar{\theta}^2 - \underline{\theta}^2) + \frac{\beta}{1-\beta}\tilde{A}(A'\bar{\theta} - x^*(\underline{\theta}))(x^*(\underline{\theta}) - A'\underline{\theta}) > 0.
\end{aligned}
$$

Here, $F_1(A')$ is a cubic polynomial. Since $F_1\left(\frac{x^*(\underline{\theta})}{\bar{\theta}}\right) < 0$ and $F_1\left(\frac{x^*(\underline{\theta})}{\underline{\theta}}\right) < 0$, whenever $I(\tilde{A}, \beta) \neq \varnothing$, there exist $A_1' \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$ and $A_2' > A_1'$ such that $F_1(A_1') > 0$, $F_1'(A_1') = 0$, $F_1(A_2') < 0$, and $F_1'(A_2') = 0$. Then, there exist $A_1'' \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, A_1'\right)$ and $A_2'' \in \left(A_1', \min\left\{A_2', \frac{x^*(\underline{\theta})}{\underline{\theta}}\right\}\right)$ such that $F_1(A_1'') = F_1(A_2'') = 0$ and $I(\tilde{A}, \beta) = (A_1'', A_2'')$. Thus, $I(\tilde{A}, \beta)$ is an open interval.

Next, take any $\tilde{A}_1, \tilde{A}_2 \in \left[\frac{x^*(\underline{\theta})}{\underline{\theta}}, 1\right]$ such that $\tilde{A}_2 > \tilde{A}_1$ and both $I(\tilde{A}_1, \beta)$ and $I(\tilde{A}_2, \beta)$ are nonempty. Suppose that $A' \in I(\tilde{A}_2, \beta)$, i.e., $\beta > \hat{\beta}(\tilde{A}_2, A')$. Note, from (24), that $\frac{\partial \hat{\beta}(\tilde{A}, A')}{\partial \tilde{A}} > 0$ because $\frac{\tilde{A}}{\tilde{A} - A'}$ decreases in $\tilde{A}$ given that $\tilde{A} > A'$. Then, we have $\beta > \hat{\beta}(\tilde{A}_2, A') > \hat{\beta}(\tilde{A}_1, A')$, so $A' \in I(\tilde{A}_1, \beta)$. Thus, $I(\tilde{A}_2, \beta) \subset I(\tilde{A}_1, \beta)$.

Now, consider the case where $\tilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$ and $A' \in \left(\frac{x^*(\tilde{\theta})}{\bar{\theta}}, \frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right)$. From proposition 9 and letting $p(A') = \frac{\frac{x^*(\tilde{\theta})}{A'} - \tilde{\theta}}{\bar{\theta} - \tilde{\theta}} = \frac{x^*(\tilde{\theta})\frac{\tilde{A}}{A'} - x^*(\underline{\theta})}{\tilde{A}\bar{\theta} - x^*(\underline{\theta})}$ be the mass of defaulting entrepreneurs in period η, we obtain:

$$
\begin{aligned}
\sum_{t=\eta}^{\infty}\beta^{t-\eta}\hat{Y}(\hat{A}^t) &= \hat{Y}(\hat{A}^\eta)+\beta\sum_{t=\eta+1}^{\infty}\beta^{t-\eta-1}\hat{Y}(\hat{A}^t)\\
&= \hat{Y}(\hat{A}^\eta)+\beta\sum_{t=\eta+1}^{\infty}\beta^{t-\eta-1}\left[-p(A')\frac{x^*(\underline{\theta})-\tilde{A}\underline{\theta}}{2}+p(A')\frac{x^*(\underline{\theta})+\tilde{A}\bar{\theta}}{2}+(1-p(A'))\left(\frac{x^*(\tilde{\theta})\frac{\tilde{A}}{A'}+\tilde{A}\bar{\theta}}{2}\right)\right]\\
&= \hat{Y}(\hat{A}^\eta)+\beta\sum_{t=\eta+1}^{\infty}\beta^{t-\eta-1}\left[-p(A')\frac{x^*(\underline{\theta})-\tilde{A}\underline{\theta}}{2}+\frac{x^*(\underline{\theta})+\tilde{A}\bar{\theta}}{2}+(1-p(A'))\left(\frac{x^*(\tilde{\theta})\frac{\tilde{A}}{A'}-x^*(\underline{\theta})}{2}\right)\right]\\
&= \frac{A'(\tilde{\theta}+\bar{\theta})}{2}-\frac{\beta p(A')}{1-\beta\triangle}\times\frac{1}{2}[x^*(\underline{\theta})-\widetilde{A}\underline{\theta}]
\end{aligned}
$$

(25)

$$
+\frac{\beta}{1-\beta}\frac{1}{2}\left[x^*(\underline{\theta})+\widetilde{A}\bar{\theta}+(1-p(A'))\left(x^*(\widetilde{\theta})\frac{\widetilde{A}}{A'}-x^*(\underline{\theta})\right)\right].
$$

Using the facts that $\hat{Y}(\tilde{A}^t)=\frac{\widetilde{A}(\tilde{\theta}+\bar{\theta})}{2}$ for all $t>\eta$ where $\tilde{\theta}=\frac{x^*(\underline{\theta})}{\tilde{A}}$, we can derive from (25) the following expression:

$$
\begin{aligned}
&\beta^{-\eta}\sum_{t=0}^{\infty}\beta^t[\hat{Y}(\hat{A}^t)-\hat{Y}(\tilde{A}^t)]\\
&= \frac{\tilde{\theta}+\bar{\theta}}{2}(A'-\tilde{A})-p(A')\times\frac{\beta[x^*(\underline{\theta})-\tilde{A}\underline{\theta}]}{2(1-\beta\Delta)}+(1-p(A'))\times\frac{\beta\left[x^*(\tilde{\theta})\frac{\tilde{A}}{A'}-x^*(\underline{\theta})\right]}{2(1-\beta)}\\
&= \frac{\tilde{\theta}+\bar{\theta}}{2}(A'-\tilde{A})+\frac{\beta\left(x^*(\tilde{\theta})\frac{\tilde{A}}{A'}-x^*(\underline{\theta})\right)}{2(1-\beta)(\tilde{A}\bar{\theta}-x^*(\underline{\theta}))}\left[\tilde{A}\bar{\theta}-x^*(\tilde{\theta})\frac{\tilde{A}}{A'}-\frac{1-\beta}{1-\beta\Delta}(x^*(\underline{\theta})-\tilde{A}\underline{\theta})\right]
\end{aligned}
$$

(26) $\quad \equiv F_2(A').$

Taking the first derivative of $F_2(A')$, we obtain:

$$F_2'(A') = \frac{\tilde{\theta}+\bar{\theta}}{2} \tag{27}$$
$$+ \frac{\beta x^*(\tilde{\theta})\frac{\widetilde{A}}{A'^2}}{2(1-\beta)(\widetilde{A}\bar{\theta}-x^*(\underline{\theta}))}\left[\frac{(1-\beta)(x^*(\underline{\theta})-\widetilde{A}\underline{\theta})}{1-\beta\triangle}+2x^*(\tilde{\theta})\frac{\widetilde{A}}{A'}-x^*(\underline{\theta})-\widetilde{A}\bar{\theta}\right].$$

Then, it can be verified from (26) and (27) that $F_2\left(\frac{x^*(\tilde{\theta})}{\bar{\theta}}\right) < 0$, $F_2\left(\frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right) < 0$, and $F_2'\left(\frac{x^*(\tilde{\theta})}{\bar{\theta}}\right) > 0$. Because $A'^2F_2(A')$ is a cubic polynomial, there can be at most two positive real values ς_1 and ς_2 such that $F_2'(\varsigma_1) = F_2'(\varsigma_2) = 0$.[5] Thus, if $F_2'\left(\frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right) < 0$, then F_2 is single-peaked in $\left(\frac{x^*(\tilde{\theta})}{\bar{\theta}}, \frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right)$, so there exists $A^* \in \left(\frac{x^*(\tilde{\theta})}{\bar{\theta}}, \frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right)$ such that $F_2'(A^*) = 0$ and $A^* = \underset{A' \in \left(\frac{x^*(\tilde{\theta})}{\bar{\theta}}, \frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right)}{\arg\max} F_2(A')$. Thus, if $F_2'\left(\frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right) < 0$ and $F_2(A^*) > 0$, then $I(\tilde{A}, \beta)$ is a nonempty open subinterval of $\left(\frac{x^*(\tilde{\theta})}{\bar{\theta}}, \frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right)$.

First, we evaluate $F_2'\left(\frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right)$ and find:

$$F_2'\left(\frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right) = \frac{\tilde{\theta}+\bar{\theta}}{2} + \frac{\beta x^*(\tilde{\theta})\frac{\tilde{A}}{A'^2}}{2(1-\beta)(\tilde{A}\bar{\theta}-x^*(\underline{\theta}))}\left[\frac{(1-\beta)(x^*(\underline{\theta})-\tilde{A}\underline{\theta})}{1-\beta\Delta}-(\tilde{A}\bar{\theta}-x^*(\underline{\theta}))\right].$$

Given that $\tilde{A}\underline{\theta} < x^*(\underline{\theta}) < \tilde{A}\bar{\theta}$ and $\frac{1-\beta}{1-\beta\Delta} \in (0,1)$, if $\tilde{A}$ is sufficiently high within the range of $\left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$, then $\frac{(1-\beta)(x^*(\underline{\theta})-\tilde{A}\underline{\theta})}{1-\beta\Delta} < \tilde{A}\bar{\theta} - x^*(\underline{\theta})$. Because $\frac{\beta}{1-\beta}$ increases with β while $\frac{1-\beta}{1-\beta\Delta}$ decreases with β, if β is also sufficiently high, then $F_2'\left(\frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right) < 0$.

[5] The equation $F_2(A') = a_1A' + a_2 + a_3A'^{-1} + a_4A'^{-2}$ holds for certain real coefficients a_1, a_2, a_3, and a_4. Additionally, $F_2'(A') = a_1 - a_3A'^{-2} - 2a_4A'^{-3} = a_1A'^{-3}\Pi_{i=1,2,3}(A'-\varsigma_i)$ holds for some $\varsigma_1, \varsigma_2, \varsigma_3 \in \mathbb{C}$ satisfying $\varsigma_1+\varsigma_2+\varsigma_3 = 0$. It should be noted that among the roots ς_1, ς_2, and ς_3, there can be at most two values, denoted as i, such that $\varsigma_i \in \mathbb{R}_{++}$ and $F_2'(\varsigma_i) = 0$.

Next, utilizing the definition of A^*, i.e., $F_2'(A^*) = 0$, we obtain from (27) that

$$x^*(\tilde{\theta})\frac{\tilde{A}}{A^*} - x^*(\underline{\theta}) = \frac{1}{2}(\tilde{A}\bar{\theta} - x^*(\underline{\theta})) - \frac{1-\beta}{1-\beta\Delta} \times \frac{1}{2}(x^*(\underline{\theta}) - \tilde{A}\underline{\theta}) - \frac{(1-\beta)(\tilde{A}\bar{\theta} - x^*(\underline{\theta}))(\bar{\theta}+\tilde{\theta})A^{*2}}{2\beta x^*(\tilde{\theta})\tilde{A}}.$$

Substituting this result into (26) with $A = A^*$ yields:

$$\begin{aligned} F_2(A^*) = & \frac{\tilde{\theta}+\bar{\theta}}{2}(A^* - \tilde{A}) \\ & + \left(\frac{\beta}{4(1-\beta)} - \frac{\beta}{4(1-\beta\Delta)} \frac{x^*(\underline{\theta}) - \tilde{A}\underline{\theta}}{\tilde{A}\bar{\theta} - x^*(\underline{\theta})} - \frac{(\tilde{\theta}+\bar{\theta})A^{*2}}{4x^*(\tilde{\theta})\tilde{A}} \right) \\ & \times \left(\tilde{A}\bar{\theta} - x^*(\tilde{\theta})\frac{\tilde{A}}{A^*} - \frac{1-\beta}{1-\beta\Delta}(x^*(\underline{\theta}) - \tilde{A}\underline{\theta}) \right). \end{aligned}$$

Observe that as $\tilde{A} \to \frac{x^*(\underline{\theta})}{\underline{\theta}}$, $F_2(A^*)$ converges to

$$\left(\frac{\beta}{4(1-\beta)} - \frac{(\tilde{\theta}+\bar{\theta})A^{*2}\underline{\theta}}{4x^*(\tilde{\theta})x^*(\underline{\theta})} \right) \frac{x^*(\underline{\theta})}{\underline{\theta}} \left(\bar{\theta} - \frac{x^*(\tilde{\theta})}{A^*} \right) + \frac{\tilde{\theta}+\bar{\theta}}{2} \left(A^* - \frac{x^*(\underline{\theta})}{\underline{\theta}} \right). \tag{28}$$

Since $\bar{\theta} - \frac{x^*(\tilde{\theta})}{A^*} > 0$ given that $A^* > \frac{x^*(\tilde{\theta})}{\bar{\theta}}$, if $\frac{\beta}{4(1-\beta)}$ is sufficiently large, then (28) is positive. Therefore, when $\tilde{A} \in \left(\frac{x^*(\underline{\theta})}{\bar{\theta}}, \frac{x^*(\underline{\theta})}{\underline{\theta}}\right)$ and β are sufficiently high, we conclude that $F_2(A^*) > 0$. Consequently, an open interval $I(\tilde{A}, \beta)$ exists within $\left(\frac{x^*(\tilde{\theta})}{\bar{\theta}}, \frac{x^*(\tilde{\theta})}{\tilde{\theta}}\right)$. ■

Online Appendix B

In this appendix, we demonstrate the existence of multiple equilibria. To accomplish this, we define a correspondence $\chi : \Theta \to \mathbb{R}_+$ as follows:

$$\chi(\theta) = \left[x^*(\theta), \min\left\{x^{**}, \frac{b(\theta)}{2}, \frac{\beta\theta}{2}\right\}\right), \tag{29}$$

where $x^{**} = \min\left\{x : x - \frac{\log\left(\frac{\underline{\theta}+\bar{\theta}}{2}\right) - \log(\underline{\theta})}{\frac{\underline{\theta}+\bar{\theta}}{2} - \underline{\theta}} x^2 \geq r\right\}$. Note that $x^*(\underline{\theta}) < x^{**}$ due to the fact that $x^*(\underline{\theta}) = \min\left\{x : x - \frac{\log\bar{\theta} - \log\underline{\theta}}{\bar{\theta} - \underline{\theta}} x^2 \geq r\right\}$ and $\frac{\log\left(\frac{\underline{\theta}+\bar{\theta}}{2}\right) - \log\underline{\theta}}{\frac{\underline{\theta}+\bar{\theta}}{2} - \underline{\theta}} > \frac{\log\bar{\theta} - \log\underline{\theta}}{\bar{\theta} - \underline{\theta}}$. Furthermore, for any $\theta \in \Theta$, we have $x^*(\theta) < \min\left\{\frac{b(\theta)}{2}, \frac{\beta\theta}{2}\right\}$, based on the definition of $x^*(\cdot)$. Consequently, $x^*(\theta) < \min\left\{x^{**}, \frac{b(\theta)}{2}, \frac{\beta\theta}{2}\right\}$, and $\chi(\theta) \neq \varnothing$ for all $\theta \in \Theta$.

Now consider the entrepreneur's strategy (x, D) that satisfies the following conditions: There exists $\hat{x} : \mathbb{H} \times \mathcal{M} \to \mathbb{R}_+$ which satisfies $\hat{x}(h, U_{[\theta', \bar{\theta}]}) \in \chi(\theta')$ for any $\theta' \in [\underline{\theta}, \bar{\theta})$ and $h \in \mathbb{H}$ such that for any $\theta \in \operatorname{supp} \hat{\Omega}_h$, $x(\theta, h) = \hat{x}(h, \hat{\Omega}_h)$ and $D(\theta, h) = \left[0, \frac{\hat{x}(h, \hat{\Omega}_h)}{\theta}\right) \cap [0, 1]$.[6] Here, we define a "$\chi^e$-strategy" as the family of the entrepreneur's strategies that satisfies the aforementioned condition. We say that such a χ^e-strategy is represented by $\hat{x}$. Since the set $\chi(\theta')$ is uncountable, there exists a continuum of χ^e-strategies. It is important to note that the χ^e-strategy does not impose any restrictions on $\hat{x}(h, \hat{\Omega}_h)$ if $\hat{\Omega}_h$ is not in the form of $U_{[\theta', \bar{\theta}]}$.

In the next proposition, we demonstrate the existence of multiple equilibria. Specifically, we show that for any χ^e-strategy, there exists a belief system that supports the entrepreneurs' strategy as an optimal choice.

[6] If $\hat{\Omega}_h = U_{[\theta', \bar{\theta}]}$ for some $\theta' \in [\underline{\theta}, \bar{\theta})$, then, in equilibrium, $\theta \geq \theta'$, thus, by the definition of $\chi(\cdot)$, $\frac{\hat{x}(h, \hat{\Omega}_h)}{\theta} < \frac{\beta\theta'}{2} \times \frac{1}{\theta} < 1$. Therefore, $\left[0, \frac{\hat{x}(h, \hat{\Omega}_h)}{\theta}\right) \subset [0, 1]$.

Proposition 11 *For any χ^e-strategy (x, D), there exists a belief system μ such that $((x, D), \mu)$ is an equilibrium.*

Proof. Consider any χ^e-strategy (x, D) represented by $\hat{x}$. Let $\widehat{\mathbb{H}} \subset \mathbb{H}$ be the set of all feasible histories generated by (x, D), i.e., the histories of entrepreneurs in some periods who play (x, D).[7] We define a function $\theta^*_{\hat{x}} : \widehat{\mathbb{H}} \to \Theta$ recursively as follows:

$$\theta^*_{\hat{x}}(s, A^{t-1}) = \begin{cases} \underline{\theta} & \text{if } t = s \\ \min\left\{\bar{\theta}, \max\left\{\theta^*_{\hat{x}}(s, A^{t-2}), \frac{\hat{x}((s,A^{t-2}),\hat{\Omega}_{s,A^{t-2}})}{A_{t-1}}\right\}\right\} & \text{for all } t > s. \end{cases}$$

Now, suppose that all entrepreneurs adopt the χ^e-strategy. Entrepreneurs who establish their company in period $s \geq 0$ play $\left(\hat{x}, \left[0, \frac{\hat{x}}{\underline{\theta}}\right)\right)$ in period s, where $\hat{x} = \hat{x}(h_{s-1}, \hat{\Omega}_{h_{s-1}}) \in \chi(\underline{\theta})$ and $h_{s-1} = (s, A^{s-1})$, because $\hat{\Omega}_{h_{s-1}} = U_{[\underline{\theta},\bar{\theta}]}$. Thus, if $\operatorname{supp} \hat{\Omega}_{h_s} \neq \varnothing$, where $h_s = (s, \{A^{s-1}, A_s\})$, we have $\hat{\Omega}_{h_s} = U_{\left[\max\left\{\theta^*_{\hat{x}}(h_{s-1}), \frac{\hat{x}}{A_s}\right\}, \bar{\theta}\right]}$. Then, using induction as explained in the proof of claim 1, we can verify that whenever $\operatorname{supp} \hat{\Omega}_{h_{t-1}} \neq \varnothing$ for any $h_{t-1} \in \widehat{\mathbb{H}}$, we have $\hat{\Omega}_{h_{t-1}} = U_{[\theta^*_{\hat{x}}(h_{t-1}),\bar{\theta}]}$.

Given the function $\theta^*_{\hat{x}}$, we construct a belief system μ such that:

$$\mu(x, h) = \begin{cases} U_{[\theta^*_{\hat{x}}(h),\bar{\theta}]} & \text{if } x \geq \hat{x}(h, \hat{\Omega}_h) \text{ and } h \in \widehat{\mathbb{H}}, \\ U_{\left[\underline{\theta}, \frac{\underline{\theta}+\bar{\theta}}{2}\right]} & \text{otherwise.} \end{cases}$$

[7] There exist histories that cannot be generated by (x, D). For example, suppose that $A_0 = 0$. Then, all entrepreneurs who were born in period 0 default and leave the economy in period 1. Thus, $(0, \{\varnothing, 0\})$ cannot be a history generated by the entrepreneur strategy. Although $(0, \{\varnothing, 0\}) \in \mathbb{H}$, it cannot be a history in period 1 for any entrepreneur, so $(0, \{\varnothing, 0\}) \notin \widehat{\mathbb{H}}$.

By construction, it is straightforward to verify that the belief system μ is consistent, given the entrepreneurs' strategy.

Now, take any $h_{t-1} = (s, A^{t-1}) \in \widehat{\mathbb{H}}$. Dropping the arguments such that $\hat{x} = \hat{x}(h_{t-1}, \hat{\Omega}_{h_{t-1}})$ and $\theta^*_{\hat{x}} = \theta^*_{\hat{x}}(h_{t-1})$, the lender's expected payoff from an entrepreneur with h_{t-1} is:

$$\int_\Theta \left(1 - \left|\left[0, \frac{\hat{x}}{\theta}\right)\right|\right) \hat{x} dU_{[\theta^*_{\hat{x}}, \bar{\theta}]} = \hat{x} - \frac{\hat{x}^2}{b(\theta^*_{\hat{x}})},$$

since $\mu(\hat{x}, h_{t-1}) = U_{[\theta_{\hat{x}^*}, \bar{\theta}]}$. Note that $x - \frac{x^2}{b(\theta^*_{\hat{x}})}$ increases in x whenever $x < \frac{b(\theta^*_{\hat{x}})}{2}$, and that $x^*(\theta^*_{\hat{x}}) - \frac{x^*(\theta^*_{\hat{x}})^2}{b(\theta^*_{\hat{x}})} = r$. Therefore, $\hat{x} - \frac{\hat{x}^2}{b(\theta^*_{\hat{x}})} \geq r$ since $x^*(\theta^*_{\hat{x}}) \leq \hat{x} < \frac{b(\theta^*_{\hat{x}})}{2}$, which implies that the entrepreneur strategy satisfies the lender's incentive compatibility condition given μ.

To conclude, we need to show that the entrepreneur strategy is an optimal strategy for entrepreneurs. The lender's expected payoff from an entrepreneur with h_{t-1} playing (x', D) in period t, where $x' < \hat{x}$, satisfies

$$\begin{aligned}
&\int_\Theta (1 - |D|) x' dU_{\left[\underline{\theta}, \frac{\underline{\theta}+\bar{\theta}}{2}\right]} \\
&\leq \max_{x < x^{**}} \int_\Theta \left(1 - \left|\left[0, \frac{x}{\theta}\right)\right|\right) x dU_{\left[\underline{\theta}, \frac{\underline{\theta}+\bar{\theta}}{2}\right]} \\
&= \max_{x < x^{**}} \left\{ x - \frac{\log\left(\frac{\underline{\theta}+\bar{\theta}}{2}\right) - \log(\underline{\theta})}{\frac{\underline{\theta}+\bar{\theta}}{2} - \underline{\theta}} x^2 \right\} < r,
\end{aligned}$$

which implies that playing $x' < \hat{x}$ in any period does not satisfy the lender's incentive compatibility condition given μ. Thus, $\hat{x}$ is the minimum incentive-compatible contract in which $\omega_\mu \geq r$ at each period. Moreover, for any $h_{t-1} \in \widehat{\mathbb{H}}$, $\hat{\Omega}_{h_{t-1}} = U_{[\theta^*_{\hat{x}}(h_{t-1}), \bar{\theta}]}$. Therefore, by lemma 2 and proposition 1, every entrepreneur with h_{t-1} offers $\hat{x}$. Additionally, note that

$\hat{x}(h_{t-1}, \hat{\Omega}_{h_{t-1}}) < \frac{\beta\theta^*_{\hat{x}}(h_{t-1})}{2}$ by construction of the correspondence χ in (29). Thus, the optimal default strategy after making contract $\hat{x}(h_{t-1}, \hat{\Omega}_{h_{t-1}})$ is $D_t = \left[0, \frac{\hat{x}(h_{t-1}, \hat{\Omega}_{h_{t-1}})}{\theta}\right)$, as explained in the proof of propositions 2 and 3. ■