# Internet Appendix for:

## Factor Model Comparisons with Conditioning Information

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This Internet Appendix provides proofs and ancilliary results for the named paper. Section I provides detailed descriptions of bias corrections and proofs of the main propositions. Section 2 derives results for the case with time-varying covariances. Section 3 establishes the equivalence between two formulations of the efficiency with respect to Z portfolio optimization. Section 4 presents a decomposition of the Sharpe ratio improvements with dynamic trading. Section 5 presents simulations and empirical results using alternative choices of lagged instruments. Section 6 presents results for mimicking portfolios for nontraded factors.

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## 1. Proofs of the Main results

Plug-in Bias Adjustments

The first bias adjustment is based on the noncental Chi-square distribution, assuming normality and a given covariance matrix. The estimated conditional mean return is $\hat{\delta}'Z_t$ and its conditional variance is Q ≡ Var( $(\hat{\delta} - \delta)'Z_t|Z$) = V $[Z_t'(Z'Z)^{-1}Z_t]$. The quadratic form $[(\hat{\delta} - \delta)'Z_t]'\, Q^{-1}\, [(\hat{\delta} - \delta)'Z_t]$ is central Chi-square with N degrees of freedom. The quadratic form $[(\hat{\delta} - \delta)'Z_t]'\, V^{-1}\, [(\hat{\delta} - \delta)'Z_t]$ has mean equal to $[Z_t'(Z'Z)^{-1}Z_t]$ and the sample squared conditional Sharpe ratio, $[\hat{\delta}'Z_t]'\, V^{-1}\, [\hat{\delta}'Z_t]$ has mean $[\delta'Z_t]'\, V^{-1}\, [\delta'Z_t] + N\, [Z_t'(Z'Z)^{-1}Z_t]$. The bias-adjusted conditional ratio is:

$$S1^{*2}(Z_t) = [\hat{\delta}'Z_t]'\, V^{-1}\, [\hat{\delta}'Z_t] - N\, [Z_t'(Z'Z)^{-1}Z_t], \qquad \text{(A.1)}$$

where the true variance matrix V is assumed in this adjustment. The adjusted UE maximum squared Sharpe ratio is then found as $[\frac{1}{T}\sum_{t=1}^{T}(1+ S1^{*2}(Z_t))^{-1}]^{-1} - 1$.

The second bias adjustment works similarly, but following Jobson and Korkie (1980) uses the assumption of a non-central *F* distribution, accommodating estimation error in the covariance matrix. If the estimated conditional mean return $\hat{\delta}Z_t$ is normal and the estimated conditional variance is $\hat{V}$, then $E\left(\hat{V}^{-1}\right) = TE\left[\left(\sum_{t=1}^{T}\hat{\varepsilon}_t\hat{\varepsilon}_t'\right)^{-1}\right] = \frac{T}{T-N-L-1}V^{-1}$ $\hat{V} \equiv \frac{1}{T}\sum_{t=1}^{T}\left(R_t - \hat{\delta}'Z_{t-1}\right)\left(R_t - \hat{\delta}'Z_{t-1}\right)' = \frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_t\hat{\varepsilon}_t'$ where $\sum_{t=1}^{T}\hat{\varepsilon}_t\hat{\varepsilon}_t'$ has a Wishart distribution with scale matrix $V$ along with degrees of freedom $T-L$. The sample squared conditional Sharpe ratio has expectation

$$E\left[\left(\hat{\delta}'Z_t\right)'\hat{V}^{-1}\left(\hat{\delta}'Z_t\right)\right] = \left\{\left(\delta'Z_t\right)'V^{-1}\left(\delta'Z_t\right) + N\left[Z_t\left(Z'Z\right)^{-1}Z_t\right]\right\}T/\left(T-N-L-1\right).$$

Solving for an unbiased estimator we obtain our second bias adjusted estimator for the conditional ratio:

$$S2^{*2}\left(Z_t\right) = \left(\hat{\delta}'Z_t\right)'\hat{V}^{-1}\left(\hat{\delta}'Z_t\right)\frac{T-N-L-1}{T} - N\left[Z_t\left(Z'Z\right)^{-1}Z_t\right] \qquad \text{(A.2)}$$

As before, the adjusted UE maximum squared Sharpe ratio is found as $[\frac{1}{T}\sum_{t=1}^{T}(1+ S2^{*2}(Z_t))^{-1}]^{-1} - 1$. Note that even if an unbiased conditional Sharpe ratio is used in the first two adjustments, the UE Sharpe ratio can retain some bias because of its nonlinear relation to the conditional ratios.

The third bias adjustment follows Noh et al. (2019) by using odd versus even months to estimate the regression coefficients. The bias of $[\hat{\delta}'Z_t]' \hat{V}^{-1} [\hat{\delta}'Z_t]$ comes in part from nonlinearity, via Jensens' inequality, and partly from the covariance between estimation errors in the two $[\hat{\delta}'Z_t]$. Our third bias adjustment method splits the time-series into subsamples with odd months and even months, and estimates the two terms using the subsamples. Then

$$S3^{*2}(Z_t) = [\hat{\delta}_{odd}'Z_t]' \hat{V}^{-1} [\hat{\delta}_{even}'Z_t]. \tag{A.3}$$

If the autocorrelation of returns are fully captured by $\delta'Z_t$, and residuals in regression (10) are not autocorrelated, the errors in odd and even months should be uncorrelated, which should reduce the finite sample bias. This approach, however may lose some efficiency.

*B. Partial derivatives for Proposition II:*

$$\frac{\partial S_\varphi^2}{\partial \alpha_1} = -\left(\frac{\alpha_2 - \varphi\alpha_3}{\alpha_1\alpha_3 - \alpha_2^2}\right)^2, \quad \frac{\partial S_\varphi^2}{\partial \alpha_2} = 2\frac{(\alpha_1 - \varphi\alpha_2)(\alpha_2 - \varphi\alpha_3)}{(\alpha_1\alpha_3 - \alpha_2^2)^2}, \quad \frac{\partial S_\varphi^2}{\partial \alpha_3} = -\left(\frac{\alpha_1 - \varphi\alpha_2}{\alpha_1\alpha_3 - \alpha_2^2}\right)^2,$$

$$\frac{\partial^2 S_\varphi^2}{\partial \alpha_1^2} = \frac{2\alpha_3(\alpha_2 - \varphi\alpha_3)^2}{(\alpha_1\alpha_3 - \alpha_2^2)^3}, \quad \frac{\partial^2 S_\varphi^2}{\partial \alpha_2^2} = 2\frac{\alpha_1 - 2\varphi\alpha_2 + \varphi^2\alpha_3}{(\alpha_1\alpha_3 - \alpha_2^2)^2} + 8\frac{\alpha_2(\alpha_1 - \varphi\alpha_2)(\alpha_2 - \varphi\alpha_3)}{(\alpha_1\alpha_3 - \alpha_2^2)^3},$$

$$\frac{\partial^2 S_\varphi^2}{\partial \alpha_3^2} = 2\frac{\alpha_1(\alpha_1 - \varphi\alpha_2)^2}{(\alpha_1\alpha_3 - \alpha_2^2)^3}, \quad \frac{\partial^2 S_\varphi^2}{\partial \alpha_1 \partial \alpha_2} = -2(\alpha_2 - \varphi\alpha_3)\frac{\alpha_2^2 + \alpha_3(\alpha_1 - 2\varphi\alpha_2)}{(\alpha_1\alpha_3 - \alpha_2^2)^3},$$

$$\frac{\partial^2 S_\varphi^2}{\partial \alpha_1 \partial \alpha_3} = 2\frac{\alpha_2(\alpha_1 - \varphi\alpha_2)(\alpha_2 - \varphi\alpha_3)}{(\alpha_1\alpha_3 - \alpha_2^2)^3} \text{ and } \frac{\partial^2 S_\varphi^2}{\partial \alpha_2 \partial \alpha_3} = 2\frac{\alpha_1 - \varphi\alpha_2}{(\alpha_1\alpha_3 - \alpha_2^2)^2}\left(\varphi - 2\frac{\alpha_2(\alpha_1 - \varphi\alpha_2)}{\alpha_1\alpha_3 - \alpha_2^2}\right)$$

Where $a = \mathbf{1}'V^{-1}\mathbf{1}$, $b_t = \mathbf{1}'V^{-1}\mu_t$, $c_{kt} = 1 + \mu_k' V^{-1}\mu_t$, and $d_{kt} = ac_{kt} - b_k b_t$ and their estimates are $\hat{a} = \mathbf{1}'\hat{V}^{-1}\mathbf{1}$, $\hat{b}_t = \mathbf{1}'\hat{V}^{-1}\hat{\mu}_t$, $\hat{c}_{kt} = 1 + \hat{\mu}_k'\hat{V}^{-1}\hat{\mu}_t$, where $\hat{d}_{kt} = \hat{a}\hat{c}_{kt} - \hat{b}_k\hat{b}_t$. Then the consistent estimates are $\hat{\alpha}_1 = \frac{1}{T}\sum_{t=1}^{T}\frac{\hat{c}_{tt}}{\hat{d}_{tt}}$, $\hat{\alpha}_2 = \frac{1}{T}\sum_{t=1}^{T}\frac{\hat{b}_t}{\hat{d}_{tt}}$, and $\hat{\alpha}_3 = \frac{\hat{a}}{T}\sum_{t=1}^{T}\frac{1}{\hat{d}_{tt}}$.

*<u>Applications of Asumptotic Variances for Optimal Zero-beta rates</u>*

**Case 1: Test for Squared Sharpe ratios of two UE portfolios.**

The true squared Sharpe ratio for a UE portfolio is

$$S_\phi^2 = \frac{\alpha_2^2 + \alpha_1\alpha_3 - 2\phi\alpha_2 + \phi^2(1-\alpha_3)}{\alpha_1(1-\alpha_3) - \alpha_2^2}. \quad \text{(A.4)}$$

The squared Sharpe ratio is a quadratic function of $\phi$. Hence, we obtain:

$$a = \frac{\alpha_2^2 + \alpha_1\alpha_3}{\alpha_1(1-\alpha_3) - \alpha_2^2}, \quad \text{(A.5)}$$

$$b = \frac{\alpha_2}{\alpha_1(1-\alpha_3) - \alpha_2^2}, \quad \text{(A.6)}$$

$$c = \frac{1-\alpha_3}{\alpha_1(1-\alpha_3) - \alpha_2^2}. \quad \text{(A.7)}$$

Since $a$, $b$, and $c$ are all continuous differentiable functions of $\alpha_1$, $\alpha_2$ and $\alpha_3$, we can expand $\hat{a}$, $\hat{a}^*$, $\hat{b}$, $\hat{b}^*$, $\hat{c}$ and $\hat{c}^*$:

$$\hat{a} - a = -\frac{\alpha_2^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}(\hat{\alpha}_1 - \alpha_1) + \frac{2\alpha_1\alpha_2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}(\hat{\alpha}_2 - \alpha_2) + \frac{\alpha_1^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}(\hat{\alpha}_3 - \alpha_3) + O(\tfrac{1}{T}),$$

$$\hat{a}^* - a^* = -\frac{(\alpha_2^*)^*}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}(\hat{\alpha}_1^* - \alpha_1^*) + \frac{2\alpha_1^*\alpha_2^*}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}(\hat{\alpha}_2^* - \alpha_2^*) + \frac{(\alpha_1^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}(\hat{\alpha}_3^* - \alpha_3^*) + O(\tfrac{1}{T}), \quad \text{(A.8)}$$

$$\hat{b} - b = -\frac{\alpha_2(1-\alpha_3)}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}(\hat{\alpha}_1 - \alpha_1) + \frac{\alpha_1(1-\alpha_3)+\alpha_2^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}(\hat{\alpha}_2 - \alpha_2) + \frac{\alpha_1\alpha_2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}(\hat{\alpha}_3 - \alpha_3) + O(\tfrac{1}{T}), \quad \text{(A.9)}$$

$$\hat{b}^* - b^* = -\frac{\alpha_2^*(1-\alpha_3^*)}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}(\hat{\alpha}_1^* - \alpha_1^*) + \frac{\alpha_1^*(1-\alpha_3^*)+(\alpha_2^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}(\hat{\alpha}_2^* - \alpha_2^*) + \frac{\alpha_1^*\alpha_2^*}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}(\hat{\alpha}_3^* - \alpha_3^*) + O(\tfrac{1}{T}), \quad \text{(A.10)}$$

$$\hat{c}-c=\frac{-(1-\alpha_3)^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}(\hat{\alpha}_1-\alpha_1)+\frac{2\alpha_2(1-\alpha_3)}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}(\hat{\alpha}_2-\alpha_2)$$
$$+\frac{\alpha_2^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}(\hat{\alpha}_3-\alpha_3)+O(\tfrac{1}{T}), \quad \text{(A.11)}$$

$$\hat{c}^*-c^*=\frac{-(1-\alpha_3^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}(\hat{\alpha}_1^*-\alpha_1^*)+\frac{2\alpha_2^*(1-\alpha_3^*)}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}(\hat{\alpha}_2^*-\alpha_2^*)$$
$$+\frac{(\alpha_2^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}(\hat{\alpha}_3^*-\alpha_3^*)+O(\tfrac{1}{T}). \quad \text{(A.12)}$$

The canonical matrices of $\alpha_1$, $\alpha_2$ and $\alpha_3$ can be expanded in the same way as $\hat{\theta}$ in the Theorem I, as shown in the Internet Appendix. Also, $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{a}^*$, $\hat{b}^*$ and $\hat{c}^*$ can be similarly expanded and the canonical matrices are:

$$C_a=-\frac{\alpha_2^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}C_{\alpha_1}+\frac{2\alpha_1\alpha_2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}C_{\alpha_2}+\frac{\alpha_1^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}C_{\alpha_3}, \quad \text{(A.13)}$$

$$C_{a^*}=-\frac{(\alpha_2^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}C_{\alpha_1^*}+\frac{2\alpha_1^*\alpha_2^*}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}C_{\alpha_2^*}+\frac{(\alpha_1^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}C_{\alpha_3^*}, \quad \text{(A.14)}$$

$$D_a=-\frac{\alpha_2^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}D_{\alpha_1}+\frac{2\alpha_1\alpha_2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}D_{\alpha_2}+\frac{\alpha_1^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}D_{\alpha_3}, \quad \text{(A.15)}$$

$$D_{a^*}=-\frac{(\alpha_2^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}D_{\alpha_1^*}+\frac{2\alpha_1^*\alpha_2^*}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}D_{\alpha_2^*}+\frac{(\alpha_1^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}D_{\alpha_3^*}, \quad \text{(A.16)}$$

$$C_b=-\frac{\alpha_2(1-\alpha_3)}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}C_{\alpha_1}+\frac{\alpha_1(1-\alpha_3)+\alpha_2^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}C_{\alpha_2}+\frac{\alpha_2\alpha_3}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}C_{\alpha_3}, \quad \text{(A.17)}$$

$$C_{b^*}=-\frac{\alpha_2^*(1-\alpha_3^*)}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}C_{\alpha_1^*}+\frac{\alpha_1^*(1-\alpha_3^*)+(\alpha_2^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}C_{\alpha_2^*}+\frac{\alpha_2^*\alpha_3^*}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}C_{\alpha_3^*}, \quad \text{(A.18)}$$

$$D_b=-\frac{\alpha_2(1-\alpha_3)}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}D_{\alpha_1}+\frac{\alpha_1(1-\alpha_3)+\alpha_2^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}D_{\alpha_2}+\frac{\alpha_2\alpha_3}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}D_{\alpha_3}, \quad \text{(A.19)}$$

$$D_{b^*}=-\frac{\alpha_2^*(1-\alpha_3^*)}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}D_{\alpha_1^*}+\frac{\alpha_1^*(1-\alpha_3^*)+(\alpha_2^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}D_{\alpha_2^*}+\frac{\alpha_2^*\alpha_3^*}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}D_{\alpha_3^*}, \quad \text{(A.20)}$$

$$C_c=\frac{-(1-\alpha_3)^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}C_{\alpha_1}+\frac{2\alpha_2(1-\alpha_3)}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}C_{\alpha_2}+\frac{\alpha_2^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}C_{\alpha_3}, \quad \text{(A.21)}$$

$$C_{c^*}=\frac{-(1-\alpha_3^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}C_{\alpha_1^*}+\frac{2\alpha_2^*(1-\alpha_3^*)}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}C_{\alpha_2^*}+\frac{(\alpha_2^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}C_{\alpha_3^*}. \quad \text{(A.22)}$$

$$D_c=\frac{-(1-\alpha_3)^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}D_{\alpha_1}+\frac{2\alpha_2(1-\alpha_3)}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}D_{\alpha_2}+\frac{\alpha_2^2}{(\alpha_1(1-\alpha_3)-\alpha_2^2)^2}D_{\alpha_3}, \quad \text{(A.23)}$$

$$D_{c^*}=\frac{-(1-\alpha_3^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}D_{\alpha_1^*}+\frac{2\alpha_2^*(1-\alpha_3^*)}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}D_{\alpha_2^*}+\frac{(\alpha_2^*)^2}{(\alpha_1^*(1-\alpha_3^*)-(\alpha_2^*)^2)^2}D_{\alpha_3^*}. \quad \text{(A.24)}$$

**Case 2: Test when one Squared Sharpe ratio is for a maximum correlation portfolio with respect to Z.**

The squared Sharpe ratio of the maximum correlation portfolio, $p$, can be written as $S_p^2 = \frac{\mu_p^2 - 2\mu_p\phi + \phi^2}{\sigma_p^2}$. Define $a \equiv \frac{\mu_p^2}{\sigma_p^2}$, $b \equiv \frac{\mu_p}{\sigma_p^2}$ and $c \equiv \frac{1}{\sigma_p^2}$. Following the derivation for case 1,

$$\hat{a} - a = \frac{2\mu_p}{\sigma_p^2}(\hat{\mu}_p - \mu_p) - \frac{(\mu_p)^2}{\sigma_p^4}(\hat{\sigma}_p^2 - \sigma_p^2) + O(\tfrac{1}{T}), \quad \text{(A.25)}$$

$$\hat{b} - b = \frac{1}{\sigma_p^2}(\hat{\mu}_p - \mu_p) - \frac{\mu_p}{\sigma_p^4}(\hat{\sigma}_p^2 - \sigma_p^2) + O(\tfrac{1}{T}), \quad \text{(A.26)}$$

$$\hat{c} - c = -\frac{1}{\sigma_p^4}(\hat{\sigma}_p^2 - \sigma_p^2) + O(\tfrac{1}{T}). \quad \text{(A.27)}$$

From equation (A.113) and (A.114),

$$C_a = \frac{2\mu_p}{\sigma_p^2} C_{\mu p} - \frac{(\mu_p)^2}{\sigma_p^4} C_{\sigma p2}, \quad \text{(A.28)}$$

$$C_b = \frac{1}{\sigma_p^2} C_{\mu p} - \frac{\mu_p}{\sigma_p^4} C_{\sigma p2}, \quad \text{(A.29)}$$

$$C_c = -\frac{1}{\sigma_p^4} C_{\sigma p2}, \quad \text{(A.30)}$$

$$D_a = \frac{2\mu_p}{\sigma_p^2} D_{\mu p} - \frac{(\mu_p)^2}{\sigma_p^4} D_{\sigma p2}, \quad \text{(A.31)}$$

$$D_b = \frac{1}{\sigma_p^2} D_{\mu p} - \frac{\mu_p}{\sigma_p^4} D_{\sigma p2}, \quad \text{(A.32)}$$

$$D_c = -\frac{1}{\sigma_p^4} D_{\sigma p2}. \quad \text{(A.33)}$$

The other squared Sharpe ratios can be written as $a^* - 2b^*\phi + c^*\phi^2$, and $\hat{a}^*$, $\hat{b}^*$ and $\hat{c}^*$ can be expanded in the same way as $\hat{\theta}$ in the Theorem I, with canonical matrices $C_{a^*}$, $D_{a^*}$, $C_{b^*}$, $D_{b^*}$, $C_{c^*}$ and $D_{C^*}$. Then the asymptotic variance of BS test with a common optimal zero-beta rate also follows the same procedure in case 1.

**Case 3: Tests where one of the Squared Sharpe ratios is for a fixed weight portfolio.**

Similar to case 2, the squared Sharpe ratio of the maximum correlation portfolio can be written as $S_p^2 = \frac{\mu_p^2 - 2\mu_p\phi + \phi^2}{\sigma_p^2}$. Define $a \equiv \frac{\mu_p^2}{\sigma_p^2}$, $b \equiv \frac{\mu_p}{\sigma_p^2}$ and $c \equiv \frac{1}{\sigma_p^2}$. The estimates for the canonical matrices are

$$C_{\mu_p} = \frac{A^{-1}}{T}\left(\textstyle\sum_{t=1}^T Z_{t-1}\right)w', \quad D_{\mu_p} = 0, \quad \text{(A.34)}$$

$$C_{\sigma_p^2} = \frac{2A^{-1}}{T}\textstyle\sum_{t=1}^T Z_{t-1}[-(\mu_t{}'w - \mu_p)]w', \quad D_{\sigma_p^2} = -ww'. \quad \text{(A.35)}$$

<u>Proof of the Theorem I.</u>

We first express $\hat{\theta}$ in terms of $\varepsilon_t$ as follows:

$$\hat{\theta} = \theta + \sum_{t=1}^{T} C_t' \left( \frac{1}{T} \sum_{i=1}^{T} \varepsilon_i Z_{i-1}' \right) A^{-1} Z_{t-1} + tr\left[ D \frac{1}{T} \sum_{t=1}^{T} \left( \varepsilon_t \varepsilon_t' - V \right) \right] + O_p\left(1/T\right) \qquad \text{(A.36)}$$

where we have used the fact that $\hat{\mu}_t - \mu_t = \left(\hat{\delta} - \delta\right)' Z_{t-1} = \left( \frac{1}{T} \sum_{i=1}^{T} \varepsilon_i Z_{i-1}' \right) A^{-1} Z_{t-1} = O_p\left(1/\sqrt{T}\right)$ whch follows from (13), and the fact that $\hat{V} - V = \frac{1}{T} \sum_{t=1}^{T} \left( \varepsilon_t \varepsilon_t' - V \right) + O_p\left(1/T\right)$ . Then

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^{T} \hat{\varepsilon}_t \hat{\varepsilon}_t' &= \frac{1}{T} \sum_{t=1}^{T} \left[ \varepsilon_t - \left(\hat{\delta} - \delta\right)' Z_{t-1} \right] \left[ \varepsilon_t - \left(\hat{\delta} - \delta\right)' Z_{t-1} \right]' \\
&= \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t \varepsilon_t' - \left(\hat{\delta} - \delta\right)' \left( \frac{1}{T} \sum_{t=1}^{T} Z_{t-1} \varepsilon_t' \right) - \left[ \left(\hat{\delta} - \delta\right)' \left( \frac{1}{T} \sum_{t=1}^{T} Z_{t-1} \varepsilon_t' \right) \right]' + \left(\hat{\delta} - \delta\right)' A \left(\hat{\delta} - \delta\right) \\
&= \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t \varepsilon_t' + O_p\left(1/T\right)
\end{aligned}$$

Using the $L \times N$ matrix (without subscript) $C = A^{-1} \sum_{t=1}^{T} Z_{t-1} C_t'$, we may rewrite (A.1) using matrix commutativity within the trace operator as follows:

$$\begin{aligned}
\hat{\theta} &= \theta + tr\left[ \sum_{t=1}^{T} A^{-1} Z_{t-1} C_t' \left( \frac{1}{T} \sum_{i=1}^{T} \varepsilon_i Z_{i-1}' \right) \right] + tr\left[ D \frac{1}{T} \sum_{t=1}^{T} \left( \varepsilon_t \varepsilon_t' - V \right) \right] + O_p\left(1/T\right) \\
&= \theta + tr\left[ C \left( \frac{1}{T} \sum_{i=1}^{T} \varepsilon_i Z_{i-1}' \right) \right] + tr\left[ \frac{1}{T} \sum_{t=1}^{T} \left( D \varepsilon_t \varepsilon_t' - DV \right) \right] + O_p\left(1/T\right) \qquad \text{(A.37)} \\
&= \theta + \frac{1}{T} \sum_{t=1}^{T} \left( Z_{t-1}' C \varepsilon_t + \varepsilon_t' D \varepsilon_t \right) - tr\left(DV\right) + O_p\left(1/T\right)
\end{aligned}$$

Where C,D and V are the true values of the coefficients.

Now we let $\Psi_t = Z_{t-1}' C \varepsilon_t + \varepsilon_t' D \varepsilon_t$. The unconditional asymptotic variance of $\hat{\theta}$ is

$\sqrt{T}\, \mathrm{Var}\{(1/T)\Sigma_t\, (Z_{t-1}' C\varepsilon_t + \varepsilon_t' D\varepsilon_t)\}$ and this may be estimated simply as:

$$\sqrt{T}\ \{\ (1/T)\Sigma_t\ \Psi_t^2\ -\ [(1/T)\Sigma_t\Psi_t]^2\ \}. \qquad (A.38)$$

This follows because $\Psi_t$ has a moving average structure of order zero, as may be seen by examining for any j≥1 the autocovariance, using the independence of $\varepsilon_t$:

$$\begin{aligned} Cov(\Psi_t, \Psi_{t-j}) &= Cov(Z_{t-1}'C\varepsilon_t + \varepsilon_t' D\varepsilon_t ; Z_{t-j-2}'C\varepsilon_{t-j-1} + \varepsilon_{t-j--1}' D\varepsilon_{t-j-1}) \\ &= Cov(Z_{t-1}'C\varepsilon_t ; Z_{t-j-2}'C\varepsilon_{t-j-1}) \\ &= E\{ Z_{t-1}'C\ E(\varepsilon_t \mid \varepsilon_{t-j-1})\ \varepsilon_{t-j-1}'\ C'\ Z_{t-j-2}\} = 0. \end{aligned}$$

Substituting consistent estimates $\hat{C}$, and $\hat{D}$ along with $\hat{\varepsilon}_t = R_t - \hat{\delta}'Z_{t-1}$ to obtain a consistent estimator of V, we have a consistent estimator for the asymptotic variance. □

Note that it is possible to find the asymptotic variance of a combination of two estimators by using the $\hat{C}$ and $\hat{D}$ matrices associated with each estimator. For example, the asymptotic variance of the sum of two estimators, $\hat{\theta}+\hat{\theta}^*$, is characterized by matrices $\hat{C}+\hat{C}^*$ and $\hat{D}+\hat{D}^*$ due to linearity of (A.1), allowing us to combine the estimators' $\hat{C}$ and $\hat{D}$ matrices. The asymptotic variance of the ratio $\hat{\theta}/\hat{\theta}^*$ is characterized by matrices $\dfrac{\hat{\theta}^*\hat{C}-\hat{\theta}\hat{C}^*}{\hat{\theta}^{*2}}$ and $\dfrac{\hat{\theta}^*\hat{D}-\hat{\theta}\hat{D}^*}{\hat{\theta}^{*2}}$, as may be seen from the following expansion:

$$
\begin{aligned}
\frac{\hat{\theta}}{\hat{\theta}^*} &= \frac{\theta + \sum_{t=1}^{T} C_t' \left(\hat{\mu}_t - \mu_t\right) + tr\left[D\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)}{\theta^* + \sum_{t=1}^{T} C_t^{*\prime} \left(\hat{\mu}_t - \mu_t\right) + tr\left[D^*\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)} \\
&= \frac{\theta}{\theta^*} \frac{1 + \frac{1}{\theta}\sum_{t=1}^{T} C_t' \left(\hat{\mu}_t - \mu_t\right) + \frac{1}{\theta} tr\left[D\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)}{1 + \frac{1}{\theta^*}\sum_{t=1}^{T} C_t^{*\prime} \left(\hat{\mu}_t - \mu_t\right) + \frac{1}{\theta^*} tr\left[D^*\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)} \\
&= \frac{\theta}{\theta^*}\left[1 + \frac{1}{\theta}\sum_{t=1}^{T} C_t' \left(\hat{\mu}_t - \mu_t\right) + \frac{1}{\theta} tr\left[D\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)\right] \times \qquad \text{(A.39)} \\
&\quad \left[1 - \frac{1}{\theta^*}\sum_{t=1}^{T} C_t^{*\prime} \left(\hat{\mu}_t - \mu_t\right) - \frac{1}{\theta^*} tr\left[D^*\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)\right] \\
&= \frac{\theta}{\theta^*} + \sum_{t=1}^{T} \frac{\theta^* C_t' - \theta C_t^{*\prime}}{\theta^{*2}} \left(\hat{\mu}_t - \mu_t\right) + tr\left[\frac{\theta^* D - \theta D^*}{\theta^{*2}}\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)
\end{aligned}
$$

The asymptotic variance of $\frac{\hat{\theta} - \hat{\theta}^*}{1 + \hat{\theta}^*}$ is characterized by the matrices $\left[\left(1 + \hat{\theta}^*\right)\hat{C} - \left(1 + \hat{\theta}\right)\hat{C}^*\right] / \left(1 + \hat{\theta}^*\right)^2$ and $\left[\left(1 + \hat{\theta}^*\right)\hat{D} - \left(1 + \hat{\theta}\right)\hat{D}^*\right] / \left(1 + \hat{\theta}^*\right)^2$, as may be seen from the following expansion:

$$
\begin{aligned}
\frac{\hat{\theta} - \hat{\theta}^*}{1 + \hat{\theta}^*} &= \frac{\theta - \theta^* + \sum_{t=1}^{T}\left(C_t' - C_t^{*\prime}\right)\left(\hat{\mu}_t - \mu_t\right) + tr\left[\left(D - D^*\right)\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)}{1 + \theta^* + \sum_{t=1}^{T} C_t^{*\prime}\left(\hat{\mu}_t - \mu_t\right) + tr\left[D^*\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)} \\
&= \left(\frac{\theta - \theta^*}{1 + \theta^*}\right) \times \left\{1 + \sum_{t=1}^{T} \frac{C_t' - C_t^{*\prime}}{\theta - \theta^*}\left(\hat{\mu}_t - \mu_t\right) + tr\left[\frac{D - D^*}{\theta - \theta^*}\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)\right\} \times \\
&\quad \left[1 - \sum_{t=1}^{T} \frac{C_t^{*\prime}}{1 + \theta^*}\left(\hat{\mu}_t - \mu_t\right) - tr\left[\frac{D^*}{1 + \theta^*}\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)\right] \qquad \text{(A.40)} \\
&= \frac{\theta - \theta^*}{1 + \theta^*}\left\{1 + \sum_{t=1}^{T}\left(\frac{C_t' - C_t^{*\prime}}{\theta - \theta^*} - \frac{C_t^{*\prime}}{1 + \theta^*}\right)\left(\hat{\mu}_t - \mu_t\right) + tr\left[\left(\frac{D - D^*}{\theta - \theta^*} - \frac{D^*}{1 + \theta^*}\right)\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)\right\} \\
&= \frac{\theta - \theta^*}{1 + \theta^*} + \sum_{t=1}^{T} \frac{\left(1 + \theta^*\right)C_t' - \left(1 + \theta\right)C_t^{*\prime}}{\left(1 + \theta^*\right)^2}\left(\hat{\mu}_t - \mu_t\right) + tr\left[\frac{\left(1 + \theta^*\right)D - \left(1 + \theta\right)D^*}{\left(1 + \theta^*\right)^2}\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)
\end{aligned}
$$

<u>Proof of Proposition II:</u>

From Ferson & Siegel (2001, pages 976-977) the weights of the UE portfolio with mean $\mu_p$, in the absence of a riskless rate, are

$$w'(Z) = \frac{\mathbf{1}'\Lambda(Z)}{\mathbf{1}'\Lambda(Z)\mathbf{1}'} + \frac{\mu_p - \alpha_2}{1-\alpha_3}\mu'(Z)\left(\Lambda(Z) - \frac{\Lambda(Z)\mathbf{1}\mathbf{1}'\Lambda(Z)}{\mathbf{1}'\Lambda(Z)\mathbf{1}'}\right) \tag{A.41}$$

where our $\alpha_3$ represents $1-\alpha_3$ from Ferson and Siegel (2001). We will write with implicit $Z_{t-1}$ (the conditioning information) dependence as

$$w' = \frac{\mathbf{1}'\Lambda}{\mathbf{1}'\Lambda\mathbf{1}'} + \frac{\mu_p - \alpha_2}{1-\alpha_3}\mu'\left(\Lambda - \frac{\Lambda\mathbf{1}\mathbf{1}'\Lambda}{\mathbf{1}'\Lambda\mathbf{1}'}\right) \tag{A.42}$$

where

$$\Lambda = \Lambda(Z) = \left[V + \mu(Z)\mu'(Z)\right]^{-1} = \left(V + \mu\mu'\right)^{-1} \tag{A.43}$$

The portfolio variance is

$$\sigma_p^2 = \left(\alpha_1 + \frac{\alpha_2^2}{1-\alpha_3}\right) - \mu_p \frac{2\alpha_2}{1-\alpha_3} + \mu_p^2 \frac{\alpha_3}{1-\alpha_3} \tag{A.44}$$

To simplify notation in what follows, we define the following:

$$a = \mathbf{1}'V^{-1}\mathbf{1},\ \ b_t = \mathbf{1}'V^{-1}\mu_t,\ \ c_{kt} = 1 + \mu_k' V^{-1}\mu_t,\ \text{ and }\ d_{kt} = ac_{kt} - b_k b_t \tag{A.45}$$

along with their estimates

$$\hat{a} = \mathbf{1}'\hat{V}^{-1}\mathbf{1},\ \ \hat{b}_t = \mathbf{1}'\hat{V}^{-1}\hat{\mu}_t,\ \ \hat{c}_{kt} = 1 + \hat{\mu}_k' \hat{V}^{-1}\hat{\mu}_t,\ \text{ and }\ \hat{d}_{kt} = \hat{a}\hat{c}_{kt} - \hat{b}_k \hat{b}_t \tag{A.46}$$

Consistent estimates of the portfolio constants are then:

$$\hat{\alpha}_1 = \frac{1}{T}\sum_{t=1}^{T}\frac{\hat{c}_{tt}}{\hat{d}_{tt}},\quad \hat{\alpha}_2 = \frac{1}{T}\sum_{t=1}^{T}\frac{\hat{b}_t}{\hat{d}_{tt}},\quad \text{and}\quad \hat{\alpha}_3 = \frac{\hat{a}}{T}\sum_{t=1}^{T}\frac{1}{\hat{d}_{tt}} \tag{A.47}$$

Here we restate the proposition for convenience.

**Proposition 11**: The estimation bias of the estimated maximized squared Sharpe Ratio at zero-beta rate $\varphi$:

$$\hat{S}_\varphi^2 = \frac{\hat{\alpha}_1 - 2\varphi\hat{\alpha}_2 + \varphi^2\hat{\alpha}_3}{\hat{\alpha}_1\hat{\alpha}_3 - \hat{\alpha}_2^2} - 1 \tag{A.48}$$

with respect to the true (but unknown) maximized squared Sharpe Ratio

$$S_\varphi^2 = \frac{\alpha_1 - 2\varphi\alpha_2 + \varphi^2\alpha_3}{\alpha_1\alpha_3 - \alpha_2^2} - 1 \tag{A.49}$$

may be expressed asymptotically by taking the expectation of its second-order Taylor Series expansion:

$$Bias = E\left(\hat{S}_\varphi^2 - S_\varphi^2\right) \cong \left(\sum_{i=1}^3 E\left(\hat{\alpha}_i - \alpha_i\right)\frac{\partial S_\varphi^2}{\partial \alpha_i}\right) + \left(\sum_{i,j=1}^3 \frac{E\left[\left(\hat{\alpha}_i - \alpha_i\right)\left(\hat{\alpha}_j - \alpha_j\right)\right]}{2}\frac{\partial^2 S_\varphi^2}{\partial \alpha_i \partial \alpha_j}\right) \tag{A.50}$$

for which the partial derivatives are derived in Proposition 2, with the expectation terms following in Propositions 3a, 3b, and 3c. Note that the estimated bias may be obtained using estimates in the formulas for derivatives and expectations that follow.

The partial derivatives for Proposition 1 are as follows:

$$\frac{\partial S_\varphi^2}{\partial \alpha_1} = -\left(\frac{\alpha_2 - \varphi\alpha_3}{\alpha_1\alpha_3 - \alpha_2^2}\right)^2, \quad \frac{\partial S_\varphi^2}{\partial \alpha_2} = 2\frac{\left(\alpha_1 - \varphi\alpha_2\right)\left(\alpha_2 - \varphi\alpha_3\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2}, \quad \frac{\partial S_\varphi^2}{\partial \alpha_3} = -\left(\frac{\alpha_1 - \varphi\alpha_2}{\alpha_1\alpha_3 - \alpha_2^2}\right)^2,$$

$$\frac{\partial^2 S_\varphi^2}{\partial \alpha_1^2} = \frac{2\alpha_3\left(\alpha_2 - \varphi\alpha_3\right)^2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}, \quad \frac{\partial^2 S_\varphi^2}{\partial \alpha_2^2} = 2\frac{\alpha_1 - 2\varphi\alpha_2 + \varphi^2\alpha_3}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2} + 8\frac{\alpha_2\left(\alpha_1 - \varphi\alpha_2\right)\left(\alpha_2 - \varphi\alpha_3\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3},$$

$$\frac{\partial^2 S_\varphi^2}{\partial \alpha_3^2} = 2\frac{\alpha_1\left(\alpha_1 - \varphi\alpha_2\right)^2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}, \quad \frac{\partial^2 S_\varphi^2}{\partial \alpha_1 \partial \alpha_2} = -2\left(\alpha_2 - \varphi\alpha_3\right)\frac{\alpha_2^2 + \alpha_3\left(\alpha_1 - 2\varphi\alpha_2\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}, \tag{A.51}$$

$$\frac{\partial^2 S_\varphi^2}{\partial\alpha_1\partial\alpha_3} = 2\frac{\alpha_2(\alpha_1-\varphi\alpha_2)(\alpha_2-\varphi\alpha_3)}{(\alpha_1\alpha_3-\alpha_2^2)^3}, \text{ and } \frac{\partial^2 S_\varphi^2}{\partial\alpha_2\partial\alpha_3} = 2\frac{\alpha_1-\varphi\alpha_2}{(\alpha_1\alpha_3-\alpha_2^2)^2}\left(\varphi-2\frac{\alpha_2(\alpha_1-\varphi\alpha_2)}{\alpha_1\alpha_3-\alpha_2^2}\right)$$

Proof: The expression for the approximate asymptotic bias represents the second-order Taylor Series. As in Siegel and Woodgate (2007) "we use the method of statistical differentials to find Taylor-series approximations to expectations of random variables, obtaining results that are asymptotically correct when the number of time periods is large and that remain statistically consistent when estimated values are substituted for unknown parameters." Note that the $\hat{\alpha}_i$ are not necessarily unbiased, so their expectations are necessary in this expression, concluding the proof. Partial differentiation shows:

$$\begin{aligned}\frac{\partial S_\varphi^2}{\partial\alpha_1} &= \frac{(\alpha_1\alpha_3-\alpha_2^2)-(\alpha_1-2\varphi\alpha_2+\varphi^2\alpha_3)\alpha_3}{(\alpha_1\alpha_3-\alpha_2^2)^2} \\ &= \frac{\alpha_1\alpha_3-\alpha_2^2-\alpha_1\alpha_3+2\varphi\alpha_2\alpha_3-\varphi^2\alpha_3^2}{(\alpha_1\alpha_3-\alpha_2^2)^2} = \frac{-\alpha_2^2+2\varphi\alpha_2\alpha_3-\varphi^2\alpha_3^2}{(\alpha_1\alpha_3-\alpha_2^2)^2} \\ &= -\left(\frac{\alpha_2-\varphi\alpha_3}{\alpha_1\alpha_3-\alpha_2^2}\right)^2\end{aligned}$$

$$\begin{aligned}\frac{\partial S_\varphi^2}{\partial\alpha_2} &= \frac{(-2\varphi)(\alpha_1\alpha_3-\alpha_2^2)-(\alpha_1-2\varphi\alpha_2+\varphi^2\alpha_3)(-2\alpha_2)}{(\alpha_1\alpha_3-\alpha_2^2)^2} \\ &= 2\frac{-\varphi\alpha_1\alpha_3+\varphi\alpha_2^2+\alpha_1\alpha_2-2\varphi\alpha_2^2+\varphi^2\alpha_2\alpha_3}{(\alpha_1\alpha_3-\alpha_2^2)^2} = 2\frac{\alpha_1\alpha_2-\varphi\alpha_1\alpha_3-\varphi\alpha_2^2+\varphi^2\alpha_2\alpha_3}{(\alpha_1\alpha_3-\alpha_2^2)^2} \\ &= 2\frac{\alpha_1(\alpha_2-\varphi\alpha_3)-\varphi\alpha_2(\alpha_2-\varphi\alpha_3)}{(\alpha_1\alpha_3-\alpha_2^2)^2} = 2\frac{(\alpha_1-\varphi\alpha_2)(\alpha_2-\varphi\alpha_3)}{(\alpha_1\alpha_3-\alpha_2^2)^2}\end{aligned}$$

$$\frac{\partial S_\varphi^2}{\partial \alpha_3} = \frac{\left(\varphi^2\right)\left(\alpha_1\alpha_3 - \alpha_2^2\right) - \left(\alpha_1 - 2\varphi\alpha_2 + \varphi^2\alpha_3\right)\left(\alpha_1\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2}$$

$$= \frac{\varphi^2\alpha_1\alpha_3 - \varphi^2\alpha_2^2 - \alpha_1^2 + 2\varphi\alpha_1\alpha_2 - \varphi^2\alpha_1\alpha_3}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2}$$

$$= \frac{-\varphi^2\alpha_2^2 - \alpha_1^2 + 2\varphi\alpha_1\alpha_2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2} = -\left(\frac{\alpha_1 - \varphi\alpha_2}{\alpha_1\alpha_3 - \alpha_2^2}\right)^2$$

$$\frac{\partial^2 S_\varphi^2}{\partial \alpha_1^2} = -\frac{\partial}{\partial \alpha_1}\left(\frac{\alpha_2 - \varphi\alpha_3}{\alpha_1\alpha_3 - \alpha_2^2}\right)^2 = -\frac{\partial}{\partial \alpha_1}\frac{\left(\alpha_2 - \varphi\alpha_3\right)^2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2}$$

$$= -\frac{-2\left(\alpha_2 - \varphi\alpha_3\right)^2\left(\alpha_1\alpha_3 - \alpha_2^2\right)\alpha_3}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^4} = \frac{2\alpha_3\left(\alpha_2 - \varphi\alpha_3\right)^2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$\frac{\partial^2 S_\varphi^2}{\partial \alpha_2^2} = 2\frac{\partial}{\partial \alpha_2}\left(\frac{\alpha_1\alpha_2 - \varphi\alpha_1\alpha_3 - \varphi\alpha_2^2 + \varphi^2\alpha_2\alpha_3}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2}\right)$$

$$= 2\frac{\left(\alpha_1 - 2\varphi\alpha_2 + \varphi^2\alpha_3\right)\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2 - \left(\alpha_1\alpha_2 - \varphi\alpha_1\alpha_3 - \varphi\alpha_2^2 + \varphi^2\alpha_2\alpha_3\right)2\left(\alpha_1\alpha_3 - \alpha_2^2\right)\left(-2\alpha_2\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^4}$$

$$= 2\frac{\left(\alpha_1 - 2\varphi\alpha_2 + \varphi^2\alpha_3\right)\left(\alpha_1\alpha_3 - \alpha_2^2\right) + 4\left(\alpha_1\alpha_2 - \varphi\alpha_1\alpha_3 - \varphi\alpha_2^2 + \varphi^2\alpha_2\alpha_3\right)\alpha_2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$= 2\frac{\left(\alpha_1 - 2\varphi\alpha_2 + \varphi^2\alpha_3\right)\left(\alpha_1\alpha_3 - \alpha_2^2\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3} + 8\frac{\left(\alpha_1\alpha_2 - \varphi\alpha_1\alpha_3 - \varphi\alpha_2^2 + \varphi^2\alpha_2\alpha_3\right)\alpha_2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$= 2\frac{\alpha_1 - 2\varphi\alpha_2 + \varphi^2\alpha_3}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2} + 8\frac{\alpha_1\alpha_2^2 - \varphi\alpha_1\alpha_2\alpha_3 - \varphi\alpha_2^3 + \varphi^2\alpha_2^2\alpha_3}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$= 2\frac{\alpha_1 - 2\varphi\alpha_2 + \varphi^2\alpha_3}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2} + 8\frac{\alpha_1\alpha_2\left(\alpha_2 - \varphi\alpha_3\right) - \varphi\alpha_2^2\left(\alpha_2 - \varphi\alpha_3\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$= 2\frac{\alpha_1 - 2\varphi\alpha_2 + \varphi^2\alpha_3}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2} + 8\frac{\alpha_2\left(\alpha_1 - \varphi\alpha_2\right)\left(\alpha_2 - \varphi\alpha_3\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$\frac{\partial^2 S_\varphi^2}{\partial \alpha_3^2} = \frac{\partial}{\partial \alpha_3}\left[-\left(\frac{\alpha_1 - \varphi\alpha_2}{\alpha_1\alpha_3 - \alpha_2^2}\right)^2\right] = -\frac{-\left(\alpha_1 - \varphi\alpha_2\right)^2 2\left(\alpha_1\alpha_3 - \alpha_2^2\right)\left(\alpha_1\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^4} = 2\frac{\alpha_1\left(\alpha_1 - \varphi\alpha_2\right)^2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$\frac{\partial^2 S_\varphi^2}{\partial \alpha_1 \partial \alpha_2} = -\frac{\partial}{\partial \alpha_2}\left(\frac{\alpha_2 - \varphi\alpha_3}{\alpha_1\alpha_3 - \alpha_2^2}\right)^2 = -\frac{\partial}{\partial \alpha_2}\frac{\left(\alpha_2 - \varphi\alpha_3\right)^2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2}$$

$$= -\frac{2\left(\alpha_2 - \varphi\alpha_3\right)\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2 - \left(\alpha_2 - \varphi\alpha_3\right)^2 2\left(\alpha_1\alpha_3 - \alpha_2^2\right)\left(-2\alpha_2\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^4}$$

$$= -2\frac{\left(\alpha_2 - \varphi\alpha_3\right)\left(\alpha_1\alpha_3 - \alpha_2^2\right) + 2\left(\alpha_2 - \varphi\alpha_3\right)^2 \alpha_2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$= -2\left(\alpha_2 - \varphi\alpha_3\right)\frac{\alpha_1\alpha_3 - \alpha_2^2 + 2\left(\alpha_2 - \varphi\alpha_3\right)\alpha_2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3} = -2\left(\alpha_2 - \varphi\alpha_3\right)\frac{\alpha_1\alpha_3 - \alpha_2^2 + 2\alpha_2^2 - 2\varphi\alpha_2\alpha_3}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$= -2\left(\alpha_2 - \varphi\alpha_3\right)\frac{\alpha_2^2 + \alpha_3\left(\alpha_1 - 2\varphi\alpha_2\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$\frac{\partial^2 S_\varphi^2}{\partial \alpha_1 \partial \alpha_3} = -\frac{\partial}{\partial \alpha_3}\left(\frac{\alpha_2 - \varphi\alpha_3}{\alpha_1\alpha_3 - \alpha_2^2}\right)^2 = -\frac{\partial}{\partial \alpha_3}\frac{\left(\alpha_2 - \varphi\alpha_3\right)^2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2}$$

$$= -\frac{2\left(\alpha_2 - \varphi\alpha_3\right)\left(-\varphi\right)\left(\alpha_1\alpha_3 - \alpha_2^2\right)^2 - \left(\alpha_2 - \varphi\alpha_3\right)^2 2\left(\alpha_1\alpha_3 - \alpha_2^2\right)\left(\alpha_1\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^4}$$

$$= 2\frac{\varphi\left(\alpha_2 - \varphi\alpha_3\right)\left(\alpha_1\alpha_3 - \alpha_2^2\right) + \alpha_1\left(\alpha_2 - \varphi\alpha_3\right)^2}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$= 2\left(\alpha_2 - \varphi\alpha_3\right)\frac{\varphi\left(\alpha_1\alpha_3 - \alpha_2^2\right) + \alpha_1\left(\alpha_2 - \varphi\alpha_3\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$= 2\frac{\left(\alpha_2 - \varphi\alpha_3\right)\left(\alpha_1\alpha_2 - \varphi\alpha_2^2\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3} = 2\frac{\alpha_2\left(\alpha_1 - \varphi\alpha_2\right)\left(\alpha_2 - \varphi\alpha_3\right)}{\left(\alpha_1\alpha_3 - \alpha_2^2\right)^3}$$

$$\frac{\partial^2 S_\varphi^2}{\partial\alpha_2\partial\alpha_3} = -\frac{\partial}{\partial\alpha_2}\left(\frac{\alpha_1-\varphi\alpha_2}{\alpha_1\alpha_3-\alpha_2^2}\right)^2 = -2\left(\frac{\alpha_1-\varphi\alpha_2}{\alpha_1\alpha_3-\alpha_2^2}\right)\frac{(-\varphi)\left(\alpha_1\alpha_3-\alpha_2^2\right)-\left(\alpha_1-\varphi\alpha_2\right)\left(-2\alpha_2\right)}{\left(\alpha_1\alpha_3-\alpha_2^2\right)^2}$$

$$= 2\left(\alpha_1-\varphi\alpha_2\right)\frac{\left(\varphi\alpha_1\alpha_3-\varphi\alpha_2^2\right)-2\alpha_2\left(\alpha_1-\varphi\alpha_2\right)}{\left(\alpha_1\alpha_3-\alpha_2^2\right)^3}$$

$$= 2\left(\alpha_1-\varphi\alpha_2\right)\frac{\varphi\left(\alpha_1\alpha_3-\alpha_2^2\right)}{\left(\alpha_1\alpha_3-\alpha_2^2\right)^3} - 4\left(\alpha_1-\varphi\alpha_2\right)\frac{\alpha_2\left(\alpha_1-\varphi\alpha_2\right)}{\left(\alpha_1\alpha_3-\alpha_2^2\right)^3} \tag{A.52}$$

$$= 2\frac{\alpha_1-\varphi\alpha_2}{\left(\alpha_1\alpha_3-\alpha_2^2\right)^2}\left(\varphi-2\frac{\alpha_2\left(\alpha_1-\varphi\alpha_2\right)}{\alpha_1\alpha_3-\alpha_2^2}\right)$$

Expectations of the form $E\left(\hat{\alpha}_i-\alpha_i\right)$ are estimated as follows:

$$E\left(\hat{\alpha}_1-\alpha_1\right) \cong \frac{1}{T(T-L)}\sum_{t=1}^{T}\frac{2a^2c_{tt}+a\left[(n-4)c_{tt}-2\ \right]d_{tt}-\left[(n-3)+(n-2)c_{tt}\right]d_{tt}^2}{d_{tt}^3}$$
$$+\frac{1}{T^2}\sum_{t=1}^{T}\frac{-4a^2c_{tt}+a\left[4-(n-5)c_{tt}\right]d_{tt}+(n-4)d_{tt}^2}{d_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1}$$

$$E\left(\hat{\alpha}_2-\alpha_2\right) \cong \frac{1}{T}\sum_{t=1}^{T}\left(b_t\frac{2a^2+(n-4)ad_{tt}-(n-2)d_{tt}^2}{d_{tt}^3(T-L)} - ab_t\frac{4a+(n-5)d_{tt}}{d_{tt}^3T}Z'_{t-1}A^{-1}Z_{t-1}\right)$$

and

$$E\left(\hat{\alpha}_3-\alpha_3\right) \cong \frac{a}{T}\sum_{t=1}^{T}\left(\frac{2a^2+(n-4)ad_{tt}-(n-2)d_{tt}^2}{(T-L)d_{tt}^3} - a\frac{4a+(n-5)d_{tt}}{Td_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1}\right) \tag{A.53}$$

To see this, begin by noting that each of these expectations has the form, expanding to second order:

$$
\begin{aligned}
&E\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\hat{x}_t}{\hat{d}_{tt}}-\frac{x_t}{d_{tt}}\right)\right]=\frac{1}{T}\sum_{t=1}^{T}\frac{x_t}{d_{tt}}E\left(\frac{1+\left(\hat{x}_t-x_t\right)/x_t}{1+\left(\hat{d}_{tt}-d_{tt}\right)/d_{tt}}-1\right)\\
&\cong\frac{1}{T}\sum_{t=1}^{T}\frac{x_t}{d_{tt}}E\left(\left(1+\frac{\hat{x}_t-x_t}{x_t}\right)\left(1-\frac{\hat{d}_{tt}-d_{tt}}{d_{tt}}+\frac{\left(\hat{d}_{tt}-d_{tt}\right)^2}{d_{tt}^2}\right)-1\right)\\
&\cong\frac{1}{T}\sum_{t=1}^{T}\frac{x_t}{d_{tt}}\left\{E\left(\frac{\hat{x}_t-x_t}{x_t}\right)-E\left[\left(\frac{\hat{x}_t-x_t}{x_t}\right)\left(\frac{\hat{d}_{tt}-d_{tt}}{d_{tt}}\right)\right]-E\left(\frac{\hat{d}_{tt}-d_{tt}}{d_{tt}}\right)+E\left(\frac{\left(\hat{d}_{tt}-d_{tt}\right)^2}{d_{tt}^2}\right)\right\}\\
&=\frac{1}{T}\sum_{t=1}^{T}\frac{E\left(\hat{x}_t-x_t\right)}{d_{tt}}-\frac{1}{T}\sum_{t=1}^{T}\frac{E\left[\left(\hat{x}_t-x_t\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{tt}^2}+\frac{1}{T}\sum_{t=1}^{T}x_t\left(\frac{E\left(\hat{d}_{tt}-d_{tt}\right)^2}{d_{tt}^3}-\frac{E\left(\hat{d}_{tt}-d_{tt}\right)}{d_{tt}^2}\right)
\end{aligned}
\tag{A.54}
$$

where $x_t = c_{tt}$ for $\alpha_1$, $x_t = b_t$ for $\alpha_2$, and $x_t = a$ for $\alpha_3$. Making use of the expectations in Lemma 9, we find

$$E\left(\hat{\alpha}_1-\alpha_1\right)=E\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\hat{c}_{tt}}{\hat{d}_{tt}}-\frac{c_{tt}}{d_{tt}}\right)\right]$$

$$\cong\frac{1}{T}\sum_{t=1}^{T}\frac{E\left(\hat{c}_{tt}-c_{tt}\right)}{d_{tt}}-\frac{1}{T}\sum_{t=1}^{T}\frac{E\left[\left(\hat{c}_{tt}-c_{tt}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{tt}^2}+\frac{1}{T}\sum_{t=1}^{T}c_{tt}\left(\frac{E\left(\hat{d}_{tt}-d_{tt}\right)^2}{d_{tt}^3}-\frac{E\left(\hat{d}_{tt}-d_{tt}\right)}{d_{tt}^2}\right)$$

$$\cong\frac{1}{T}\sum_{t=1}^{T}\left(\frac{(n+1)(c_{tt}-1)}{(T-L)d_{tt}}+\frac{n}{Td_{tt}}Z'_{t-1}A^{-1}Z_{t-1}\right)$$

$$-\frac{1}{T}\sum_{t=1}^{T}\left(\frac{2a\left(c_{tt}-1\right)^2-2b_t^2\left(c_{tt}-2\right)}{(T-L)d_{tt}^2}+\frac{4\left(d_{tt}-a\right)}{Td_{tt}^2}Z'_{t-1}A^{-1}Z_{t-1}\right)$$

$$+\frac{1}{T}\sum_{t=1}^{T}c_{tt}\left(\frac{2a^2+(n-4)ad_{tt}+(3-2n)d_{tt}^2}{(T-L)d_{tt}^3}+\frac{-4a^2-(n-5)ad_{tt}}{Td_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1}\right)$$

$$=\frac{1}{T}\sum_{t=1}^{T}\frac{(n+1)(c_{tt}-1)d_{tt}^2-2a\left(c_{tt}-1\right)^2d_{tt}+2b_t^2\left(c_{tt}-2\right)d_{tt}+2a^2c_{tt}+(n-4)ac_{tt}d_{tt}+(3-2n)c_{tt}d_{tt}^2}{(T-L)d_{tt}^3}$$

$$+\frac{1}{T}\sum_{t=1}^{T}\frac{nd_{tt}^2-4\left(d_{tt}-a\right)d_{tt}-4a^2c_{tt}-(n-5)ac_{tt}d_{tt}}{Td_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1}$$

$$=\frac{1}{T}\sum_{t=1}^{T}\frac{2a^2c_{tt}+\left[-2a\left(c_{tt}-1\right)^2+2b_t^2\left(c_{tt}-2\right)+(n-4)ac_{tt}\right]d_{tt}+\left[(3-2n)c_{tt}+(n+1)(c_{tt}-1)\right]d_{tt}^2}{(T-L)d_{tt}^3}$$

$$+\frac{1}{T}\sum_{t=1}^{T}\frac{-4a^2c_{tt}+\left[4a-(n-5)ac_{tt}\right]d_{tt}+nd_{tt}^2-4d_{tt}^2}{Td_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1}$$

(A.55)

continuing, we find

$$
\begin{aligned}
E\left(\hat{\alpha}_1-\alpha_1\right) &\cong \frac{1}{T}\sum_{t=1}^{T}\frac{2a^2c_{tt}+\left[-2c_{tt}\left(ac_{tt}-b_t^2\right)-2a-4b_t^2+nac_{tt}\right]d_{tt}+\left[\left(3-2n\right)c_{tt}+\left(n+1\right)\left(c_{tt}-1\right)\right]d_{tt}^2}{\left(T-L\right)d_{tt}^3} \\
&+\frac{1}{T}\sum_{t=1}^{T}\frac{-4a^2c_{tt}+a\left[4-\left(n-5\right)c_{tt}\right]d_{tt}+\left(n-4\right)d_{tt}^2}{Td_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1} \\
&=\frac{1}{T}\sum_{t=1}^{T}\frac{2a^2c_{tt}+\left[-2a+4\left(ac_{tt}-b_t^2\right)+\left(n-4\right)ac_{tt}\right]d_{tt}+\left[-2c_{tt}+\left(3-2n\right)c_{tt}+\left(n+1\right)\left(c_{tt}-1\right)\right]d_{tt}^2}{\left(T-L\right)d_{tt}^3} \\
&+\frac{1}{T}\sum_{t=1}^{T}\frac{-4a^2c_{tt}+a\left[4-\left(n-5\right)c_{tt}\right]d_{tt}+\left(n-4\right)d_{tt}^2}{Td_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1} \\
&=\frac{1}{T\left(T-L\right)}\sum_{t=1}^{T}\frac{2a^2c_{tt}+a\left[\left(n-4\right)c_{tt}-2\right]d_{tt}-\left[\left(n-3\right)+\left(n-2\right)c_{tt}\right]d_{tt}^2}{d_{tt}^3} \\
&+\frac{1}{T^2}\sum_{t=1}^{T}\frac{-4a^2c_{tt}+a\left[4-\left(n-5\right)c_{tt}\right]d_{tt}+\left(n-4\right)d_{tt}^2}{d_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1}
\end{aligned}
\tag{A.56}
$$

Next, we have

$$
\begin{aligned}
E\left(\hat{\alpha}_2-\alpha_2\right) &= E\left[\frac{1}{T}\sum\nolimits_{t=1}^{T}\left(\frac{\hat{b}_t}{\hat{d}_{tt}}-\frac{b_t}{d_{tt}}\right)\right] \\
&\cong \frac{1}{T}\sum\nolimits_{t=1}^{T}\frac{E\left(\hat{b}_t-b_t\right)}{d_{tt}}-\frac{1}{T}\sum\nolimits_{t=1}^{T}\frac{E\left[\left(\hat{b}_t-b_t\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{tt}^2}+\frac{1}{T}\sum\nolimits_{t=1}^{T}b_t\left(\frac{E\left(\hat{d}_{tt}-d_{tt}\right)^2}{d_{tt}^3}-\frac{E\left(\hat{d}_{tt}-d_{tt}\right)}{d_{tt}^2}\right) \\
&\cong \frac{1}{T}\sum\nolimits_{t=1}^{T}\left(\frac{(n+1)}{(T-L)d_{tt}}b_t\right)-\frac{1}{T}\sum\nolimits_{t=1}^{T}\left(\frac{2b_td_{tt}}{(T-L)d_{tt}^2}\right) \\
&\quad+\frac{1}{T}\sum\nolimits_{t=1}^{T}b_t\left(\frac{2a^2+(n-4)ad_{tt}+(3-2n)d_{tt}^2}{(T-L)d_{tt}^3}+\frac{-4a^2-(n-5)ad_{tt}}{Td_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1}\right) \\
&= \frac{1}{T(T-L)}\sum\nolimits_{t=1}^{T}\left(\frac{(n+1)b_t}{d_{tt}}-\frac{2b_td_{tt}}{d_{tt}^2}+b_t\frac{2a^2+(n-4)ad_{tt}+(3-2n)d_{tt}^2}{d_{tt}^3}\right) \\
&\quad+\frac{1}{T^2}\sum\nolimits_{t=1}^{T}b_t\frac{-4a^2-(n-5)ad_{tt}}{d_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1} \\
&= \frac{1}{T(T-L)}\sum\nolimits_{t=1}^{T}\left(\frac{(n+1)b_td_{tt}^2-2b_td_{tt}^2+2a^2b_t+(n-4)ab_td_{tt}-(2n-3)b_td_{tt}^2}{d_{tt}^3}\right) \\
&\quad-\frac{1}{T^2}\sum\nolimits_{t=1}^{T}b_t\frac{4a^2+(n-5)ad_{tt}}{d_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1} \\
&= \frac{1}{T(T-L)}\sum\nolimits_{t=1}^{T}b_t\frac{2a^2+(n-4)ad_{tt}+\left[(n-1)-(2n-3)\right]d_{tt}^2}{d_{tt}^3} \\
&\quad-\frac{1}{T^2}\sum\nolimits_{t=1}^{T}b_t\frac{4a^2+(n-5)ad_{tt}}{d_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1} \\
&= \frac{1}{T(T-L)}\sum\nolimits_{t=1}^{T}b_t\frac{2a^2+(n-4)ad_{tt}-(n-2)d_{tt}^2}{d_{tt}^3}-\frac{1}{T^2}\sum\nolimits_{t=1}^{T}ab_t\frac{4a+(n-5)d_{tt}}{d_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1} \\
&= \frac{1}{T}\sum\nolimits_{t=1}^{T}\left(b_t\frac{2a^2+(n-4)ad_{tt}-(n-2)d_{tt}^2}{d_{tt}^3(T-L)}-ab_t\frac{4a+(n-5)d_{tt}}{d_{tt}^3T}Z'_{t-1}A^{-1}Z_{t-1}\right)
\end{aligned}
\tag{A.57}
$$

and finally

$$
\begin{aligned}
&E\left(\hat{\alpha}_3-\alpha_3\right)=E\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\hat{a}}{\hat{d}_{tt}}-\frac{a}{d_{tt}}\right)\right]\\
&\cong\frac{1}{T}\sum_{t=1}^{T}\frac{E\left(\hat{a}-\hat{a}\right)}{d_{tt}}-\frac{1}{T}\sum_{t=1}^{T}\frac{E\left[\left(\hat{a}-\hat{a}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{tt}^2}+\frac{1}{T}\sum_{t=1}^{T}a\left(\frac{E\left(\hat{d}_{tt}-d_{tt}\right)^2}{d_{tt}^3}-\frac{E\left(\hat{d}_{tt}-d_{tt}\right)}{d_{tt}^2}\right)\\
&\cong\frac{1}{T}\sum_{t=1}^{T}\left(\frac{a\left(n+1\right)}{\left(T-L\right)d_{tt}}\right)-\frac{1}{T}\sum_{t=1}^{T}\left(\frac{2ad_{tt}}{\left(T-L\right)d_{tt}^2}\right)\\
&\quad+\frac{1}{T}\sum_{t=1}^{T}a\left(\frac{2a^2+\left(n-4\right)ad_{tt}+\left(3-2n\right)d_{tt}^2}{\left(T-L\right)d_{tt}^3}+\frac{-4a^2-\left(n-5\right)ad_{tt}}{Td_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1}\right)\\
&=\frac{1}{T}\sum_{t=1}^{T}a\left(\frac{\left(n+1\right)d_{tt}^2-2d_{tt}^2+2a^2+\left(n-4\right)ad_{tt}+\left(3-2n\right)d_{tt}^2}{\left(T-L\right)d_{tt}^3}\right)\\
&\quad+\frac{1}{T}\sum_{t=1}^{T}\left(a^2\frac{-4a-\left(n-5\right)d_{tt}}{Td_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1}\right)\\
&=\frac{a}{T}\sum_{t=1}^{T}\left(\frac{2a^2+\left(n-4\right)ad_{tt}-\left(n-2\right)d_{tt}^2}{\left(T-L\right)d_{tt}^3}-a\frac{4a+\left(n-5\right)d_{tt}}{Td_{tt}^3}Z'_{t-1}A^{-1}Z_{t-1}\right)
\end{aligned}
\tag{A.58}
$$

Expectations of the form $E\left[\left(\hat{\alpha}_i-\alpha_i\right)^2\right]$ are as follows:

$$
\begin{aligned}
&E\left[\left(\hat{\alpha}_1-\alpha_1\right)^2\right]\cong\frac{2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\left(\frac{\left(c_{kt}-1\right)^2}{d_{kk}d_{tt}}-2c_{tt}\frac{a\left(1-c_{kt}^2\right)+b_k^2c_{tt}+b_t^2c_{kk}+2\left(c_{kt}-1\right)d_{kt}}{d_{kk}d_{tt}^2}\right)\\
&\quad+\frac{2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\left(c_{kk}c_{tt}\frac{a^2\left(1+c_{kk}c_{tt}-c_{kt}^2\right)+2a\left(c_{kt}-1\right)d_{kt}}{d_{kk}^2d_{tt}^2}\right)\\
&\quad+\frac{4}{T^3}\sum_{k,t=1}^{T}\left(\frac{c_{kt}-1}{d_{kk}d_{tt}}-2c_{tt}\frac{d_{kt}-a}{d_{kk}d_{tt}^2}+ac_{kk}c_{tt}\frac{d_{kt}-a}{d_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}
\end{aligned}
$$

$$E\left[\left(\hat{\alpha}_2-\alpha_2\right)^2\right]\cong\frac{a}{T^3}\sum_{k,t=1}^{T}\frac{d_{kk}d_{tt}+4b_kb_t\left(d_{kt}-a\right)}{d_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}$$
$$+\frac{1}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\left(\frac{2ab_kb_t\left(a+ac_{kk}c_{tt}-b_kb_tc_{kt}\right)+2a\,b_kb_t\left(c_{kt}-2\right)d_{kt}-4b_kb_t^3d_{kk}+\left(ac_{kt}-a-3b_kb_t\right)d_{kk}d_{tt}}{d_{kk}^2d_{tt}^2}\right)$$

$$E\left[\left(\hat{\alpha}_3-\alpha_3\right)^2\right]\cong\frac{4a^3}{T^3}\sum_{k,t=1}^{T}\frac{d_{kt}-a}{d_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}$$
$$+\frac{2a^2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\frac{-ab_k\,b_tc_{kt}+a^2+a^2c_{kk}c_{tt}-b_k^2d_{tt}-b_t^2d_{kk}-2a\,d_{kt}+a\,c_{kt}d_{kt}-d_{kk}d_{tt}}{d_{kk}^2d_{tt}^2}\qquad\text{(A.59)}$$

Proof: Begin by noting that each of these expectations has the form

$$E\left\{\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\hat{x}_t}{\hat{d}_{tt}}-\frac{x_t}{d_{tt}}\right)\right]^2\right\}=\frac{1}{T^2}\sum_{k,t=1}^{T}\frac{x_kx_t}{d_{kk}d_{tt}}E\left[\left(\frac{1+\left(\hat{x}_k-x_k\right)/x_k}{1+\left(\hat{d}_{kk}-d_{kk}\right)/d_{kk}}-1\right)\left(\frac{1+\left(\hat{x}_t-x_t\right)/x_t}{1+\left(\hat{d}_{tt}-d_{tt}\right)/d_{tt}}-1\right)\right]$$
$$\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\frac{x_kx_t}{d_{kk}d_{tt}}E\left\{\left[\left(1+\frac{\hat{x}_k-x_k}{x_k}\right)\left(1-\frac{\hat{d}_{kk}-d_{kk}}{d_{kk}}\right)-1\right]\left[\left(1+\frac{\hat{x}_t-x_t}{x_t}\right)\left(1-\frac{\hat{d}_{tt}-d_{tt}}{d_{tt}}\right)-1\right]\right\}$$
$$\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\frac{x_kx_t}{d_{kk}d_{tt}}E\left[\left(\frac{\hat{x}_k-x_k}{x_k}-\frac{\hat{d}_{kk}-d_{kk}}{d_{kk}}\right)\left(\frac{\hat{x}_t-x_t}{x_t}-\frac{\hat{d}_{tt}-d_{tt}}{d_{tt}}\right)\right]$$
$$\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\frac{x_kx_t}{d_{kk}d_{tt}}\left(\frac{E\left[\left(\hat{x}_k-x_k\right)\left(\hat{x}_t-x_t\right)\right]}{x_kx_t}-\frac{E\left[\left(\hat{x}_k-x_k\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{x_kd_{tt}}\right)$$
$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\frac{x_kx_t}{d_{kk}d_{tt}}\left(-\frac{E\left[\left(\hat{x}_t-x_t\right)\left(\hat{d}_{kk}-d_{kk}\right)\right]}{x_td_{kk}}+\frac{E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}}\right)$$

(A.60)

where $x_t=c_{tt}$ for $\alpha_1$, $x_t=b_t$ for $\alpha_2$, and $x_t=a$ for $\alpha_3$. Noting that the summation indices $(k,t)$ may be exchanged within an additive term of the summation, this may be written as

$$E\left\{\left[\frac{1}{T}\sum\nolimits_{t=1}^{T}\left(\frac{\hat{x}_t}{\hat{d}_{tt}}-\frac{x_t}{d_{tt}}\right)\right]^2\right\}$$

$$\cong\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\frac{x_k x_t}{d_{kk}d_{tt}}\left(\frac{E\left[\left(\hat{x}_k-x_k\right)\left(\hat{x}_t-x_t\right)\right]}{x_k x_t}-\frac{2E\left[\left(\hat{x}_k-x_k\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{x_k d_{tt}}+\frac{E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}}\right)$$

$$=\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{E\left[\left(\hat{x}_k-x_k\right)\left(\hat{x}_t-x_t\right)\right]}{d_{kk}d_{tt}}-\frac{2x_t E\left[\left(\hat{x}_k-x_k\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}^2}+\frac{x_k x_t E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}^2 d_{tt}^2}\right)$$

(A.61)

We find, using the expectations from Lemma 9, as follows:

$$
\begin{aligned}
&E\left[\left(\hat{\alpha}_1-\alpha_1\right)^2\right]\\
&\cong \frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{E\left[\left(\hat{c}_{kk}-c_{kk}\right)\left(\hat{c}_{tt}-c_{tt}\right)\right]}{d_{kk}d_{tt}}-\frac{2c_{tt}E\left[\left(\hat{c}_{kk}-c_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}^2}+\frac{c_{kk}c_{tt}E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}^2d_{tt}^2}\right)\\
&\cong \frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{\frac{2\left(c_{kt}-1\right)^2}{T-L}+\frac{4\left(c_{kt}-1\right)}{T}Z'_{k-1}A^{-1}Z_{t-1}}{d_{kk}d_{tt}}\right)\\
&\quad+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-\frac{2c_{tt}\left[\frac{2a\left(c_{kt}-1\right)^2-4b_kb_t\left(c_{kt}-1\right)+2b_k^2c_{tt}}{T-L}+\frac{4\left(d_{kt}-a\right)}{T}Z'_{k-1}A^{-1}Z_{t-1}\right]}{d_{kk}d_{tt}^2}\right)\\
&\quad+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{c_{kk}c_{tt}\left[\frac{-2ab_k\,b_tc_{kt}+2a^2+2a^2c_{kk}c_{tt}-2b_k^2d_{tt}-2b_t^2d_{kk}-4a\,d_{kt}+2a\,c_{kt}d_{kt}}{T-L}+4\frac{ad_{kt}-a^2}{T}Z'_{k-1}A^{-1}Z_{t-1}\right]}{d_{kk}^2d_{tt}^2}\right)\\
&\cong \frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2\left(c_{kt}-1\right)^2}{\left(T-L\right)d_{kk}d_{tt}}+\frac{4\left(c_{kt}-1\right)}{Td_{kk}d_{tt}}Z'_{k-1}A^{-1}Z_{t-1}\right)\\
&\quad+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-4c_{tt}\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k^2c_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}-\frac{8c_{tt}\left(d_{kt}-a\right)}{Td_{kk}d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right)\\
&\quad+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(2c_{kk}c_{tt}\frac{-ab_k\,b_tc_{kt}+a^2+a^2c_{kk}c_{tt}-b_k^2d_{tt}-b_t^2d_{kk}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}+4c_{kk}c_{tt}\frac{ad_{kt}-a^2}{Td_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right)
\end{aligned}
\tag{A.62}
$$

continuing, we find

$$E\left[\left(\hat{\alpha}_1-\alpha_1\right)^2\right]\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2\left(c_{kt}-1\right)^2}{\left(T-L\right)d_{kk}d_{tt}}-4c_{tt}\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k^2c_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right)$$
$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(2c_{kk}c_{tt}\frac{-ab_k\,b_tc_{kt}+a^2+a^2c_{kk}c_{tt}-b_k^2d_{tt}-b_t^2d_{kk}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)\tag{A.63}$$
$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{4\left(c_{kt}-1\right)}{Td_{kk}d_{tt}}-\frac{8c_{tt}\left(d_{kt}-a\right)}{Td_{kk}d_{tt}^2}+4c_{kk}c_{tt}\frac{ad_{kt}-a^2}{Td_{kk}^2d_{tt}^2}\right)Z_{k-1}'A^{-1}Z_{t-1}$$

Making use of the fact that

$$\begin{aligned}a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k^2c_{tt}&=ac_{kt}^2-2ac_{kt}+a-2b_kb_tc_{kt}+2b_kb_t+b_k^2c_{tt}\\&=\left(-ac_{kt}^2+2ac_{kt}^2\right)+a-2b_kb_tc_{kt}+b_k^2c_{tt}-2\left(ac_{kt}-b_kb_t\right)\\&=-ac_{kt}^2+a+b_k^2c_{tt}+2c_{kt}\left(ac_{kt}-b_kb_t\right)-2d_{kt}\\&=a-ac_{kt}^2+b_k^2c_{tt}+2c_{kt}d_{kt}-2d_{kt}\\&=a-ac_{kt}^2+b_k^2c_{tt}+2\left(c_{kt}-1\right)d_{kt}\end{aligned}\tag{A.64}$$

and also replacing $-b_k^2d_{tt}-b_t^2d_{kk}$ with $-2b_t^2d_{kk}$ (which is permissible because its multipliers are exchangeable in $(k,t)$) we find

$$
\begin{aligned}
E\left[\left(\hat{\alpha}_1-\alpha_1\right)^2\right] &\cong \frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2\left(c_{kt}-1\right)^2}{\left(T-L\right)d_{kk}d_{tt}}-4c_{tt}\frac{a-ac_{kt}^2+b_k^2c_{tt}+2\left(c_{kt}-1\right)d_{kt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right)\\
&+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(2c_{kk}c_{tt}\frac{-ab_k b_t c_{kt}+a^2+a^2c_{kk}c_{tt}-2b_t^2d_{kk}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)\\
&+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{4\left(c_{kt}-1\right)}{Td_{kk}d_{tt}}-\frac{8c_{tt}\left(d_{kt}-a\right)}{Td_{kk}d_{tt}^2}+4c_{kk}c_{tt}\frac{ad_{kt}-a^2}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}\\
&\cong \frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2\left(c_{kt}-1\right)^2}{\left(T-L\right)d_{kk}d_{tt}}-4c_{tt}\frac{a-ac_{kt}^2+b_k^2c_{tt}+2\left(c_{kt}-1\right)d_{kt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right)\\
&+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(2c_{kk}c_{tt}\frac{-ab_k b_t c_{kt}+a^2+a^2c_{kk}c_{tt}+a\left(c_{kt}-2\right)d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}+2c_{kk}c_{tt}\frac{-2b_t^2}{\left(T-L\right)d_{kk}d_{tt}^2}\right)\\
&+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{4\left(c_{kt}-1\right)}{Td_{kk}d_{tt}}-\frac{8c_{tt}\left(d_{kt}-a\right)}{Td_{kk}d_{tt}^2}+4c_{kk}c_{tt}\frac{ad_{kt}-a^2}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}
\end{aligned}
\tag{A.65}
$$

and

$$
\begin{aligned}
&E\left[\left(\hat{\alpha}_1-\alpha_1\right)^2\right]\\
&\cong \frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2\left(c_{kt}-1\right)^2}{\left(T-L\right)d_{kk}d_{tt}}-4c_{tt}\frac{a-ac_{kt}^2+b_k^2c_{tt}+2\left(c_{kt}-1\right)d_{kt}}{\left(T-L\right)d_{kk}d_{tt}^2}+2c_{kk}c_{tt}\frac{-2b_t^2}{\left(T-L\right)d_{kk}d_{tt}^2}\right)\\
&+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(2c_{kk}c_{tt}\frac{-ab_k b_t c_{kt}+a^2+a^2c_{kk}c_{tt}+a\left(c_{kt}-2\right)d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)\\
&+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{4\left(c_{kt}-1\right)}{Td_{kk}d_{tt}}-\frac{8c_{tt}\left(d_{kt}-a\right)}{Td_{kk}d_{tt}^2}+4c_{kk}c_{tt}\frac{ad_{kt}-a^2}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}\\
&=\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2\left(c_{kt}-1\right)^2}{\left(T-L\right)d_{kk}d_{tt}}-4c_{tt}\frac{a\left(1-c_{kt}^2\right)+b_k^2c_{tt}+b_t^2c_{kk}+2\left(c_{kt}-1\right)d_{kt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right)\\
&+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(2c_{kk}c_{tt}\frac{-ab_k b_t c_{kt}+a^2+a^2c_{kk}c_{tt}+a\left(c_{kt}-2\right)d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)\\
&+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{4\left(c_{kt}-1\right)}{Td_{kk}d_{tt}}-\frac{8c_{tt}\left(d_{kt}-a\right)}{Td_{kk}d_{tt}^2}+4c_{kk}c_{tt}\frac{ad_{kt}-a^2}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}
\end{aligned}
\tag{A.66}
$$

We next use

$$-ab_k b_t c_{kt} = a^2 c_{kt}^2 - ab_k b_t c_{kt} - a^2 c_{kt}^2 = ac_{kt}\left(ac_{kt} - b_k b_t\right) - a^2 c_{kt}^2 = ac_{kt} d_{kt} - a^2 c_{kt}^2$$

to find

$$\begin{aligned}
E\left[\left(\hat{\alpha}_1 - \alpha_1\right)^2\right] &\cong \frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2\left(c_{kt}-1\right)^2}{\left(T-L\right)d_{kk}d_{tt}} - 4c_{tt}\frac{a\left(1-c_{kt}^2\right)+b_k^2 c_{tt} + b_t^2 c_{kk} + 2\left(c_{kt}-1\right)d_{kt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right) \\
&+ \frac{1}{T^2}\sum_{k,t=1}^{T}\left(2c_{kk}c_{tt}\frac{ac_{kt}d_{kt} - a^2 c_{kt}^2 + a^2 + a^2 c_{kk}c_{tt} + a\left(c_{kt}-2\right)d_{kt}}{\left(T-L\right)d_{kk}^2 d_{tt}^2}\right) \\
&+ \frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{4\left(c_{kt}-1\right)}{Td_{kk}d_{tt}} - \frac{8c_{tt}\left(d_{kt}-a\right)}{Td_{kk}d_{tt}^2} + 4c_{kk}c_{tt}\frac{ad_{kt}-a^2}{Td_{kk}^2 d_{tt}^2}\right)Z_{k-1}' A^{-1} Z_{t-1} \\
&= \frac{2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\left(\frac{\left(c_{kt}-1\right)^2}{d_{kk}d_{tt}} - 2c_{tt}\frac{a\left(1-c_{kt}^2\right)+b_k^2 c_{tt} + b_t^2 c_{kk} + 2\left(c_{kt}-1\right)d_{kt}}{d_{kk}d_{tt}^2}\right) \\
&+ \frac{2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\left(c_{kk}c_{tt}\frac{a^2\left(1+c_{kk}c_{tt}-c_{kt}^2\right)+2a\left(c_{kt}-1\right)d_{kt}}{d_{kk}^2 d_{tt}^2}\right) \\
&+ \frac{4}{T^3}\sum_{k,t=1}^{T}\left(\frac{c_{kt}-1}{d_{kk}d_{tt}} - 2c_{tt}\frac{d_{kt}-a}{d_{kk}d_{tt}^2} + ac_{kk}c_{tt}\frac{d_{kt}-a}{d_{kk}^2 d_{tt}^2}\right)Z_{k-1}' A^{-1} Z_{t-1}
\end{aligned} \tag{A.67}$$

Next is

$$
\begin{aligned}
&E\left[\left(\hat{\alpha}_2-\alpha_2\right)^2\right] \\
&\cong \frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{E\left[\left(\hat{b}_k-b_k\right)\left(\hat{b}_t-b_t\right)\right]}{d_{kk}d_{tt}}-\frac{2b_tE\left[\left(\hat{b}_k-b_k\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}^2}+\frac{b_kb_tE\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}^2d_{tt}^2}\right) \\
&\cong \frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{\frac{a\left(c_{kt}-1\right)+b_kb_t}{T-L}+\frac{a}{T}Z'_{k-1}A^{-1}Z_{t-1}}{d_{kk}d_{tt}}\right)+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-\frac{2b_t\frac{2b_kd_{tt}}{T-L}}{d_{kk}d_{tt}^2}\right) \\
&\quad+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{b_kb_t\left[\frac{-2ab_k\,b_tc_{kt}+2a^2+2a^2c_{kk}c_{tt}-2b_k^2d_{tt}-2b_t^2d_{kk}-4a\,d_{kt}+2a\,c_{kt}d_{kt}}{T-L}+4\frac{ad_{kt}-a^2}{T}Z'_{k-1}A^{-1}Z_{t-1}\right]}{d_{kk}^2d_{tt}^2}\right) \\
&=\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{a\left(c_{kt}-1\right)+b_kb_t}{\left(T-L\right)d_{kk}d_{tt}}-\frac{4b_kb_td_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}+\frac{a}{Td_{kk}d_{tt}}Z'_{k-1}A^{-1}Z_{t-1}+4b_kb_t\frac{ad_{kt}-a^2}{Td_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right) \\
&\quad+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(b_kb_t\frac{-2ab_k\,b_tc_{kt}+2a^2+2a^2c_{kk}c_{tt}-2b_k^2d_{tt}-2b_t^2d_{kk}-4a\,d_{kt}+2a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right) \\
&=\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{a\left(c_{kt}-1\right)d_{kk}d_{tt}+b_kb_td_{kk}d_{tt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}+\frac{-4b_kb_td_{tt}d_{kk}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{ad_{kk}d_{tt}+4ab_kb_t\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1} \\
&\quad+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(b_kb_t\frac{-2ab_k\,b_tc_{kt}+2a^2+2a^2c_{kk}c_{tt}-2b_k^2d_{tt}-2b_t^2d_{kk}-4a\,d_{kt}+2a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right) \\
&=\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{a\left(c_{kt}-1\right)d_{kk}d_{tt}-3b_kb_td_{tt}d_{kk}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right) \\
&+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{-2ab_k\,b_tb_kb_tc_{kt}+2a^2b_kb_t+2a^2b_kb_tc_{kk}c_{tt}-2b_k^2b_kb_td_{tt}-2b_t^2b_kb_td_{kk}-4a\,b_kb_td_{kt}+2a\,b_kb_tc_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right) \\
&+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{ad_{kk}d_{tt}+4ab_kb_t\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1} \\
&=\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{-2ab_k^2b_t^2c_{kt}+2a^2b_kb_t+2a^2b_kb_tc_{kk}c_{tt}-4a\,b_kb_td_{kt}+2a\,b_kb_tc_{kt}d_{kt}-2b_k^3b_td_{tt}-2b_kb_t^3d_{kk}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right) \\
&+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{\left(ac_{kt}-a-3b_kb_t\right)d_{kk}d_{tt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{ad_{kk}d_{tt}+4ab_kb_t\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}
\end{aligned}
$$

(A.68)

exchanging summation indices to replace $b_k^3 b_t d_{tt}$ with $b_k b_t^3 d_{kk}$, we find

$$E\left[\left(\hat{\alpha}_2-\alpha_2\right)^2\right]\cong\frac{a}{T^3}\sum_{k,t=1}^{T}\frac{d_{kk}d_{tt}+4b_kb_t\left(d_{kt}-a\right)}{d_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}$$
$$+\frac{1}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\left(\frac{2ab_kb_t\left(a+ac_{kk}c_{tt}-b_kb_tc_{kt}\right)+2ab_kb_t\left(c_{kt}-2\right)d_{kt}-4b_kb_t^3d_{kk}+\left(ac_{kt}-a-3b_kb_t\right)d_{kk}d_{tt}}{d_{kk}^2d_{tt}^2}\right)$$

(A.69)

Next is

$$
\begin{aligned}
&E\left[\left(\hat{\alpha}_3-\alpha_3\right)^2\right]\\
&\cong \frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{E\left[\left(\hat{a}-a\right)^2\right]}{d_{kk}d_{tt}}-\frac{2aE\left[\left(\hat{a}-a\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}^2}+\frac{a^2E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}^2d_{tt}^2}\right)\\
&\cong \frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{\frac{2a^2}{T-L}}{d_{kk}d_{tt}}\right)+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(-\frac{2a\frac{2ad_{tt}}{T-L}}{d_{kk}d_{tt}^2}\right)\\
&\quad+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{a^2\left[\frac{-2ab_k b_t c_{kt}+2a^2+2a^2c_{kk}c_{tt}-2b_k^2d_{tt}-2b_t^2d_{kk}-4a\,d_{kt}+2a\,c_{kt}d_{kt}}{T-L}+4\frac{ad_{kt}-a^2}{T}Z'_{k-1}A^{-1}Z_{t-1}\right]}{d_{kk}^2d_{tt}^2}\right)\\
&=\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{2a^2}{\left(T-L\right)d_{kk}d_{tt}}\right)+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(-\frac{4a^2}{\left(T-L\right)d_{kk}d_{tt}}\right)+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{a^2\left[4\frac{ad_{kt}-a^2}{T}Z'_{t-1}A^{-1}Z_{k-1}\right]}{d_{kk}^2d_{tt}^2}\right)\\
&\quad+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{a^2\left[\frac{-2ab_k b_t c_{kt}+2a^2+2a^2c_{kk}c_{tt}-2b_k^2d_{tt}-2b_t^2d_{kk}-4a\,d_{kt}+2a\,c_{kt}d_{kt}}{T-L}\right]}{d_{kk}^2d_{tt}^2}\right)\\
&=\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{-2a^2}{\left(T-L\right)d_{kk}d_{tt}}\right)+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{4a^3d_{kt}-4a^4}{Td_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right)\\
&\quad+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{a^2\left[-2ab_k b_t c_{kt}+2a^2+2a^2c_{kk}c_{tt}-2b_k^2d_{tt}-2b_t^2d_{kk}-4a\,d_{kt}+2a\,c_{kt}d_{kt}\right]}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)\\
&=\frac{4a^3}{T^3}\sum\nolimits_{k,t=1}^{T}\frac{d_{kt}-a}{d_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\\
&\quad+\frac{2a^2}{T^2\left(T-L\right)}\sum\nolimits_{k,t=1}^{T}\frac{-ab_k b_t c_{kt}+a^2+a^2c_{kk}c_{tt}-b_k^2d_{tt}-b_t^2d_{kk}-2a\,d_{kt}+a\,c_{kt}d_{kt}-d_{kk}d_{tt}}{d_{kk}^2d_{tt}^2}
\end{aligned}
\tag{A.70}
$$

Expectations of the form $E\left[\left(\hat{\alpha}_i-\alpha_i\right)\left(\hat{\alpha}_j-\alpha_j\right)\right]$ with $i\neq j$ are as estimated as follows:

$$E\left[\left(\hat{\alpha}_1-\alpha_1\right)\left(\hat{\alpha}_2-\alpha_2\right)\right]\cong-\frac{2\alpha_2\left(\alpha_3+a\alpha_1\right)}{\left(T-L\right)a}+\frac{2}{T^3}\sum_{k,t=1}^{T}\left(\frac{b_k}{d_{kk}d_{tt}}-\frac{2b_t\left(d_{kt}-a\right)}{d_{kk}d_{tt}^2}+\frac{2ab_kc_{tt}\left(d_{kt}-a\right)}{d_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}$$
$$+\frac{2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\left(\frac{b_kc_{kt}}{d_{kk}d_{tt}}-b_t\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k\left(b_k+b_t\right)c_{tt}+b_t^2c_{kk}}{d_{kk}d_{tt}^2}\right)$$
$$+\frac{2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\left(b_kc_{tt}\frac{-ab_kb_tc_{kt}+a^2+a^2c_{kk}c_{tt}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{d_{kk}^2d_{tt}^2}\right)$$

$$E\left[\left(\hat{\alpha}_1-\alpha_1\right)\left(\hat{\alpha}_3-\alpha_3\right)\right]\cong\frac{-2\alpha_3}{\left(T-L\right)a}-\frac{2a}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k^2c_{tt}+b_t^2\left(c_{kk}+c_{tt}\right)}{d_{kk}d_{tt}^2}$$
$$+\frac{2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\left(a^2c_{tt}\frac{-b_kb_tc_{kt}+a+ac_{kk}c_{tt}+\left(c_{kt}-2\right)d_{kt}}{d_{kk}^2d_{tt}^2}\right)-\frac{4a}{T^3}\sum_{k,t=1}^{T}\frac{\left(a-d_{kt}\right)b_k^2}{d_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}$$

and

$$E\left[\left(\hat{\alpha}_2-\alpha_2\right)\left(\hat{\alpha}_3-\alpha_3\right)\right]\cong\frac{-2\alpha_2\alpha_3}{\left(T-L\right)}+\frac{4a^2}{T^3}\sum_{k,t=1}^{T}\frac{b_t\left(d_{kt}-a\right)}{d_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}$$
$$+\frac{2a}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\left(ab_t\frac{a+ac_{kk}c_{tt}-b_kb_tc_{kt}+\left(c_{kt}-2\right)d_{kt}}{d_{kk}^2d_{tt}^2}-\frac{b_t^2\left(b_k+b_t\right)}{d_{kk}d_{tt}^2}\right) \tag{A.71}$$

To see this note that each of these expectations has the form

$$E\left\{\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\hat{x}_t}{\hat{d}_{tt}}-\frac{x_t}{d_{tt}}\right)\right]\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\hat{y}_t}{\hat{d}_{tt}}-\frac{y_t}{d_{tt}}\right)\right]\right\}$$
$$=\frac{1}{T^2}\sum_{k,t=1}^{T}\frac{x_k y_t}{d_{kk}d_{tt}}E\left[\left(\frac{1+(\hat{x}_k-x_k)/x_k}{1+\left(\hat{d}_{kk}-d_{kk}\right)/d_{kk}}-1\right)\left(\frac{1+(\hat{y}_t-y_t)/y_t}{1+\left(\hat{d}_{tt}-d_{tt}\right)/d_{tt}}-1\right)\right]$$
$$\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\frac{x_k y_t}{d_{kk}d_{tt}}E\left\{\left[\left(1+\frac{\hat{x}_k-x_k}{x_k}\right)\left(1-\frac{\hat{d}_{kk}-d_{kk}}{d_{kk}}\right)-1\right]\left[\left(1+\frac{\hat{y}_t-y_t}{y_t}\right)\left(1-\frac{\hat{d}_{tt}-d_{tt}}{d_{tt}}\right)-1\right]\right\}$$
$$\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\frac{x_k y_t}{d_{kk}d_{tt}}E\left[\left(\frac{\hat{x}_k-x_k}{x_k}-\frac{\hat{d}_{kk}-d_{kk}}{d_{kk}}\right)\left(\frac{\hat{y}_t-y_t}{y_t}-\frac{\hat{d}_{tt}-d_{tt}}{d_{tt}}\right)\right]$$
$$\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\frac{x_k y_t}{d_{kk}d_{tt}}\left(\frac{E\left[(\hat{x}_k-x_k)(\hat{y}_t-y_t)\right]}{x_k y_t}-\frac{E\left[(\hat{x}_k-x_k)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{x_k d_{tt}}\right)$$
$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\frac{y_k x_t}{d_{kk}d_{tt}}\left(-\frac{E\left[(\hat{y}_k-y_k)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{y_k d_{tt}}+\frac{E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}}\right) \qquad \text{(A.72)}$$

where $x_t$ or $y_t$ is $c_{tt}$ for $\alpha_1$, $x_t$ or $y_t$ is $b_t$ for $\alpha_2$, and $x_t$ or $y_t$ is $a$ for $\alpha_3$, and we exchanged the summation indices in the final summation. Continuing, we have

$$E\left\{\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\hat{x}_t}{\hat{d}_{tt}}-\frac{x_t}{d_{tt}}\right)\right]\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\hat{y}_t}{\hat{d}_{tt}}-\frac{y_t}{d_{tt}}\right)\right]\right\}$$
$$\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{E\left[(\hat{x}_k-x_k)(\hat{y}_t-y_t)\right]}{d_{kk}d_{tt}}-y_t\frac{E\left[(\hat{x}_k-x_k)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}^2}\right) \qquad \text{(A.73)}$$
$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-x_t\frac{E\left[(\hat{y}_k-y_k)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}^2}+y_k x_t\frac{E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}^2 d_{tt}^2}\right)$$

Evaluating these expectations we find

$$E\left[\left(\hat{\alpha}_1-\alpha_1\right)\left(\hat{\alpha}_2-\alpha_2\right)\right]\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{E\left[\left(\hat{b}_t-b_t\right)\left(\hat{c}_{kk}-c_{kk}\right)\right]}{d_{kk}d_{tt}}-b_t\frac{E\left[\left(\hat{c}_{kk}-c_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-c_{tt}\frac{E\left[\left(\hat{b}_k-b_k\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}^2}+b_kc_{tt}\frac{E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}^2d_{tt}^2}\right)$$

$$\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{\frac{2b_k\left(c_{kt}-1\right)}{T-L}+\frac{2b_k}{T}Z'_{k-1}A^{-1}Z_{t-1}}{d_{kk}d_{tt}}\right)$$

$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-b_t\frac{\frac{2a\left(c_{kt}-1\right)^2-4b_kb_t\left(c_{kt}-1\right)+2b_k^2c_{tt}}{T-L}+\frac{4\left(d_{kt}-a\right)}{T}Z'_{k-1}A^{-1}Z_{t-1}}{d_{kk}d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-c_{tt}\frac{\frac{2b_kd_{tt}}{T-L}}{d_{kk}d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(b_kc_{tt}\frac{\frac{-2ab_k\,b_tc_{kt}+2a^2+2a^2c_{kk}c_{tt}-2b_k^2d_{tt}-2b_t^2d_{kk}-4a\,d_{kt}+2a\,c_{kt}d_{kt}}{T-L}+4\frac{ad_{kt}-a^2}{T}Z'_{k-1}A^{-1}Z_{t-1}}{d_{kk}^2d_{tt}^2}\right)$$

$$=\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2b_k}{Td_{kk}d_{tt}}-\frac{4b_t\left(d_{kt}-a\right)}{Td_{kk}d_{tt}^2}+\frac{4ab_kc_{tt}\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}$$

$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2b_k\left(c_{kt}-1\right)}{\left(T-L\right)d_{kk}d_{tt}}-b_t\frac{2a\left(c_{kt}-1\right)^2-4b_kb_t\left(c_{kt}-1\right)+2b_k^2c_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}-c_{tt}\frac{2b_kd_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(b_kc_{tt}\frac{-2ab_k\,b_tc_{kt}+2a^2+2a^2c_{kk}c_{tt}-2b_k^2d_{tt}-2b_t^2d_{kk}-4a\,d_{kt}+2a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)$$

$$=\frac{2}{T^3}\sum_{k,t=1}^{T}\left(\frac{b_k}{d_{kk}d_{tt}}-\frac{2b_t\left(d_{kt}-a\right)}{d_{kk}d_{tt}^2}+\frac{2ab_kc_{tt}\left(d_{kt}-a\right)}{d_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}$$

$$+\frac{2}{T^2}\sum_{k,t=1}^{T}\left(\frac{b_k\left(c_{kt}-1\right)-b_kc_{tt}}{\left(T-L\right)d_{kk}d_{tt}}-b_t\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k^2c_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right)$$

$$+\frac{2}{T^2}\sum_{k,t=1}^{T}\left(b_kc_{tt}\frac{-ab_k\,b_tc_{kt}+a^2+a^2c_{kk}c_{tt}-b_k^2d_{tt}-b_t^2d_{kk}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)$$

(A.74)

Manipulating the final summation (including an exchange of the indices) we have

$$
\begin{aligned}
&\frac{2}{T^2}\sum_{k,t=1}^{T}\left(b_k c_{tt}\frac{-ab_k b_t c_{kt}+a^2+a^2 c_{kk}c_{tt}-b_k^2 d_{tt}-b_t^2 d_{kk}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2 d_{tt}^2}\right)\\
&=\frac{2}{T^2}\sum_{k,t=1}^{T}\left(b_k c_{tt}\frac{-ab_k b_t c_{kt}+a^2+a^2 c_{kk}c_{tt}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2 d_{tt}^2}+b_k c_{tt}\frac{-b_k^2 d_{tt}-b_t^2 d_{kk}}{\left(T-L\right)d_{kk}^2 d_{tt}^2}\right)\\
&=\frac{2}{T^2}\sum_{k,t=1}^{T}\left(b_k c_{tt}\frac{-ab_k b_t c_{kt}+a^2+a^2 c_{kk}c_{tt}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2 d_{tt}^2}+\frac{-b_k^3 c_{tt}d_{tt}-b_k b_t^2 c_{tt}d_{kk}}{\left(T-L\right)d_{kk}^2 d_{tt}^2}\right) \qquad \text{(A.75)}\\
&=\frac{2}{T^2}\sum_{k,t=1}^{T}\left(b_k c_{tt}\frac{-ab_k b_t c_{kt}+a^2+a^2 c_{kk}c_{tt}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2 d_{tt}^2}+\frac{-b_t^3 c_{kk}d_{kk}-b_k b_t^2 c_{tt}d_{kk}}{\left(T-L\right)d_{kk}^2 d_{tt}^2}\right)\\
&=\frac{2}{T^2}\sum_{k,t=1}^{T}\left(b_k c_{tt}\frac{-ab_k b_t c_{kt}+a^2+a^2 c_{kk}c_{tt}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2 d_{tt}^2}+\frac{-b_t^3 c_{kk}-b_k b_t^2 c_{tt}}{\left(T-L\right)d_{kk} d_{tt}^2}\right)
\end{aligned}
$$

which leads to

$$
\begin{aligned}
E\left[\left(\hat{\alpha}_1-\alpha_1\right)\left(\hat{\alpha}_2-\alpha_2\right)\right] &\cong \frac{2}{T^3}\sum_{k,t=1}^{T}\left(\frac{b_k}{d_{kk}d_{tt}}-\frac{2b_t\left(d_{kt}-a\right)}{d_{kk}d_{tt}^2}+\frac{2ab_kc_{tt}\left(d_{kt}-a\right)}{d_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1} \\
&+\frac{2}{T^2}\sum_{k,t=1}^{T}\left(\frac{b_k\left(c_{kt}-1\right)-b_kc_{tt}}{\left(T-L\right)d_{kk}d_{tt}}-b_t\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k^2c_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right) \\
&+\frac{2}{T^2}\sum_{k,t=1}^{T}\left(b_kc_{tt}\frac{-ab_k\,b_tc_{kt}+a^2+a^2c_{kk}c_{tt}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}+\frac{-b_t^3c_{kk}-b_kb_t^2c_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right) \\
&=\frac{2}{T^3}\sum_{k,t=1}^{T}\left(\frac{b_k}{d_{kk}d_{tt}}-\frac{2b_t\left(d_{kt}-a\right)}{d_{kk}d_{tt}^2}+\frac{2ab_kc_{tt}\left(d_{kt}-a\right)}{d_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1} \\
&+\frac{2}{T^2}\sum_{k,t=1}^{T}\left(\frac{b_k\left(c_{kt}-1\right)-b_kc_{tt}}{\left(T-L\right)d_{kk}d_{tt}}-b_t\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k^2c_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}-b_t\frac{+b_t^2c_{kk}+b_kb_tc_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right) \\
&+\frac{2}{T^2}\sum_{k,t=1}^{T}\left(b_kc_{tt}\frac{-ab_k\,b_tc_{kt}+a^2+a^2c_{kk}c_{tt}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right) \\
&=\frac{2}{T^3}\sum_{k,t=1}^{T}\left(\frac{b_k}{d_{kk}d_{tt}}-\frac{2b_t\left(d_{kt}-a\right)}{d_{kk}d_{tt}^2}+\frac{2ab_kc_{tt}\left(d_{kt}-a\right)}{d_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1} \\
&+\frac{2}{T^2}\sum_{k,t=1}^{T}\left(\frac{b_k\left(c_{kt}-1\right)-b_kc_{tt}}{\left(T-L\right)d_{kk}d_{tt}}-b_t\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k\left(b_k+b_t\right)c_{tt}+b_t^2c_{kk}}{\left(T-L\right)d_{kk}d_{tt}^2}\right) \qquad \text{(A.76)} \\
&+\frac{2}{T^2}\sum_{k,t=1}^{T}\left(b_kc_{tt}\frac{-ab_k\,b_tc_{kt}+a^2+a^2c_{kk}c_{tt}-2a\,d_{kt}+a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)
\end{aligned}
$$

We next make use of the facts that $\Sigma\left(c_{tt}/d_{tt}\right)=T\alpha_1$, and $\Sigma b_t/d_{tt}=T\alpha_2$, and $\Sigma\left(1/d_{tt}\right)=T\alpha_3/a$ so that

$$
\begin{aligned}
&\frac{2}{T^2}\sum_{k,t=1}^{T}\frac{-b_k-b_kc_{tt}}{\left(T-L\right)d_{kk}d_{tt}}=-\frac{2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\frac{b_k}{d_{kk}d_{tt}}-\frac{2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\frac{b_kc_{tt}}{d_{kk}d_{tt}} \\
&=-\frac{2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\frac{b_k}{d_{kk}d_{tt}}-\frac{2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\frac{b_kc_{tt}}{d_{kk}d_{tt}} \\
&=-\frac{2}{T^2\left(T-L\right)}\left(\sum_{k=1}^{T}\frac{b_k}{d_{kk}}\right)\left(\sum_{t=1}^{T}\frac{1}{d_{tt}}\right)-\frac{2}{T^2\left(T-L\right)}\left(\sum_{k=1}^{T}\frac{b_k}{d_{kk}}\right)\left(\sum_{t=1}^{T}\frac{c_{tt}}{d_{tt}}\right) \qquad \text{(A.77)} \\
&=-\frac{2\alpha_2\alpha_3}{\left(T-L\right)a}-\frac{2\alpha_1\alpha_2}{\left(T-L\right)}=-\frac{2}{\left(T-L\right)}\left(\frac{\alpha_2\alpha_3}{a}+\alpha_1\alpha_2\right)=-\frac{2\alpha_2\left(\alpha_3+a\alpha_1\right)}{a\left(T-L\right)}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
E\left[\left(\hat{\alpha}_1-\alpha_1\right)\left(\hat{\alpha}_2-\alpha_2\right)\right] &\cong \frac{2}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{-b_k-b_kc_{tt}}{\left(T-L\right)d_{kk}d_{tt}}\right)\\
&+\frac{2}{T^3}\sum\nolimits_{k,t=1}^{T}\left(\frac{b_k}{d_{kk}d_{tt}}-\frac{2b_t\left(d_{kt}-a\right)}{d_{kk}d_{tt}^2}+\frac{2ab_kc_{tt}\left(d_{kt}-a\right)}{d_{kk}^2d_{tt}^2}\right)Z_{k-1}'A^{-1}Z_{t-1}\\
&+\frac{2}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{b_kc_{kt}}{\left(T-L\right)d_{kk}d_{tt}}-b_t\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k\left(b_k+b_t\right)c_{tt}+b_t^2c_{kk}}{\left(T-L\right)d_{kk}d_{tt}^2}\right)\\
&+\frac{2}{T^2}\sum\nolimits_{k,t=1}^{T}\left(b_kc_{tt}\frac{-ab_kb_tc_{kt}+a^2+a^2c_{kk}c_{tt}-2ad_{kt}+ac_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)\\
&=-\frac{2\alpha_2\left(\alpha_3+a\alpha_1\right)}{\left(T-L\right)a}+\frac{2}{T^3}\sum\nolimits_{k,t=1}^{T}\left(\frac{b_k}{d_{kk}d_{tt}}-\frac{2b_t\left(d_{kt}-a\right)}{d_{kk}d_{tt}^2}+\frac{2ab_kc_{tt}\left(d_{kt}-a\right)}{d_{kk}^2d_{tt}^2}\right)Z_{k-1}'A^{-1}Z_{t-1}\\
&+\frac{2}{T^2\left(T-L\right)}\sum\nolimits_{k,t=1}^{T}\left(\frac{b_kc_{kt}}{d_{kk}d_{tt}}-b_t\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k\left(b_k+b_t\right)c_{tt}+b_t^2c_{kk}}{d_{kk}d_{tt}^2}\right)\\
&+\frac{2}{T^2\left(T-L\right)}\sum\nolimits_{k,t=1}^{T}\left(b_kc_{tt}\frac{-ab_kb_tc_{kt}+a^2+a^2c_{kk}c_{tt}-2ad_{kt}+ac_{kt}d_{kt}}{d_{kk}^2d_{tt}^2}\right) \qquad \text{(A78)}
\end{aligned}
$$

Next is

$$E\left[\left(\hat{\alpha}_1-\alpha_1\right)\left(\hat{\alpha}_3-\alpha_3\right)\right]\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{E\left[\left(\hat{a}-a\right)\left(\hat{c}_{kk}-c_{kk}\right)\right]}{d_{kk}d_{tt}}-a\frac{E\left[\left(\hat{c}_{kk}-c_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-c_{tt}\frac{E\left[\left(\hat{a}-a\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}^2}+ac_{tt}\frac{E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}^2d_{tt}^2}\right)$$

$$\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2b_k^2}{\left(T-L\right)d_{kk}d_{tt}}-a\frac{\dfrac{2a\left(c_{kt}-1\right)^2-4b_kb_t\left(c_{kt}-1\right)+2b_k^2c_{tt}}{T-L}+\dfrac{4\left(d_{kt}-a\right)}{T}Z'_{k-1}A^{-1}Z_{t-1}}{d_{kk}d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-c_{tt}\frac{2ad_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(ac_{tt}\frac{\dfrac{-2ab_kb_tc_{kt}+2a^2+2a^2c_{kk}c_{tt}-2b_k^2d_{tt}-2b_t^2d_{kk}-4ad_{kt}+2ac_{kt}d_{kt}}{T-L}+4\dfrac{ad_{kt}-a^2}{T}Z'_{k-1}A^{-1}Z_{t-1}}{d_{kk}^2d_{tt}^2}\right)$$

$$=\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2b_k^2}{\left(T-L\right)d_{kk}d_{tt}}-a\frac{2a\left(c_{kt}-1\right)^2-4b_kb_t\left(c_{kt}-1\right)+2b_k^2c_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}-c_{tt}\frac{2ad_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(ac_{tt}\frac{-2ab_kb_tc_{kt}+2a^2+2a^2c_{kk}c_{tt}-2b_k^2d_{tt}-2b_t^2d_{kk}-4ad_{kt}+2ac_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-\frac{4a\left(d_{kt}-a\right)}{Td_{kk}d_{tt}^2}+4ac_{tt}\frac{ad_{kt}-a^2}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}$$

(A.79)

continuing, we find

$$E\left[\left(\hat{\alpha}_1-\alpha_1\right)\left(\hat{\alpha}_3-\alpha_3\right)\right]\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2b_k^2}{\left(T-L\right)d_{kk}d_{tt}}-a\frac{2a\left(c_{kt}-1\right)^2-4b_kb_t\left(c_{kt}-1\right)+2b_k^2c_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}-a\frac{2c_{tt}d_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(ac_{tt}\frac{-2ab_kb_tc_{kt}+2a^2+2a^2c_{kk}c_{tt}-4ad_{kt}+2ac_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}+ac_{tt}\frac{-2b_k^2d_{tt}-2b_t^2d_{kk}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)$$

$$+\frac{4a}{T^2}\sum_{k,t=1}^{T}\left(\frac{a-d_{kt}}{Td_{kk}d_{tt}^2}+\frac{ac_{tt}\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}$$

(A.80)

Selectively exchanging the indices, we find

$$E\left[\left(\hat{\alpha}_1-\alpha_1\right)\left(\hat{\alpha}_3-\alpha_3\right)\right]\cong\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{2b_k^2-2ac_{kk}}{(T-L)d_{kk}d_{tt}}-a\frac{2a\left(c_{kt}-1\right)^2-4b_kb_t\left(c_{kt}-1\right)+2b_k^2c_{tt}}{(T-L)d_{kk}d_{tt}^2}\right)$$
$$+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(ac_{tt}\frac{-2ab_kb_tc_{kt}+2a^2+2a^2c_{kk}c_{tt}-4ad_{kt}+2ac_{kt}d_{kt}}{(T-L)d_{kk}^2d_{tt}^2}+\frac{-2ac_{kk}b_t^2d_{kk}-2ac_{tt}b_t^2d_{kk}}{(T-L)d_{kk}^2d_{tt}^2}\right)$$
$$+\frac{4a}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{\left(a-d_{kt}\right)d_{kk}}{Td_{kk}^2d_{tt}^2}+\frac{ac_{tt}\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}$$
$$=\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{-2d_{kk}}{(T-L)d_{kk}d_{tt}}-a\frac{2a\left(c_{kt}-1\right)^2-4b_kb_t\left(c_{kt}-1\right)+2b_k^2c_{tt}}{(T-L)d_{kk}d_{tt}^2}+\frac{-2ac_{kk}b_t^2-2ac_{tt}b_t^2}{(T-L)d_{kk}d_{tt}^2}\right)$$
$$+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(ac_{tt}\frac{-2ab_kb_tc_{kt}+2a^2+2a^2c_{kk}c_{tt}-4ad_{kt}+2ac_{kt}d_{kt}}{(T-L)d_{kk}^2d_{tt}^2}\right)$$
$$+\frac{4a}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{\left(a-d_{kt}\right)\left(ac_{kk}-b_k^2\right)-\left(a-d_{kt}\right)ac_{kk}}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}$$
$$=\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{-2}{(T-L)d_{tt}}-2a\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k^2c_{tt}+b_t^2\left(c_{kk}+c_{tt}\right)}{(T-L)d_{kk}d_{tt}^2}\right)$$
$$+\frac{2}{T^2}\sum\nolimits_{k,t=1}^{T}\left(a^2c_{tt}\frac{-b_kb_tc_{kt}+a+ac_{kk}c_{tt}-2d_{kt}+c_{kt}d_{kt}}{(T-L)d_{kk}^2d_{tt}^2}\right)+\frac{4a}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{-\left(a-d_{kt}\right)b_k^2}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1} \qquad \text{(A.81)}$$

continuing, we find

$$E\left[\left(\hat{\alpha}_1-\alpha_1\right)\left(\hat{\alpha}_3-\alpha_3\right)\right]\cong\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{-2}{(T-L)d_{tt}}-2a\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k^2c_{tt}+b_t^2\left(c_{kk}+c_{tt}\right)}{(T-L)d_{kk}d_{tt}^2}\right)$$
$$+\frac{2}{T^2}\sum\nolimits_{k,t=1}^{T}\left(a^2c_{tt}\frac{-b_kb_tc_{kt}+a+ac_{kk}c_{tt}+\left(c_{kt}-2\right)d_{kt}}{(T-L)d_{kk}^2d_{tt}^2}\right)-\frac{4a}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{\left(a-d_{kt}\right)b_k^2}{Td_{kk}^2d_{tt}^2}\right)Z'_{k-1}A^{-1}Z_{t-1}$$

(A.82)

Recalling that $\alpha_3=\frac{a}{T}\sum\nolimits_{t=1}^{T}\frac{1}{d_{tt}}$, we have

$$E\left[\left(\hat{\alpha}_1-\alpha_1\right)\left(\hat{\alpha}_3-\alpha_3\right)\right]\cong\frac{-2\alpha_3}{\left(T-L\right)a}-\frac{2a}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\frac{a\left(c_{kt}-1\right)^2-2b_kb_t\left(c_{kt}-1\right)+b_k^2c_{tt}+b_t^2\left(c_{kk}+c_{tt}\right)}{d_{kk}d_{tt}^2}$$
$$+\frac{2}{T^2\left(T-L\right)}\sum_{k,t=1}^{T}\left(a^2c_{tt}\frac{-b_kb_tc_{kt}+a+ac_{kk}c_{tt}+\left(c_{kt}-2\right)d_{kt}}{d_{kk}^2d_{tt}^2}\right)-\frac{4a}{T^3}\sum_{k,t=1}^{T}\frac{\left(a-d_{kt}\right)b_k^2}{d_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}$$

(A.83)

Next is

$$E\left[\left(\hat{\alpha}_2-\alpha_2\right)\left(\hat{\alpha}_3-\alpha_3\right)\right]\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{E\left[\left(\hat{a}-a\right)\left(\hat{b}_k-b_k\right)\right]}{d_{kk}d_{tt}}-a\frac{E\left[\left(\hat{b}_k-b_k\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}^2}\right)$$
$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-b_t\frac{E\left[\left(\hat{a}-a\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}d_{tt}^2}+ab_t\frac{E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]}{d_{kk}^2d_{tt}^2}\right)$$
$$\cong\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{2ab_k}{\left(T-L\right)d_{kk}d_{tt}}-\frac{2ab_kd_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right)+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-\frac{2ab_td_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}\right)$$
$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(ab_t\frac{-2ab_kb_tc_{kt}+2a^2+2a^2c_{kk}c_{tt}-2b_k^2d_{tt}-2b_t^2d_{kk}-4a\,d_{kt}+2a\,c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}+4ab_t\frac{ad_{kt}-a^2}{Td_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right)$$
$$=\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-\frac{2ab_td_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}+\frac{4a^2b_t\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right)$$
$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{-2a^2b_kb_t^2c_{kt}+2a^3b_t+2a^3b_tc_{kk}c_{tt}-4a^2b_td_{kt}+2a^2b_tc_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}+\frac{-2ab_k^2b_td_{tt}-2ab_t^3d_{kk}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)$$
$$=\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-\frac{2ab_td_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}+\frac{4a^2b_t\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right)$$
$$+\frac{1}{T^2}\sum_{k,t=1}^{T}\left(\frac{-2a^2b_kb_t^2c_{kt}+2a^3b_t+2a^3b_tc_{kk}c_{tt}-4a^2b_td_{kt}+2a^2b_tc_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}+\frac{-2ab_t^2b_kd_{kk}-2ab_t^3d_{kk}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)$$

(A.84)

where indices were exchanged in one of the last terms above. Continuing, we find

$$E\left[\left(\hat{\alpha}_2-\alpha_2\right)\left(\hat{\alpha}_3-\alpha_3\right)\right]\cong\frac{2a^2}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{-b_k\,b_t^2c_{kt}+ab_t+ab_tc_{kk}c_{tt}-2b_td_{kt}+b_tc_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(-\frac{2ab_td_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}+\frac{-2ab_t^2b_kd_{kk}-2ab_t^3d_{kk}}{\left(T-L\right)d_{kk}^2d_{tt}^2}+\frac{4a^2b_t\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right)$$

$$=\frac{2a^2b_t}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{-b_k\,b_tc_{kt}+a+ac_{kk}c_{tt}-2d_{kt}+c_{kt}d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(2\frac{-ab_td_{tt}-ab_t^2b_k-ab_t^3}{\left(T-L\right)d_{kk}d_{tt}^2}+\frac{4a^2b_t\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right)$$

$$=\frac{2a^2b_t}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{a+ac_{kk}c_{tt}-b_kb_tc_{kt}+\left(c_{kt}-2\right)d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(2ab_t\frac{-d_{tt}-b_tb_k-b_t^2}{\left(T-L\right)d_{kk}d_{tt}^2}+\frac{4a^2b_t\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right)$$

$$=\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(2a^2b_t\frac{a+ac_{kk}c_{tt}-b_kb_tc_{kt}+\left(c_{kt}-2\right)d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}-2ab_t\frac{d_{tt}+b_tb_k+b_t^2}{\left(T-L\right)d_{kk}d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{4a^2b_t\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right)$$

$$=\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(-2ab_t\frac{d_{tt}}{\left(T-L\right)d_{kk}d_{tt}^2}+2a^2b_t\frac{a+ac_{kk}c_{tt}-b_kb_tc_{kt}+\left(c_{kt}-2\right)d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}-2ab_t\frac{b_tb_k+b_t^2}{\left(T-L\right)d_{kk}d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{4a^2b_t\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right)$$

$$=\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(-2a\frac{b_t}{\left(T-L\right)d_{kk}d_{tt}}+2a^2b_t\frac{a+ac_{kk}c_{tt}-b_kb_tc_{kt}+\left(c_{kt}-2\right)d_{kt}}{\left(T-L\right)d_{kk}^2d_{tt}^2}-2ab_t\frac{b_tb_k+b_t^2}{\left(T-L\right)d_{kk}d_{tt}^2}\right)$$

$$+\frac{1}{T^2}\sum\nolimits_{k,t=1}^{T}\left(\frac{4a^2b_t\left(d_{kt}-a\right)}{Td_{kk}^2d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}\right)$$

(A.85)

We next make use of the facts that $\Sigma b_t/d_{tt}=T\alpha_2$ and $\Sigma\left(1/d_{tt}\right)=\alpha_3T/a$ so that

$$\frac{1}{T^2}\sum_{k,t=1}^{T}\left(-2a\frac{b_t}{(T-L)d_{kk}d_{tt}}\right)=\frac{-2a}{T^2(T-L)}\left(\sum_{k=1}^{T}\frac{1}{d_{kk}}\right)\left(\sum_{t=1}^{T}\frac{b_t}{d_{tt}}\right)$$
$$=\frac{-2a}{T^2(T-L)}\left(\frac{\alpha_3 T}{a}\right)(T\alpha_2)=\frac{-2\alpha_2\alpha_3}{(T-L)} \tag{A.86}$$

which we use to find

$$E\left[(\hat{\alpha}_2-\alpha_2)(\hat{\alpha}_3-\alpha_3)\right]\cong\frac{-2\alpha_2\alpha_3}{(T-L)}+\frac{4a^2}{T^3}\sum_{k,t=1}^{T}\frac{b_t(d_{kt}-a)}{d_{kk}^2 d_{tt}^2}Z'_{k-1}A^{-1}Z_{t-1}$$
$$+\frac{2a}{T^2(T-L)}\sum_{k,t=1}^{T}\left(ab_t\frac{a+ac_{kk}c_{tt}-b_k b_t c_{kt}+(c_{kt}-2)d_{kt}}{d_{kk}^2 d_{tt}^2}-\frac{b_t^2(b_k+b_t)}{d_{kk}d_{tt}^2}\right) \tag{A.87}$$

completing the proof. □

**Lemmas for Proposition 11**

Lemma 1: Let $u$ be a vector. Then

$$(I+uu')^{-1}=I-\frac{uu'}{1+u'u} \tag{A.88}$$

Proof: We multiply:

$$(I+uu')\left(I-\frac{uu'}{1+u'u}\right)=\left(I-\frac{uu'}{1+u'u}\right)+uu'\left(I-\frac{uu'}{1+u'u}\right)$$
$$=I-\frac{uu'}{1+u'u}+uu'-\frac{u(u'u)u'}{1+u'u}=I+uu'\left(-\frac{1}{1+u'u}+1-\frac{u'u}{1+u'u}\right)=I \tag{A.89}$$

completing the proof. □

Lemma 2: Let $u$ be a vector and let $V$ be a fixed positive semidefinite matrix. Then

$$(V+uu')^{-1}=V^{-1}-\frac{V^{-1}uu'V^{-1}}{1+u'V^{-1}u} \tag{A.90}$$

Proof: Let $V = AA'$ be the Cholesky decomposition. Then we have:

$$\left(V+uu'\right)^{-1}=\left(AA'+uu'\right)^{-1}=\left(A'\right)^{-1}\left[I+\left(A^{-1}u\right)\left(A^{-1}u\right)'\right]^{-1}A^{-1} \tag{A.91}$$

Applying Lemma 1, we find

$$\begin{aligned}\left(V+uu'\right)^{-1}&=\left(A'\right)^{-1}\left(I-\frac{\left(A^{-1}u\right)\left(A^{-1}u\right)'}{1+\left(A^{-1}u\right)'\left(A^{-1}u\right)}\right)A^{-1}\\ &=\left(A'\right)^{-1}A^{-1}-\frac{\left(A'\right)^{-1}\left(A^{-1}u\right)\left(A^{-1}u\right)'A^{-1}}{1+\left(A^{-1}u\right)'\left(A^{-1}u\right)}=\left(AA'\right)^{-1}-\frac{\left(A'\right)^{-1}A^{-1}uu'\left(A^{-1}\right)'A^{-1}}{1+u'\left(A^{-1}\right)'\left(A^{-1}\right)u}\\ &=\left(AA'\right)^{-1}-\frac{\left(AA'\right)^{-1}uu'\left(AA'\right)^{-1}}{1+u'\left(AA'\right)^{-1}u}=V^{-1}-\frac{V^{-1}uu'V^{-1}}{1+u'V^{-1}u}\end{aligned} \tag{A.92}$$

completing the proof. □

<u>Lemma 3</u>: Here are some expressions with $\Lambda(Z)=\left[V(Z)+\mu(Z)\mu'(Z)\right]^{-1}$. For simplicity we will omit the argument $Z$.

$$\Lambda=V^{-1}-\frac{V^{-1}\mu\mu'V^{-1}}{1+\mu'V^{-1}\mu},\quad \Lambda\mu=\frac{V^{-1}\mu}{1+\mu'V^{-1}\mu},\quad \text{and}\quad \mu'\Lambda\mu=1-\frac{1}{1+\mu'V^{-1}\mu} \tag{A.93}$$

<u>Proof</u>: Using Lemma 2, we find

$$\Lambda=V^{-1}-\frac{V^{-1}\mu\mu'V^{-1}}{1+\mu'V^{-1}\mu} \tag{A.94}$$

Next, we compute:

$$\begin{aligned}\Lambda\mu&=\left(V^{-1}-\frac{V^{-1}\mu\mu'V^{-1}}{1+\mu'V^{-1}\mu}\right)\mu=V^{-1}\mu-\frac{V^{-1}\mu\mu'V^{-1}\mu}{1+\mu'V^{-1}\mu}=\left(1-\frac{\mu'V^{-1}\mu}{1+\mu'V^{-1}\mu}\right)V^{-1}\mu\\ &=\left(\frac{1+\mu'V^{-1}\mu-\mu'V^{-1}\mu}{1+\mu'V^{-1}\mu}\right)V^{-1}\mu=\frac{V^{-1}\mu}{1+\mu'V^{-1}\mu}\end{aligned} \tag{A.95}$$

using this we find

$$\mu'\Lambda\mu = \mu'\frac{V^{-1}\mu}{1+\mu'V^{-1}\mu} = \frac{1+\mu'V^{-1}\mu-1}{1+\mu'V^{-1}\mu} = 1-\frac{1}{1+\mu'V^{-1}\mu} \tag{A.96}$$

completing the proof. □

<u>Lemma 4</u>: Let $W$ be a small perturbation of $V$. Then

$$\left(V+W\right)^{-1} = V^{-1} - V^{-1}WV^{-1} + V^{-1}WV^{-1}WV^{-1} + O\left(\|W\|^3\right) \tag{A.97}$$

and

$$\hat{V}^{-1} \cong V^{-1} - V^{-1}\left(\hat{V}-V\right)V^{-1} + V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1} \tag{A.98}$$

<u>Proof</u>: we multiply

$$\begin{aligned}
&\left(V+W\right)\left(V^{-1}-V^{-1}WV^{-1}+V^{-1}WV^{-1}WV^{-1}\right)\\
&=V\left(V^{-1}-V^{-1}WV^{-1}+V^{-1}WV^{-1}WV^{-1}\right)+W\left(V^{-1}-V^{-1}WV^{-1}+V^{-1}WV^{-1}WV^{-1}\right)\\
&=\left(I-WV^{-1}+WV^{-1}WV^{-1}\right)+\left(WV^{-1}-WV^{-1}WV^{-1}+WV^{-1}WV^{-1}WV^{-1}\right)\\
&=I+WV^{-1}-WV^{-1}+WV^{-1}WV^{-1}-WV^{-1}WV^{-1}+WV^{-1}WV^{-1}WV^{-1}\\
&=I+WV^{-1}WV^{-1}WV^{-1}=I+O\left(\|W\|^3\right)
\end{aligned} \tag{A.99}$$

Note that we may apply this result to obtain

$$\hat{V}^{-1} = \left[V+\left(\hat{V}-V\right)\right]^{-1} \cong V^{-1} - V^{-1}\left(\hat{V}-V\right)V^{-1} + V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1} \tag{A.100}$$

completing the proof. □

<u>Lemma 5</u>: $E\left[\left(\hat{\mu}_k-\mu_k\right)'Q\left(\hat{\mu}_t-\mu_t\right)\right] = \frac{tr\left(VQ\right)}{T}Z'_{k-1}A^{-1}Z_{t-1}$ and, in particular,

$$E\left[\left(\hat{\mu}_k-\mu_k\right)'V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right]=\frac{n}{T}Z'_{k-1}A^{-1}Z_{t-1},\quad E\left[\left(\hat{\mu}_k-\mu_k\right)\left(\hat{\mu}_t-\mu_t\right)'\right]=\frac{VZ'_{k-1}A^{-1}Z_{t-1}}{T} \qquad \text{(A.101)}$$

and

$$E\left[\left(\hat{\mu}_k-\mu_k\right)'V^{-1}u_1u_2'V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right]=\frac{u_1'V^{-1}u_2}{T}Z'_{k-1}A^{-1}Z_{t-1} \qquad \text{(A.102)}$$

<u>Proof</u>: Recall that $\hat{\delta}-\delta=A^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}Z_{t-1}\varepsilon_t'\right)$, so that

$$\hat{\mu}_t-\mu_t=\left(\hat{\delta}-\delta\right)'Z_{t-1}=\left(\frac{1}{T}\sum_{k=1}^{T}\varepsilon_kZ'_{k-1}\right)A^{-1}Z_{t-1} \qquad \text{(A.103)}$$

Using commutativity within the trace operator, independence of $(\varepsilon_t,\varepsilon_k)$ when $k\neq t$, and taking advantage of commutativity of scalars, we find:

$$\begin{aligned}
&E\left[\left(\hat{\mu}_k-\mu_k\right)'Q\left(\hat{\mu}_t-\mu_t\right)\right]\\
&=E\left\{tr\left[\left(\hat{\mu}_k-\mu_k\right)'Q\left(\hat{\mu}_t-\mu_t\right)\right]\right\}=E\left\{tr\left[\left(\hat{\mu}_t-\mu_t\right)\left(\hat{\mu}_k-\mu_k\right)'Q\right]\right\}\\
&=tr\left\{E\left[\left(\hat{\mu}_t-\mu_t\right)\left(\hat{\mu}_k-\mu_k\right)'\right]Q\right\}=tr\left\{E\left[\left(\frac{1}{T}\sum_{i=1}^{T}\varepsilon_iZ'_{i-1}\right)A^{-1}Z_{t-1}Z'_{k-1}A^{-1}\left(\frac{1}{T}\sum_{i=1}^{T}Z_{i-1}\varepsilon_i'\right)\right]Q\right\}\\
&=tr\left[E\left(\frac{1}{T^2}\sum_{i=1}^{T}\varepsilon_iZ'_{i-1}A^{-1}Z_{t-1}Z'_{k-1}A^{-1}Z_{i-1}\varepsilon_i'\right)Q\right]\\
&=E\left(\frac{1}{T^2}\sum_{i=1}^{T}tr\left(\varepsilon_iZ'_{i-1}A^{-1}Z_{t-1}Z'_{k-1}A^{-1}Z_{i-1}\varepsilon_i'Q\right)\right)=E\left(\frac{1}{T^2}\sum_{i=1}^{T}tr\left(\varepsilon_i'Q\varepsilon_i\right)Z'_{i-1}A^{-1}Z_{t-1}Z'_{k-1}A^{-1}Z_{i-1}\right)\\
&=\frac{1}{T^2}\sum_{i=1}^{T}tr\left[E\left(\varepsilon_i\varepsilon_i'Q\right)\right]Z'_{i-1}A^{-1}Z_{t-1}Z'_{k-1}A^{-1}Z_{i-1}\\
&=\frac{1}{T^2}\sum_{i=1}^{T}tr\left(VQ\right)Z'_{i-1}A^{-1}Z_{t-1}Z'_{k-1}A^{-1}Z_{i-1}=\frac{tr\left(VQ\right)}{T^2}\sum_{i=1}^{T}Z'_{i-1}A^{-1}Z_{t-1}Z'_{k-1}A^{-1}Z_{i-1}\\
&=\frac{tr\left(VQ\right)}{T^2}Z'_{k-1}A^{-1}\left(\sum_{i=1}^{T}Z_{i-1}Z'_{i-1}\right)A^{-1}Z_{t-1}=\frac{tr\left(VQ\right)}{T}Z'_{k-1}A^{-1}AA^{-1}Z_{t-1}\\
&=\frac{tr\left(VQ\right)}{T}Z'_{k-1}A^{-1}Z_{t-1}
\end{aligned}$$

(A.104)

where we note that $Z'_{k-1}A^{-1}Z_{t-1} = Z'_{t-1}A^{-1}Z_{k-1}$ because each is a transposed scalar of the other. In particular, we have

$$E\left[\left(\hat{\mu}_k - \mu_k\right)' V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right] = \frac{tr\left(VV^{-1}\right)}{T} Z'_{k-1}A^{-1}Z_{t-1} = \frac{tr\left(I_n\right)}{T} Z'_{k-1}A^{-1}Z_{t-1} = \frac{n}{T} Z'_{k-1}A^{-1}Z_{t-1} \quad \text{(A.105)}$$

Moreover, for any fixed vectors $x$ and $y$, we have

$$\begin{aligned} x'E\left[\left(\hat{\mu}_k - \mu_k\right)\left(\hat{\mu}_t - \mu_t\right)'\right]y &= E\left[x'\left(\hat{\mu}_k - \mu_k\right)\left(\hat{\mu}_t - \mu_t\right)' y\right] = E\left[\left(\hat{\mu}_k - \mu_k\right)' xy'\left(\hat{\mu}_t - \mu_t\right)\right] \\ &= \frac{tr\left(Vxy'\right)}{T} Z'_{k-1}A^{-1}Z_{t-1} = \frac{x'Vy}{T} Z'_{k-1}A^{-1}Z_{t-1} = x'\frac{VZ'_{k-1}A^{-1}Z_{t-1}}{T} y \end{aligned} \quad \text{(A.106)}$$

which states equality for all $x$ and $y$, implying that

$$E\left[\left(\hat{\mu}_k - \mu_k\right)\left(\hat{\mu}_t - \mu_t\right)'\right] = \frac{VZ'_{k-1}A^{-1}Z_{t-1}}{T} \quad \text{(A.107)}$$

and

$$E\left[\left(\hat{\mu}_k - \mu_k\right)' V^{-1}u_1u_2'V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right] = \frac{tr\left(VV^{-1}u_1u_2'V^{-1}\right)}{T} Z'_{k-1}A^{-1}Z_{t-1} = \frac{u_1'V^{-1}u_2}{T} Z'_{k-1}A^{-1}Z_{t-1} \quad \text{(A.108)}$$

and completing the proof. □

<u>Lemma 6</u>: For a fixed matrix $Q$, we have

$$E\left[\left(\hat{V} - V\right)Q\left(\hat{V} - V\right)\right] = \frac{VQ'V + V\left[tr\left(VQ\right)\right]}{T - L} \quad \text{(A.109)}$$

and, in particular,

$$E\left[\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)\right]=\frac{(n+1)V}{T-L} \tag{A.110}$$

as well as

$$E\left[u_1'V^{-1}\left(\hat{V}-V\right)V^{-1}u_2u_3'V^{-1}\left(\hat{V}-V\right)V^{-1}u_4\right]=\frac{u_1'V^{-1}u_3u_2'V^{-1}u_4+u_1'V^{-1}u_4u_2'V^{-1}u_3}{T-L} \tag{A.111}$$

which implies that:

$$\begin{aligned}
\mu_t'V^{-1}E\left[\left(\hat{V}-V\right)V^{-1}\mu_t\mu_t'V^{-1}\left(\hat{V}-V\right)\right]V^{-1}\mu_t &= \frac{2\left(c_{tt}-1\right)^2}{T-L}\\
\mathbf{1}'V^{-1}E\left[\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\left(\hat{V}-V\right)\right]V^{-1}\mathbf{1} &= \frac{2a^2}{T-L}\\
\mu_t'V^{-1}E\left[\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\left(\hat{V}-V\right)\right]V^{-1}\mathbf{1} &= \frac{2ab_t}{T-L}\\
\mu_t'V^{-1}E\left[\left(\hat{V}-V\right)V^{-1}\mu_t\mu_t'V^{-1}\left(\hat{V}-V\right)\right]V^{-1}\mathbf{1} &= \frac{2b_t\left(c_{tt}-1\right)}{T-L}\\
\mathbf{1}'V^{-1}E\left[\left(\hat{V}-V\right)V^{-1}\mu_k\mu_t'V^{-1}\left(\hat{V}-V\right)\right]V^{-1}\mathbf{1} &= \frac{a\left(c_{kt}-1\right)+b_kb_t}{T-L}\\
\mu_t'V^{-1}E\left[\left(\hat{V}-V\right)V^{-1}\mu_t\mathbf{1}'V^{-1}\left(\hat{V}-V\right)\right]V^{-1}\mathbf{1} &= \frac{2b_t^2}{T-L}\\
\mu_t'V^{-1}E\left[\left(\hat{V}-V\right)V^{-1}\mu_t\mu_k'V^{-1}\left(\hat{V}-V\right)\right]V^{-1}\mu_k &= \frac{2\left(c_{tk}-1\right)^2}{T-L}
\end{aligned} \tag{A.112}$$

where many others of the form $E\left[u_1'V^{-1}\left(\hat{V}-V\right)V^{-1}u_2u_3'V^{-1}\left(\hat{V}-V\right)V^{-1}u_4\right]$ may be transformed (commutativity of scalar multiplication, transposing a scalar) into one from the above list. Those having four repeated instances with each $u_i \in \{\mu_t, \mathbf{1}\}$ are covered in the first two in the above list. Those having exactly three repeated instances with each $u_i \in \{\mu_t, \mathbf{1}\}$ along with one of the other type are covered in the next two in the list. Those having two of one type and two of the other with each $u_i \in \{\mu_k, \mu_t, \mathbf{1}\}$ are covered by the fifth and sixth in the list.

Proof: While Haff (1979) Theorem 3.1(iii) can be used to prove this in the case of a positive semidefinite $Q$, we need the more general result for cases such as $Q = V^{-1}\mu_t \mathbf{1}' V^{-1} = \left(V^{-1}\mu_t\right)\left(V^{-1}\mathbf{1}\right)'$ with $\mu_t$ and $\mathbf{1}$ noncollinear, for which $Q$ is not positive semidefinite because when $Q = xy'$ with noncollinear $x$ and $y$, we can find a vector $u$ such that $u'Qu = u'xy'u < 0$, for example by choosing $u = x + a\left(y - \frac{x'y}{x'x}x\right)$ with scalar $a$, for which

$$\begin{aligned} u'Qu = u'xy'u &= \left[x + a\left(y - \frac{x'y}{x'x}x\right)\right]' xy' \left[x + a\left(y - \frac{x'y}{x'x}x\right)\right] \\ &= x'xy'x + ax'x\left(y'y - \frac{x'y}{x'x}y'x\right) + a\left(y'x - \frac{x'y}{x'x}x'x\right)y'x + a^2\left(y - \frac{x'y}{x'x}x\right)' xy' \left(y - \frac{x'y}{x'x}x\right) \\ &= x'xy'x + a\left[\|x\|^2\|y\|^2 - (x'y)^2\right] \end{aligned} \tag{A.113}$$

in which the coefficient of $a$ is positive and cannot be zero because $x$ and $y$ are assumed to be noncollinear (by the Cauchy-Schwartz Inequality). Choosing a negative $a$ with large magnitude shows that such a $Q$ is not positive semidefinite.

We next prove a related result when $V = I$:

$$E(YQY) = p(p+1)Q + p\left[tr(Q)\right]I - p(Q - Q') \tag{A.114}$$

for an $n \times n$ matrix $Y$ with the Wishart distribution $W(n, p, I)$, and $Q$ an arbitrary $n \times n$ matrix of constants. We may represent $Y$ using $X_{it} \overset{iid}{\sim} N(0,1)$ for $i = 1...n$ and $t = 1,..., p$, defining $Y_{ij} = \sum_{t=1}^{p} X_{it}X_{jt}$, for which the matrix $Y$ has a Wishart distribution $W(n, p, I)$. Note that

$$\left[E(YQY)\right]_{ij} = E\left(\sum_{k,l=1}^{n} Y_{ik}Q_{kl}Y_{lj}\right) = \sum_{k,l=1}^{n} Q_{kl}E\left(Y_{ik}Y_{lj}\right) \tag{A.115}$$

so we need $E\left(Y_{ik}Y_{lj}\right)$. There are four cases:

- All four subscripts identical so that $i = j = k = l$: In this case, $Y_{ii} = \sum_{t=1}^{p} X_{it}^2$ and

$$E\left(Y_{ii}^2\right) = E\left[\left(\sum_{t=1}^{p} X_{it}^2\right)\left(\sum_{u=1}^{p} X_{iu}^2\right)\right] = \sum_{u,t=1}^{p} E\left(X_{it}^2 X_{iu}^2\right) \tag{A.116}$$

  When $t \neq u$, $X_{it}^2$ and $X_{iu}^2$ are independent and we have

$$E\left(Y_{ii}^2\right) = \sum_{u,t=1}^{p} E\left(X_{it}^2 X_{iu}^2\right) = pE\left(X_{1t}^4\right) + p(p-1)E\left(X_{1t}^2 X_{2t}^2\right) = 3p + p(p-1) = p(p+2) \tag{A.117}$$

- Exactly three identical, for example, $i = j = k$ with $l \neq i$

$$\begin{aligned} E\left(Y_{ik}Y_{lj}\right) = E\left(Y_{ii}Y_{li}\right) &= E\left[\left(\sum_{t=1}^{p} X_{it}^2\right)\left(\sum_{t=1}^{p} X_{lt}X_{it}\right)\right] = E\left(\sum_{t,u=1}^{p} X_{it}^2 X_{lu} X_{iu}\right) \\ &= \sum_{t,u=1}^{p} E\left(X_{it}^2 X_{lu} X_{iu}\right) = 0 \end{aligned} \tag{A.118}$$

  because $X_{iu}$ is independent of both $X_{it}$ and $X_{lu}$

- Exactly two identical pairs of the form $E\left(Y_{ik}Y_{ki}\right) = E\left(Y_{ik}^2\right)$ with $i \neq k$. We find

$$E\left(Y_{ik}Y_{ki}\right) = E\left[\left(\sum_{t=1}^{p} X_{it}X_{kt}\right)\left(\sum_{t=1}^{p} X_{kt}X_{it}\right)\right] = \sum_{t,u=1}^{p} E\left(X_{it}X_{kt}X_{ku}X_{iu}\right) \tag{A.119}$$

  When $t \neq u$ we have $\sum_{t,u=1}^{p} E\left(X_{it}X_{kt}\right)E\left(X_{ku}X_{iu}\right) = 0$ by independence.

  We therefore have

$$E\left(Y_{ik}Y_{ki}\right) = E\left(Y_{ik}^2\right) = \sum_{t=1}^{p} E\left(X_{it}X_{kt}X_{kt}X_{it}\right) = \sum_{t=1}^{p} E\left(X_{it}^2 X_{kt}^2\right) = p \tag{A.120}$$

- Exactly two identical pairs of the form $E\left(Y_{ii}Y_{jj}\right)$ with $i \neq j$, for which

$$E\left(Y_{ii}Y_{jj}\right) = E\left[\left(\sum_{t=1}^{p} X_{it}^2\right)\left(\sum_{t=1}^{p} X_{jt}^2\right)\right] = \left[E\left(\sum_{t=1}^{p} X_{it}^2\right)\right]\left[E\left[\sum_{t=1}^{p} X_{jt}^2\right]\right] = p^2 \tag{A.121}$$

- All other cases have expectation zero by independence of any single, non-repeated, subscript term.

There are two cases for evaluating $\left[E(YQY)\right]_{ij} = \sum_{k,l=1}^{n} Q_{kl} E\left(Y_{ik} Y_{lj}\right)$

- If $i = j$ then

$$\begin{aligned}\left[E(YQY)\right]_{ii} &= \sum_{k,l=1}^{n} Q_{kl} E\left(Y_{ik} Y_{li}\right) = \sum_{k=1}^{n} Q_{kk} E\left(Y_{ik} Y_{ki}\right) + \sum_{k \neq l} Q_{kl} E\left(Y_{ik} Y_{li}\right) \\ &= \sum_{k \neq i} Q_{kk} E\left(Y_{ik} Y_{ki}\right) + Q_{ii} E\left(Y_{ii}^{2}\right) + \sum_{k \neq l} Q_{kl} E\left(Y_{ik} Y_{li}\right) \\ &= p \sum_{k \neq i} Q_{kk} + p(p+2) Q_{ii} + 0 = p \sum_{k \neq i} Q_{kk} + p Q_{ii} + p(p+1) Q_{ii} \\ &= p\left[tr(Q)\right] + p(p+1) Q_{ii}\end{aligned} \quad \text{(A.122)}$$

- If $i \neq j$ then the only nonzero expectations $E\left(Y_{ik} Y_{lj}\right)$ occur either when $k = i$ and $l = j$, or when $k = j$ and $l = i$. We find

$$\begin{aligned}\left[E(YQY)\right]_{ij} &= \sum_{k,l=1}^{n} Q_{kl} E\left(Y_{ik} Y_{lj}\right) = Q_{ij} E\left(Y_{ii} Y_{jj}\right) + Q_{ji} E\left(Y_{ij}^{2}\right) \\ &= Q_{ij} E\left(Y_{ii} Y_{jj}\right) + Q_{ji} E\left(Y_{ij}^{2}\right) = p^{2} Q_{ij} + p Q_{ji}\end{aligned} \quad \text{(A.123)}$$

and it will be helpful to have an alternative representation:

$$\left[E(YQY)\right]_{ij} = p^{2} Q_{ij} + p Q_{ji} = p(p+1) Q_{ij} - p Q_{ij} + p Q_{ji} = p(p+1) Q_{ij} - p\left(Q_{ij} - Q_{ji}\right) \quad \text{(A.124)}$$

Putting these together, we find

$$E(YQY) = p(p+1) Q + p\left[tr(Q)\right] I - p\left(Q - Q'\right) \quad \text{(A.125)}$$

for $Y \sim W(n, p, I)$ with general $Q$.

Now we consider general $V$ and note that $(T-L)\hat{V} \sim W(n, T-L, V)$. Choose $n \times n$ matrix $A$ using the Cholesky Decomposition such that $AVA' = I$, $V = A^{-1} A'^{-1}$ and $V^{-1} = A'A$. Then

$\left(T-L\right)A\hat{V}A' \sim W\left(n,T-L,AVA'\right)=W\left(n,T-L,I\right)$ and $\left(T-L\right)A\hat{V}A' \overset{d}{=} Y$ with $p=T-L$ and $Y \sim W\left(n,p,I\right)$. We then have

$$
\begin{aligned}
E\left(\hat{V}Q\hat{V}\right) &= \frac{1}{\left(T-L\right)^2}E\left\{A^{-1}\left[\left(T-L\right)A\hat{V}A'\left(A'^{-1}QA^{-1}\right)\left(T-L\right)A\hat{V}A'\right]A'^{-1}\right\} \\
&= \frac{1}{\left(T-L\right)^2}E\left\{A^{-1}\left[Y\left(A'^{-1}QA^{-1}\right)Y\right]A'^{-1}\right\} \\
&= \frac{1}{\left(T-L\right)^2}A^{-1}\left\{\left(T-L\right)\left(T-L+1\right)A'^{-1}QA^{-1}+\left(T-L\right)\left[tr\left(A'^{-1}QA^{-1}\right)\right]I-\left(T-L\right)\left[A'^{-1}QA^{-1}-\left(A'^{-1}QA^{-1}\right)'\right]\right\}A'^{-1} \\
&= \frac{1}{\left(T-L\right)}\left[\left(T-L+1\right)A^{-1}A'^{-1}QA^{-1}A'^{-1}+tr\left(QA^{-1}A'^{-1}\right)A^{-1}A'^{-1}-\left(A^{-1}A'^{-1}QA^{-1}A'^{-1}-A^{-1}A'^{-1}Q'A^{-1}A'^{-1}\right)\right] \\
&= \frac{1}{\left(T-L\right)}\left[\left(T-L+1\right)VQV+tr\left(QV\right)V-V\left(Q-Q'\right)V\right]
\end{aligned}
$$

(A.126)

and therefore

$$
\begin{aligned}
E\left[\left(\hat{V}-V\right)Q\left(\hat{V}-V\right)\right] &= E\left(\hat{V}Q\hat{V}\right)-VQV \\
&= \frac{1}{\left(T-L\right)}\left[\left(T-L+1\right)VQV+tr\left(QV\right)V-V\left(Q-Q'\right)V\right]-VQV \\
&= VQV+\frac{1}{\left(T-L\right)}VQV+\frac{1}{\left(T-L\right)}tr\left(QV\right)V-\frac{1}{\left(T-L\right)}V\left(Q-Q'\right)V-VQV \\
&= \frac{VQV+V\left[tr\left(QV\right)\right]-V\left(Q-Q'\right)V}{\left(T-L\right)}=\frac{VQ'V+V\left[tr\left(QV\right)\right]}{\left(T-L\right)}
\end{aligned}
$$

(A.127)

completing the proof of the first result. The special cases then follow immediately from this general result. For example,

$$E\left[u_1'V^{-1}\left(\hat{V}-V\right)V^{-1}u_2u_3'V^{-1}\left(\hat{V}-V\right)V^{-1}u_4\right]=u_1'V^{-1}E\left[\left(\hat{V}-V\right)V^{-1}u_2u_3'V^{-1}\left(\hat{V}-V\right)\right]V^{-1}u_4$$
$$=u_1'V^{-1}\frac{V\left[V^{-1}u_2u_3'V^{-1}\right]'V+V\left[tr\left(VV^{-1}u_2u_3'V^{-1}\right)\right]}{T-L}V^{-1}u_4$$
$$=\frac{u_1'\left[V^{-1}u_2u_3'V^{-1}\right]'u_4+u_1'V^{-1}u_4\left[tr\left(u_2u_3'V^{-1}\right)\right]}{T-L} \tag{A.128}$$
$$=\frac{u_1'V^{-1}u_3u_2'V^{-1}u_4+u_1'V^{-1}u_4u_2'V^{-1}u_3}{T-L}$$

completing the proof. □

<u>Lemma 7</u>: Rewriting the portfolio constants using the definitions above we find their consistent estimators as:

$$\hat{\alpha}_1=\frac{1}{T}\sum_{t=1}^{T}\frac{1+\hat{\mu}_t'\hat{V}^{-1}\hat{\mu}_t}{\mathbf{1}'\hat{V}^{-1}\mathbf{1}\left(1+\hat{\mu}_t'\hat{V}^{-1}\hat{\mu}_t\right)-\left(\mathbf{1}'\hat{V}^{-1}\hat{\mu}_t\right)^2}=\frac{1}{T}\sum_{t=1}^{T}\frac{\hat{c}_{tt}}{\hat{d}_{tt}}$$

$$\hat{\alpha}_2=\frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{1}'\hat{V}^{-1}\hat{\mu}_t}{\mathbf{1}'\hat{V}^{-1}\mathbf{1}\left(1+\hat{\mu}_t'\hat{V}^{-1}\hat{\mu}_t\right)-\left(\mathbf{1}'\hat{V}^{-1}\hat{\mu}_t\right)^2}=\frac{1}{T}\sum_{t=1}^{T}\frac{\hat{b}_t}{\hat{d}_{tt}} \tag{A.129}$$

$$\hat{\alpha}_3=\frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{1}'\hat{V}^{-1}\mathbf{1}}{\mathbf{1}'\hat{V}^{-1}\mathbf{1}\left(1+\hat{\mu}_t'\hat{V}^{-1}\hat{\mu}_t\right)-\left(\mathbf{1}'\hat{V}^{-1}\hat{\mu}_t\right)^2}=\frac{\hat{a}}{T}\sum_{t=1}^{T}\frac{1}{\hat{d}_{tt}}$$

<u>Proof</u>: Using Lemma 2, we find

$$\Lambda=V^{-1}-\frac{V^{-1}\mu\mu'V^{-1}}{1+\mu'V^{-1}\mu} \tag{A.130}$$

from which we find expressions for the portfolio constants in terms of $\mu$ and $V$

$$\alpha_1 = E\left(\frac{1}{\mathbf{1}'\Lambda\mathbf{1}}\right) = E\left(\frac{1}{\mathbf{1}'\left(V^{-1} - \frac{V^{-1}\mu\mu'V^{-1}}{1+\mu'V^{-1}\mu}\right)\mathbf{1}}\right) = E\left(\frac{1}{\mathbf{1}'V^{-1}\mathbf{1} - \frac{\mathbf{1}'V^{-1}\mu\mu'V^{-1}\mathbf{1}}{1+\mu'V^{-1}\mu}}\right)$$

$$= E\left(\frac{1+\mu'V^{-1}\mu}{\mathbf{1}'V^{-1}\mathbf{1}\left(1+\mu'V^{-1}\mu\right) - \left(\mathbf{1}'V^{-1}\mu\right)^2}\right)$$

$$\alpha_2 = E\left(\frac{\mathbf{1}'\Lambda\mu}{\mathbf{1}'\Lambda\mathbf{1}}\right) = E\left(\frac{\mathbf{1}'V^{-1}\mu - \frac{\mathbf{1}'V^{-1}\mu\mu'V^{-1}\mu}{1+\mu'V^{-1}\mu}}{\mathbf{1}'V^{-1}\mathbf{1} - \frac{\mathbf{1}'V^{-1}\mu\mu'V^{-1}\mathbf{1}}{1+\mu'V^{-1}\mu}}\right) = E\left(\frac{\frac{\mathbf{1}'V^{-1}\mu\left(1+\mu'V^{-1}\mu\right) - \mathbf{1}'V^{-1}\mu\mu'V^{-1}\mu}{1+\mu'V^{-1}\mu}}{\frac{\mathbf{1}'V^{-1}\mathbf{1}\left(1+\mu'V^{-1}\mu\right) - \mathbf{1}'V^{-1}\mu\mu'V^{-1}\mathbf{1}}{1+\mu'V^{-1}\mu}}\right)$$

$$= E\left(\frac{\mathbf{1}'V^{-1}\mu\left(1+\mu'V^{-1}\mu\right) - \mathbf{1}'V^{-1}\mu\mu'V^{-1}\mu}{\mathbf{1}'V^{-1}\mathbf{1}\left(1+\mu'V^{-1}\mu\right) - \mathbf{1}'V^{-1}\mu\mu'V^{-1}\mathbf{1}}\right) = E\left(\frac{\mathbf{1}'V^{-1}\mu}{\mathbf{1}'V^{-1}\mathbf{1}\left(1+\mu'V^{-1}\mu\right) - \left(\mathbf{1}'V^{-1}\mu\right)^2}\right)$$

$$
\begin{aligned}
\alpha_3 &= 1 - E\left[\mu'\left(\Lambda - \frac{\Lambda\mathbf{1}\mathbf{1}'\Lambda}{\mathbf{1}'\Lambda\mathbf{1}}\right)\mu\right] = 1 - E\left(\mu'\Lambda\mu - \frac{\mu'\Lambda\mathbf{1}\mathbf{1}'\Lambda\mu}{\mathbf{1}'\Lambda\mathbf{1}}\right) \\
&= 1 - E\left(\mu'V^{-1}\mu - \frac{\mu'V^{-1}\mu\mu'V^{-1}\mu}{1+\mu'V^{-1}\mu} - \frac{\left(\mathbf{1}'V^{-1}\mu - \frac{\mathbf{1}'V^{-1}\mu\mu'V^{-1}\mu}{1+\mu'V^{-1}\mu}\right)^2}{\mathbf{1}'V^{-1}\mathbf{1} - \frac{\mathbf{1}'V^{-1}\mu\mu'V^{-1}\mathbf{1}}{1+\mu'V^{-1}\mu}}\right) \\
&= 1 - E\left(\frac{\mu'V^{-1}\mu}{1+\mu'V^{-1}\mu} - \frac{\left(\frac{\mathbf{1}'V^{-1}\mu}{1+\mu'V^{-1}\mu}\right)^2}{\frac{\mathbf{1}'V^{-1}\mathbf{1}\left(1+\mu'V^{-1}\mu\right)-\left(\mathbf{1}'V^{-1}\mu\right)^2}{1+\mu'V^{-1}\mu}}\right) \\
&= 1 - E\left(1 - \frac{1}{1+\mu'V^{-1}\mu} - \left(\frac{1}{1+\mu'V^{-1}\mu}\right)\frac{\left(\mathbf{1}'V^{-1}\mu\right)^2}{\mathbf{1}'V^{-1}\mathbf{1}\left(1+\mu'V^{-1}\mu\right)-\left(\mathbf{1}'V^{-1}\mu\right)^2}\right) \\
&= E\left[\left(\frac{1}{1+\mu'V^{-1}\mu}\right)\frac{\mathbf{1}'V^{-1}\mathbf{1}\left(1+\mu'V^{-1}\mu\right)-\left(\mathbf{1}'V^{-1}\mu\right)^2+\left(\mathbf{1}'V^{-1}\mu\right)^2}{\mathbf{1}'V^{-1}\mathbf{1}\left(1+\mu'V^{-1}\mu\right)-\left(\mathbf{1}'V^{-1}\mu\right)^2}\right] \\
&= E\left(\left(\frac{1}{1+\mu'V^{-1}\mu}\right)\frac{\mathbf{1}'V^{-1}\mathbf{1}\left(1+\mu'V^{-1}\mu\right)}{\mathbf{1}'V^{-1}\mathbf{1}\left(1+\mu'V^{-1}\mu\right)-\left(\mathbf{1}'V^{-1}\mu\right)^2}\right) \\
&= E\left(\frac{\mathbf{1}'V^{-1}\mathbf{1}}{\mathbf{1}'V^{-1}\mathbf{1}\left(1+\mu'V^{-1}\mu\right)-\left(\mathbf{1}'V^{-1}\mu\right)^2}\right)
\end{aligned}
\tag{A.131}
$$

The estimates make use of the estimated values $\hat{V}$ and $\hat{\mu}_t$. □

Lemma 8: Second-order expansions may be found as follows, omitting terms involving both $\hat{\mu}-\mu$ and $\hat{V}-V$ because they are independent of one another and will have expectation zero. Keeping only relevant terms up to second order, we find :

$$\hat{a} - a \cong -\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1} + \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}$$

$$\left(\hat{a}-a\right)^2 \cong \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}$$

$$\hat{b}_t - b_t \cong -\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + \mathbf{1}'V^{-1}\left(\hat{\mu}_t - \mu_t\right) + \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t$$

$$\left(\hat{b}_k - b_k\right)\left(\hat{b}_t - b_t\right) \cong \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + \left(\hat{\mu}_k - \mu_k\right)'V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\left(\hat{\mu}_t - \mu_t\right)$$

$$\left(\hat{a}-a\right)\left(\hat{b}_t - b_t\right) \cong \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t$$

$$\begin{aligned}\hat{c}_{tt} - c_{tt} &\cong -\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + 2\mu_t'V^{-1}\left(\hat{\mu}_t - \mu_t\right) \\ &\quad + \mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + \left(\hat{\mu}_t - \mu_t\right)'V^{-1}\left(\hat{\mu}_t - \mu_t\right)\end{aligned}$$

$$\begin{aligned}&\left(\hat{c}_{kk} - c_{kk}\right)\left(\hat{c}_{tt} - c_{tt}\right) \\ &\quad \cong \mu_k'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + 4\left(\hat{\mu}_k - \mu_k\right)'V^{-1}\mu_k\mu_t'V^{-1}\left(\hat{\mu}_t - \mu_t\right)\end{aligned}$$

$$\left(\hat{a}-a\right)\left(\hat{c}_{tt} - c_{tt}\right) \cong \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t$$

$$\left(\hat{b}_k - b_k\right)\left(\hat{c}_{tt} - c_{tt}\right) \cong \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + 2\left(\hat{\mu}_k - \mu_k\right)'V^{-1}\mathbf{1}\mu_t'V^{-1}\left(\hat{\mu}_t - \mu_t\right)$$

$$\begin{aligned}&\hat{d}_{tt} - d_{tt} \cong -a\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + \left(-c_{tt}\mathbf{1} + 2b_t\mu_t\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1} \\ &+2\left(a\mu_t - b_t\mathbf{1}\right)'V^{-1}\left(\hat{\mu}_t - \mu_t\right) + a\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t \\ &+\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\left(c_{tt}\mathbf{1} - 2b_t\mu_t\right) \\ &+\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\mathbf{1}\mu_t' - \mu_t\mathbf{1}'\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + \left(\hat{\mu}_t - \mu_t\right)'\left(aV^{-1} - V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\right)\left(\hat{\mu}_t - \mu_t\right)\end{aligned}$$

$$\begin{aligned}&\left(\hat{d}_{kk} - d_{kk}\right)\left(\hat{d}_{tt} - d_{tt}\right) \cong a^2\mu_k'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t \\ &\quad - a\mu_k'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\left(-c_{tt}\mathbf{1} + 2b_t\mu_t\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1} \\ &\quad - a\left(-c_{kk}\mathbf{1} + 2b_k\mu_k\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t \\ &\quad + \left(-c_{kk}\mathbf{1} + 2b_k\mu_k\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\left(-c_{tt}\mathbf{1} + 2b_t\mu_t\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1} \\ &\quad + 4\left(\hat{\mu}_k - \mu_k\right)'V^{-1}\left(a\mu_k - b_k\mathbf{1}\right)\left(a\mu_t - b_t\mathbf{1}\right)'V^{-1}\left(\hat{\mu}_t - \mu_t\right)\end{aligned}$$

$$\left(\hat{d}_{tt}-d_{tt}\right)^2 \cong a^2\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t$$
$$-a\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}$$
$$-a\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t$$
$$+\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}$$
$$+4\left(\hat{\mu}_t-\mu_t\right)' V^{-1}\left(a\mu_t-b_t\mathbf{1}\right)\left(a\mu_t'-b_t\mathbf{1}'\right)V^{-1}\left(\hat{\mu}_t-\mu_t\right)$$

$$\left(\hat{a}-a\right)\left(\hat{d}_{tt}-d_{tt}\right)$$
$$\cong a\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t+\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\left(c_{tt}\mathbf{1}-2b_t\mu_t\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}$$

$$\left(\hat{b}_k-b_k\right)\left(\hat{d}_{tt}-d_{tt}\right)\cong a\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t$$
$$-\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}$$
$$+2\left(\hat{\mu}_k-\mu_k\right)' V^{-1}\mathbf{1}\left(a\mu_t-b_t\mathbf{1}\right)' V^{-1}\left(\hat{\mu}_t-\mu_t\right)$$

$$\left(\hat{b}_t-b_t\right)\left(\hat{d}_{tt}-d_{tt}\right)\cong a\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t$$
$$-\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}$$
$$+2\left(\hat{\mu}_t-\mu_t\right)' V^{-1}\mathbf{1}\left(a\mu_t-b_t\mathbf{1}\right)' V^{-1}\left(\hat{\mu}_t-\mu_t\right)$$

$$\left(\hat{c}_{kk}-c_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\cong a\mu_k' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t$$
$$-\mu_k' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}$$
$$+4\left(\hat{\mu}_k-\mu_k\right)' V^{-1}\mu_k\left(a\mu_t-b_t\mathbf{1}\right)' V^{-1}\left(\hat{\mu}_t-\mu_t\right)$$

$$\left(\hat{c}_{tt}-c_{tt}\right)\left(\hat{d}_{tt}-d_{tt}\right)\cong a\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t$$
$$-\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1} \tag{A.132}$$
$$+4\left(\hat{\mu}_t-\mu_t\right)' V^{-1}\mu_t\left(a\mu_t-b_t\mathbf{1}\right)' V^{-1}\left(\hat{\mu}_t-\mu_t\right)$$

Proof: Using the expansion $\hat{V}^{-1} \cong V^{-1} - V^{-1}\left(\hat{V}-V\right)V^{-1} + V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1}$ from Lemma 4, we find

$$\begin{aligned}\hat{a}-a &= \mathbf{1}'\hat{V}^{-1}\mathbf{1} - \mathbf{1}'V^{-1}\mathbf{1} \cong \mathbf{1}'\left[V^{-1} - V^{-1}\left(\hat{V}-V\right)V^{-1} + V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\right]\mathbf{1} - \mathbf{1}'V^{-1}\mathbf{1} \\ &= -\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1} + \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\end{aligned}$$

$$\left(\hat{a}-a\right)^2 \cong \left[-\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right]^2 = \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}$$

$$\begin{aligned}\hat{b}_t - b_t &= \mathbf{1}'\hat{V}^{-1}\hat{\mu}_t - \mathbf{1}'V^{-1}\mu_t \cong \mathbf{1}'\left(\hat{V}^{-1} - V^{-1}\right)\mu_t + \mathbf{1}'V^{-1}\left(\hat{\mu}_t - \mu_t\right) \\ &\cong \mathbf{1}'\left[-V^{-1}\left(\hat{V}-V\right)V^{-1} + V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\right]\mu_t + \mathbf{1}'V^{-1}\left(\hat{\mu}_t - \mu_t\right) \\ &= -\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + \mathbf{1}'V^{-1}\left(\hat{\mu}_t - \mu_t\right) + \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\end{aligned}$$

$$\begin{aligned}&\left(\hat{b}_k - b_k\right)\left(\hat{b}_t - b_t\right) \\ &\cong \left[-\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k + \mathbf{1}'V^{-1}\left(\hat{\mu}_k - \mu_k\right)\right]\left[-\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + \mathbf{1}'V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right] \\ &\cong \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + \left(\hat{\mu}_k - \mu_k\right)'V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\left(\hat{\mu}_t - \mu_t\right)\end{aligned}$$

$$\begin{aligned}\left(\hat{a}-a\right)\left(\hat{b}_t - b_t\right) &\cong \left[-\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right]\left[-\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right] \\ &= \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\end{aligned}$$

$$\begin{aligned}\hat{c}_{tt} - c_{tt} &= \hat{\mu}_t'\hat{V}^{-1}\hat{\mu}_t - \mu_t'V^{-1}\mu_t \\ &\cong \mu_t'\left(\hat{V}^{-1} - V^{-1}\right)\mu_t + 2\mu_t'V^{-1}\left(\hat{\mu}_t - \mu_t\right) + \left(\hat{\mu}_t - \mu_t\right)'V^{-1}\left(\hat{\mu}_t - \mu_t\right) \\ &\cong \mu_t'\left[-V^{-1}\left(\hat{V}-V\right)V^{-1} + V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\right]\mu_t \\ &\quad +2\mu_t'V^{-1}\left(\hat{\mu}_t - \mu_t\right) + \left(\hat{\mu}_t - \mu_t\right)'V^{-1}\left(\hat{\mu}_t - \mu_t\right) \\ &= -\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + 2\mu_t'V^{-1}\left(\hat{\mu}_t - \mu_t\right) \\ &\quad +\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t + \left(\hat{\mu}_t - \mu_t\right)'V^{-1}\left(\hat{\mu}_t - \mu_t\right)\end{aligned}$$

$$
\begin{aligned}
&\left(\hat{c}_{kk}-c_{kk}\right)\left(\hat{c}_{tt}-c_{tt}\right) \\
&\quad\cong\left[-\mu_k' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k+2\mu_k' V^{-1}\left(\hat{\mu}_k-\mu_k\right)\right]\left[-\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t+2\mu_t' V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right] \\
&\quad\cong\mu_k' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t+4\left(\hat{\mu}_k-\mu_k\right)' V^{-1}\mu_k\mu_t' V^{-1}\left(\hat{\mu}_t-\mu_t\right)
\end{aligned}
$$

$$
\begin{aligned}
\left(\hat{a}-a\right)\left(\hat{c}_{tt}-c_{tt}\right)&\cong\left[-\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right]\left[-\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right] \\
&=\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t
\end{aligned}
$$

$$
\begin{aligned}
&\left(\hat{b}_k-b_k\right)\left(\hat{c}_{tt}-c_{tt}\right) \\
&\cong\left[-\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k+\mathbf{1}' V^{-1}\left(\hat{\mu}_k-\mu_k\right)\right]\left[-\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t+2\mu_t' V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right] \\
&\cong\left[-\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\right]\left[-\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right]+\left[\mathbf{1}' V^{-1}\left(\hat{\mu}_k-\mu_k\right)\right]\left[2\mu_t' V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right] \\
&=\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t+2\left(\hat{\mu}_k-\mu_k\right)' V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{\mu}_t-\mu_t\right)
\end{aligned}
$$

$$
\begin{aligned}
&\hat{d}_{tt} - d_{tt} = a\left(\hat{c}_{tt} - c_{tt}\right) + c_{tt}\left(\hat{a} - a\right) + \left(\hat{a} - a\right)\left(\hat{c}_{tt} - c_{tt}\right) - 2b_t\left(\hat{b}_t - b_t\right) - \left(\hat{b}_t - b_t\right)^2 \\
&\cong a\left[-\mu_t' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + 2\mu_t' V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right] \\
&+a\left[\mu_t' V^{-1}\left(\hat{V} - V\right)V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + \left(\hat{\mu}_t - \mu_t\right)' V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right] \\
&+c_{tt}\left[-\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1} + \mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1}\right] \\
&+\left[-\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1}\right]\left[-\mu_t' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + 2\mu_t' V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right] \\
&-2b_t\left[-\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + \mathbf{1}' V^{-1}\left(\hat{\mu}_t - \mu_t\right) + \mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t\right] \\
&-\left[-\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + \mathbf{1}' V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right]^2 \\
&\cong -a\mu_t' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + 2a\mu_t' V^{-1}\left(\hat{\mu}_t - \mu_t\right) \\
&+a\mu_t' V^{-1}\left(\hat{V} - V\right)V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + a\left(\hat{\mu}_t - \mu_t\right)' V^{-1}\left(\hat{\mu}_t - \mu_t\right) \\
&-c_{tt}\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1} + c_{tt}\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1} \\
&+\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t \\
&+2b_t\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t - 2b_t\mathbf{1}' V^{-1}\left(\hat{\mu}_t - \mu_t\right) - 2b_t\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t \\
&-\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t - \left(\hat{\mu}_t - \mu_t\right)' V^{-1}\mathbf{1}\mathbf{1}' V^{-1}\left(\hat{\mu}_t - \mu_t\right)
\end{aligned}
\tag{A.133}
$$

Continuing, we find

$$
\begin{aligned}
&\hat{d}_{tt} - d_{tt} \\
&\cong -a\mu_t' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t - c_{tt}\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1} + 2b_t\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t \\
&+2a\mu_t' V^{-1}\left(\hat{\mu}_t - \mu_t\right) - 2b_t\mathbf{1}' V^{-1}\left(\hat{\mu}_t - \mu_t\right) + a\mu_t' V^{-1}\left(\hat{V} - V\right)V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t \\
&+c_{tt}\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1} + \mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t \\
&-2b_t\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t - \mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t \\
&+a\left(\hat{\mu}_t - \mu_t\right)' V^{-1}\left(\hat{\mu}_t - \mu_t\right) - \left(\hat{\mu}_t - \mu_t\right)' V^{-1}\mathbf{1}\mathbf{1}' V^{-1}\left(\hat{\mu}_t - \mu_t\right) \\
&= -a\mu_t' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + \left(-c_{tt}\mathbf{1} + 2b_t\mu_t\right)' V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1} + 2\left(a\mu_t - b_t\mathbf{1}\right)' V^{-1}\left(\hat{\mu}_t - \mu_t\right) \\
&+a\mu_t' V^{-1}\left(\hat{V} - V\right)V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + \mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\left(\hat{V} - V\right)V^{-1}\left(c_{tt}\mathbf{1} - 2b_t\mu_t\right) \\
&+\mathbf{1}' V^{-1}\left(\hat{V} - V\right)V^{-1}\left(\mathbf{1}\mu_t' - \mu_t\mathbf{1}'\right)V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + \left(\hat{\mu}_t - \mu_t\right)'\left(aV^{-1} - V^{-1}\mathbf{1}\mathbf{1}' V^{-1}\right)\left(\hat{\mu}_t - \mu_t\right)
\end{aligned}
$$

$$\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)$$
$$\cong\left[-a\mu_k'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k+\left(-c_{kk}\mathbf{1}+2b_k\mu_k\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}+2\left(a\mu_k'-b_k\mathbf{1}'\right)V^{-1}\left(\hat{\mu}_k-\mu_k\right)\right]$$
$$\left[-a\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t+\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}+2\left(a\mu_t'-b_t\mathbf{1}'\right)V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right]$$
$$\cong\left[-a\mu_k'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k+\left(-c_{kk}\mathbf{1}+2b_k\mu_k\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right]$$
$$\left[-a\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t+\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right]$$
$$+\left[2\left(a\mu_k'-b_k\mathbf{1}'\right)V^{-1}\left(\hat{\mu}_k-\mu_k\right)\right]\left[2\left(a\mu_t'-b_t\mathbf{1}'\right)V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right]$$
$$=a^2\mu_k'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t-a\mu_k'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}$$
$$-a\left(-c_{kk}\mathbf{1}+2b_k\mu_k\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t$$
$$+\left(-c_{kk}\mathbf{1}+2b_k\mu_k\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}$$
$$+4\left(\hat{\mu}_k-\mu_k\right)'V^{-1}\left(a\mu_k-b_k\mathbf{1}\right)\left(a\mu_t-b_t\mathbf{1}\right)'V^{-1}\left(\hat{\mu}_t-\mu_t\right)$$

$$\left(\hat{a}-a\right)\left(\hat{d}_{tt}-d_{tt}\right)$$
$$\cong\left[-\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right]\left[-a\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t+\left(-c_{tt}\mathbf{1}'+2b_t\mu_t'\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right]$$
$$=a\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t$$
$$+\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\left(c_{tt}\mathbf{1}-2b_t\mu_t\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}$$

$$
\begin{aligned}
&\left(\hat{b}_k - b_k\right)\left(\hat{d}_{tt} - d_{tt}\right) \\
&\cong \left[-\mathbf{1}'V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_k + \mathbf{1}'V^{-1}\left(\hat{\mu}_k - \mu_k\right)\right] \\
&\quad \left[-a\mu_t'V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + \left(-c_{tt}\mathbf{1} + 2b_t\mu_t\right)'V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1} + 2\left(a\mu_t' - b_t\mathbf{1}'\right)V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right] \\
&\cong -\mathbf{1}'V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_k\left[-a\mu_t'V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + \left(-c_{tt}\mathbf{1} + 2b_t\mu_t\right)'V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1}\right] \\
&\quad + 2\left(\hat{\mu}_k - \mu_k\right)'V^{-1}\mathbf{1}\left(a\mu_t - b_t\mathbf{1}\right)'V^{-1}\left(\hat{\mu}_t - \mu_t\right) \\
&= a\mathbf{1}'V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_k\mu_t'V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t \\
&\quad - \mathbf{1}'V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_k\left(-c_{tt}\mathbf{1} + 2b_t\mu_t\right)'V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1} \\
&\quad + 2\left(\hat{\mu}_k - \mu_k\right)'V^{-1}\mathbf{1}\left(a\mu_t - b_t\mathbf{1}\right)'V^{-1}\left(\hat{\mu}_t - \mu_t\right)
\end{aligned} \tag{A.134}
$$

The expression for $\left(\hat{b}_t - b_t\right)\left(\hat{d}_{tt} - d_{tt}\right)$ follows directly.

$$
\begin{aligned}
&\left(\hat{c}_{kk} - c_{kk}\right)\left(\hat{d}_{tt} - d_{tt}\right) \\
&\cong \left[-\mu_k'V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_k + 2\mu_k'V^{-1}\left(\hat{\mu}_k - \mu_k\right)\right] \\
&\quad \left[-a\mu_t'V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t + \left(-c_{tt}\mathbf{1} + 2b_t\mu_t\right)'V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1} + 2\left(a\mu_t' - b_t\mathbf{1}'\right)V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right] \\
&\cong a\mu_k'V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_k\mu_t'V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t \\
&\quad - \mu_k'V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_k\left(-c_{tt}\mathbf{1} + 2b_t\mu_t\right)'V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1} \\
&\quad + 4\left(\hat{\mu}_k - \mu_k\right)'V^{-1}\mu_k\left(a\mu_t - b_t\mathbf{1}\right)'V^{-1}\left(\hat{\mu}_t - \mu_t\right)
\end{aligned} \tag{A.135}
$$

The expression for $\left(\hat{c}_{tt} - c_{tt}\right)\left(\hat{d}_{tt} - d_{tt}\right)$ follows directly, completing the proof. □

Lemma 9: Expectations are as follows:

$$E\left(\hat{a} - a\right) \cong \frac{a\left(n+1\right)}{T - L}$$

$$E\left[\left(\hat{a} - a\right)^2\right] \cong \frac{2a^2}{T - L}$$

$$E\left(\hat{b}_t - b_t\right) \cong \frac{(n+1)}{T-L} b_t$$

$$E\left[\left(\hat{b}_k - b_k\right)\left(\hat{b}_t - b_t\right)\right] \cong \frac{a\left(c_{kt}-1\right)+b_k b_t}{T-L} + \frac{a}{T} Z'_{k-1} A^{-1} Z_{t-1}$$

$$E\left[\left(\hat{a} - a\right)\left(\hat{b}_t - b_t\right)\right] \cong \frac{2ab_t}{T-L}$$

$$E\left(\hat{c}_{tt} - c_{tt}\right) \cong \frac{(n+1)\left(c_{tt}-1\right)}{T-L} + \frac{n}{T} Z'_{t-1} A^{-1} Z_{t-1}$$

$$E\left[\left(\hat{c}_{kk} - c_{kk}\right)\left(\hat{c}_{tt} - c_{tt}\right)\right] \cong \frac{2\left(c_{kt}-1\right)^2}{T-L} + \frac{4\left(c_{kt}-1\right)}{T} Z'_{t-1} A^{-1} Z_{k-1}$$

$$E\left[\left(\hat{a} - a\right)\left(\hat{c}_{tt} - c_{tt}\right)\right] \cong \frac{2b_t^2}{T-L}$$

$$E\left[\left(\hat{b}_k - b_k\right)\left(\hat{c}_{tt} - c_{tt}\right)\right] \cong \frac{2b_t\left(c_{kt}-1\right)}{T-L} + \frac{2b_t}{T} Z'_{k-1} A^{-1} Z_{t-1}$$

$$E\left(\hat{d}_{tt} - d_{tt}\right) \cong \frac{-na + (2n+1) d_{tt}}{T-L} + \frac{a(n-1)}{T} Z'_{t-1} A^{-1} Z_{t-1}$$

$$E\left[\left(\hat{d}_{kk} - d_{kk}\right)\left(\hat{d}_{tt} - d_{tt}\right)\right]$$
$$\cong \frac{-2ab_k b_t c_{kt} + 2a^2 + 2a^2 c_{kk} c_{tt} - 2b_k^2 d_{tt} - 2b_t^2 d_{kk} - 4a\, d_{kt} + 2a\, c_{kt} d_{kt}}{T-L} + 4\frac{ad_{kt} - a^2}{T} Z'_{t-1} A^{-1} Z_{k-1}$$

$$E\left[\left(\hat{d}_{tt} - d_{tt}\right)^2\right] \cong \frac{2a^2 - 4a\, d_{tt} + 4d_{tt}^2}{T-L} + 4\frac{ad_{tt} - a^2}{T} Z'_{t-1} A^{-1} Z_{t-1}$$

$$\frac{E\left(\hat{d}_{tt}-d_{tt}\right)^{2}}{d_{tt}^{3}}-\frac{E\left(\hat{d}_{tt}-d_{tt}\right)}{d_{tt}^{2}}$$
$$\cong\frac{2a^{2}+(n-4)ad_{tt}+(3-2n)d_{tt}^{2}}{(T-L)d_{tt}^{3}}+\frac{-4a^{2}-(n-5)ad_{tt}}{Td_{tt}^{3}}Z'_{t-1}A^{-1}Z_{t-1}$$

$$E\left[(\hat{a}-a)\left(\hat{d}_{tt}-d_{tt}\right)\right]\cong\frac{2ad_{tt}}{T-L}$$

$$E\left[\left(\hat{b}_{k}-b_{k}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]\cong\frac{2b_{k}d_{tt}}{T-L}$$

$$E\left[\left(\hat{b}_{t}-b_{t}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]\cong\frac{2b_{t}d_{tt}}{T-L}$$

$$E\left[\left(\hat{c}_{kk}-c_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]\cong\frac{2a\left(c_{kt}-1\right)^{2}-4b_{k}b_{t}\left(c_{kt}-1\right)+2b_{k}^{2}c_{tt}}{T-L}+\frac{4\left(d_{kt}-a\right)}{T}Z'_{k-1}A^{-1}Z_{t-1}$$

$$E\left[\left(\hat{c}_{tt}-c_{tt}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]\cong\frac{2a\left(c_{tt}-1\right)^{2}-2b_{t}^{2}\left(c_{tt}-2\right)}{T-L}+\frac{4\left(d_{tt}-a\right)}{T}Z'_{t-1}A^{-1}Z_{t-1} \qquad \text{(A.136)}$$

<u>Proof</u>: Making use of unbiasedness of $\hat{\mu}_t$ and of $\hat{V}$, along with Lemmas 5 and 6, and commutativity within the trace operator, together with the expansions from Lemma 8, we find

$$E(\hat{a}-a)\cong\mathbf{1}'V^{-1}E\left[\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)\right]V^{-1}\mathbf{1}=\mathbf{1}'V^{-1}\frac{(n+1)V}{T-L}V^{-1}\mathbf{1}=\frac{a(n+1)}{T-L}$$

$$E\left[(\hat{a}-a)^{2}\right]\cong E\left[\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right]=\frac{2a^{2}}{T-L}$$

$$E\left(\hat{b}_{t}-b_{t}\right)\cong\mathbf{1}'V^{-1}E\left[\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)\right]V^{-1}\mu_{t}=\mathbf{1}'V^{-1}\frac{(n+1)V}{T-L}V^{-1}\mu_{t}=\frac{(n+1)}{T-L}b_{t}$$

$$E\left[\left(\hat{b}_k - b_k\right)\left(\hat{b}_t - b_t\right)\right]$$

$$\cong E\left[\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k \mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right] + E\left[\left(\hat{\mu}_k - \mu_k\right)' V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right]$$

$$= \frac{a\left(c_{kt}-1\right)+b_k b_t}{T-L} + \frac{a}{T} Z'_{t-1}A^{-1}Z_{k-1}$$

$$E\left[\left(\hat{a}-a\right)\left(\hat{b}_t - b_t\right)\right] \cong E\left[\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right] = \frac{2ab_t}{T-L}$$

$$E\left(\hat{c}_{tt} - c_{tt}\right) \cong \mu_t' V^{-1} E\left[\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)\right]V^{-1}\mu_t + E\left[\left(\hat{\mu}_t - \mu_t\right)' V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right]$$

$$= \mu_t' V^{-1}\frac{(n+1)V}{T-L}V^{-1}\mu_t + \frac{n}{T}Z'_{t-1}A^{-1}Z_{t-1} = \frac{(n+1)\left(c_{tt}-1\right)}{T-L} + \frac{n}{T}Z'_{t-1}A^{-1}Z_{t-1}$$

$$E\left[\left(\hat{c}_{kk}-c_{kk}\right)\left(\hat{c}_{tt}-c_{tt}\right)\right]$$

$$\cong E\left[\mu_k' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k \mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right] + 4E\left[\left(\hat{\mu}_k - \mu_k\right)' V^{-1}\mu_k \mu_t' V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right]$$

$$= \frac{2\left(c_{kt}-1\right)^2}{T-L} + 4\frac{\mu_k' V^{-1}\mu_t}{T}Z'_{t-1}A^{-1}Z_{k-1} = \frac{2\left(c_{kt}-1\right)^2}{T-L} + \frac{4\left(c_{kt}-1\right)}{T}Z'_{t-1}A^{-1}Z_{k-1}$$

$$E\left[\left(\hat{a}-a\right)\left(\hat{c}_{tt}-c_{tt}\right)\right] \cong E\left[\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right]$$

$$= E\left[\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right] = \frac{2b_t^2}{T-L}$$

$$E\left[\left(\hat{b}_k - b_k\right)\left(\hat{c}_{tt}-c_{tt}\right)\right]$$

$$\cong E\left[\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k \mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right] + 2E\left[\left(\hat{\mu}_k - \mu_k\right)' V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right]$$

$$= \frac{\mathbf{1}'V^{-1}\mu_t \mu_k' V^{-1}\mu_t + \mathbf{1}'V^{-1}\mu_t \mu_k' V^{-1}\mu_t}{T-L} + 2\frac{\mathbf{1}'V^{-1}\mu_t}{T}Z'_{k-1}A^{-1}Z_{t-1} = \frac{2b_t\left(c_{kt}-1\right)}{T-L} + \frac{2b_t}{T}Z'_{k-1}A^{-1}Z_{t-1}$$

$$
\begin{aligned}
E\left(\hat{d}_{tt}-d_{tt}\right) &\cong a\mu_t' V^{-1} E\left[\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)\right]V^{-1}\mu_t \\
&+ \mathbf{1}'V^{-1}E\left[\left(\hat{V}-V\right)V^{-1}\left(\hat{V}-V\right)\right]V^{-1}\left(c_{tt}\mathbf{1}-2b_t\mu_t\right) \\
&+ E\left[\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\left(\mathbf{1}\mu_t'-\mu_t\mathbf{1}'\right)V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right] \\
&+ E\left[\left(\hat{\mu}_t-\mu_t\right)'\left(aV^{-1}-V^{-1}\mathbf{1}\mathbf{1}'V^{-1}\right)\left(\hat{\mu}_t-\mu_t\right)\right] \\
&= a\mu_t'V^{-1}\frac{(n+1)V}{T-L}V^{-1}\mu_t + \mathbf{1}'V^{-1}\frac{(n+1)V}{T-L}V^{-1}\left(c_{tt}\mathbf{1}-2b_t\mu_t\right) \\
&+ \frac{2b_t^2}{T-L} - \frac{a\left(c_{tt}-1\right)+b_t^2}{T-L} + \frac{an}{T}Z_{t-1}'A^{-1}Z_{t-1} - \frac{\mathbf{1}'V^{-1}\mathbf{1}}{T}Z_{t-1}'A^{-1}Z_{t-1} \\
&= \frac{a\left(c_{tt}-1\right)(n+1)+\left(ac_{tt}-2b_t^2\right)(n+1)+2b_t^2-a\left(c_{tt}-1\right)-b_t^2}{T-L} + \frac{an-a}{T}Z_{t-1}'A^{-1}Z_{t-1} \\
&= \frac{\left(-a+2ac_{tt}-2b_t^2\right)(n+1)-\left(ac_{tt}-b_t^2\right)+a}{T-L} + \frac{a(n-1)}{T}Z_{t-1}'A^{-1}Z_{t-1} \\
&= \frac{\left(-a+2d_{tt}\right)(n+1)-d_{tt}+a}{T-L} + \frac{a(n-1)}{T}Z_{t-1}'A^{-1}Z_{t-1} \\
&= \frac{-na+(2n+1)d_{tt}}{T-L} + \frac{a(n-1)}{T}Z_{t-1}'A^{-1}Z_{t-1}
\end{aligned}
$$

$$
\begin{aligned}
E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right] &\cong a^2 E\left[\mu_k' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k \mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right] \\
&- aE\left[\mu_k' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right] \\
&- aE\left[\left(-c_{kk}\mathbf{1}+2b_k\mu_k\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right] \\
&+ E\left[\left(-c_{kk}\mathbf{1}+2b_k\mu_k\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right] \\
&+ 4E\left[\left(\hat{\mu}_k-\mu_k\right)' V^{-1}\left(a\mu_k-b_k\mathbf{1}\right)\left(a\mu_t-b_t\mathbf{1}\right)' V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right] \\
= a^2\frac{2\left(c_{kt}-1\right)^2}{T-L} &+ ac_{tt}E\left[\mu_k' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right] \\
&- 2ab_t E\left[\mu_k' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right] \\
&+ ac_{kk}E\left[\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right] \\
&- 2ab_k E\left[\mu_k' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right] \\
&+ c_{kk}c_{tt} E\left[\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right] \\
&- 2b_t c_{kk}E\left[\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right] \\
&- 2b_k c_{tt}E\left[\mu_k' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mathbf{1}' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right] \\
&+ 4b_k b_t E\left[\mu_k' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right] \\
&+ 4a^2 E\left[\left(\hat{\mu}_k-\mu_k\right)' V^{-1}\mu_k\mu_t' V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right] - 4ab_t E\left[\left(\hat{\mu}_k-\mu_k\right)' V^{-1}\mu_k\mathbf{1}' V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right] \\
&- 4ab_k E\left[\left(\hat{\mu}_k-\mu_k\right)' V^{-1}\mathbf{1}\mu_t' V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right] + 4b_k b_t E\left[\left(\hat{\mu}_k-\mu_k\right)' V^{-1}\mathbf{1}\mathbf{1}' V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right]
\end{aligned} \tag{A.137}
$$

continuing, we find

$$
\begin{aligned}
E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right] &\cong a^2\frac{2\left(c_{kt}^2-2c_{kt}+1\right)}{T-L}+ac_{tt}\frac{2b_k^2}{T-L}-2ab_t\frac{2b_kc_{kt}-2b_k}{T-L} \\
&+ac_{kk}\frac{2b_t^2}{T-L}-2ab_k\frac{2b_tc_{kt}-2b_t}{T-L}+c_{kk}c_{tt}\frac{2a^2}{T-L}-2b_tc_{kk}\frac{2ab_t}{T-L}-2b_kc_{tt}\frac{2ab_k}{T-L} \\
&+4b_kb_t\frac{ac_{kt}-a+b_kb_t}{T-L}+4a^2\frac{\left(c_{kt}-1\right)}{T}Z'_{t-1}A^{-1}Z_{k-1}-4ab_t\frac{b_k}{T}Z'_{t-1}A^{-1}Z_{k-1} \\
&-4ab_k\frac{b_t}{T}Z'_{t-1}A^{-1}Z_{k-1}+4b_kb_t\frac{a}{T}Z'_{t-1}A^{-1}Z_{k-1} \\
&=\frac{2a^2c_{tk}^2-4a^2c_{tk}+2a^2}{T-L}+\frac{2ab_k^2c_{tt}}{T-L}-\frac{4ab_tb_kc_{kt}-4ab_tb_k}{T-L} \\
&+\frac{2ab_t^2c_{kk}}{T-L}-\frac{4ab_kb_tc_{kt}-4ab_kb_t}{T-L}+\frac{2a^2c_{kk}c_{tt}}{T-L}-\frac{4ab_t^2c_{kk}}{T-L}-\frac{4ab_k^2c_{tt}}{T-L} \\
&+\frac{4ab_kb_tc_{kt}-4ab_kb_t+4b_k^2b_t^2}{T-L}+\frac{\left(4a^2c_{kt}-4a^2\right)}{T}Z'_{t-1}A^{-1}Z_{k-1}-\frac{4ab_kb_t}{T}Z'_{t-1}A^{-1}Z_{k-1} \\
&-\frac{4ab_kb_t}{T}Z'_{t-1}A^{-1}Z_{k-1}+\frac{4ab_kb_t}{T}Z'_{t-1}A^{-1}Z_{k-1} \\
&=\frac{2a^2+4ab_kb_t-4ab_kb_t+4ab_kb_t+4b_k^2b_t^2+2ab_k^2c_{tt}-4ab_k^2c_{tt}}{T-L} \\
&+\frac{4ab_kb_tc_{kt}-4ab_kb_tc_{kt}-4ab_kb_tc_{kt}-4ab_t^2c_{kk}+2ab_t^2c_{kk}-4a^2c_{kt}+2a^2c_{kt}^2+2a^2c_{kk}c_{tt}}{T-L} \\
&+4\frac{a^2c_{kt}-a^2-ab_kb_t}{T}Z'_{t-1}A^{-1}Z_{k-1} \\
&=\frac{2a^2+4ab_kb_t+4b_k^2b_t^2-2ab_k^2c_{tt}-2ab_t^2c_{kk}-4ab_kb_tc_{kt}-4a^2c_{kt}+2a^2c_{kt}^2+2a^2c_{kk}c_{tt}}{T-L} \\
&+4\frac{a^2c_{kt}-a^2-ab_kb_t}{T}Z'_{t-1}A^{-1}Z_{k-1}
\end{aligned}
\tag{A.138}
$$

continuing, we find

$$E\left[\left(\hat{d}_{kk}-d_{kk}\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]$$

$$\cong \frac{-2ab_k b_t c_{kt}+2a^2+2a^2 c_{kk}c_{tt}-2b_k^2\left(ac_{tt}-b_t^2\right)}{T-L}$$

$$+\frac{-2b_t^2\left(ac_{kk}-b_k^2\right)-4a\left(ac_{kt}-b_k b_t\right)+2a c_{kt}\left(ac_{kt}-b_k b_t\right)}{T-L}$$

$$+4\frac{a\left(ac_{kt}-b_k b_t\right)-a^2}{T}Z'_{t-1}A^{-1}Z_{k-1}$$

$$=\frac{-2ab_k b_t c_{kt}+2a^2+2a^2 c_{kk}c_{tt}-2b_k^2 d_{tt}-2b_t^2 d_{kk}-4a d_{kt}+2a c_{kt}d_{kt}}{T-L}+4\frac{ad_{kt}-a^2}{T}Z'_{t-1}A^{-1}Z_{k-1}$$

$$E\left[\left(\hat{d}_{tt}-d_{tt}\right)^2\right]$$

$$\cong \frac{-2ab_t b_t c_{tt}+2a^2+2a^2 c_{tt}^2-4b_t^2 d_{tt}-4a d_{tt}+2a c_{tt}d_{tt}}{T-L}+4\frac{ad_{tt}-a^2}{T}Z'_{t-1}A^{-1}Z_{t-1}$$

$$=\frac{2a c_{tt}\left(ac_{tt}-b_t^2\right)+2a^2-2b_t^2 d_{tt}-4a d_{tt}+2d_{tt}\left(ac_{tt}-b_t^2\right)}{T-L}+4\frac{ad_{tt}-a^2}{T}Z'_{t-1}A^{-1}Z_{t-1}$$

$$=\frac{2a^2+2d_{tt}\left(ac_{tt}-b_t^2\right)-4a d_{tt}+2d_{tt}^2}{T-L}+4\frac{ad_{tt}-a^2}{T}Z'_{t-1}A^{-1}Z_{t-1}$$

$$=\frac{2a^2-4a d_{tt}+4d_{tt}^2}{T-L}+4\frac{ad_{tt}-a^2}{T}Z'_{t-1}A^{-1}Z_{t-1}$$

$$\frac{E\left(\hat{d}_{tt}-d_{tt}\right)^{2}}{d_{tt}^{3}}-\frac{E\left(\hat{d}_{tt}-d_{tt}\right)}{d_{tt}^{2}}$$

$$\cong\frac{2a^{2}-4ad_{tt}+4d_{tt}^{2}}{\left(T-L\right)d_{tt}^{3}}+4\frac{ad_{tt}-a^{2}}{Td_{tt}^{3}}Z'_{t-1}A^{-1}Z_{t-1}-\frac{-na+\left(2n+1\right)d_{tt}}{\left(T-L\right)d_{tt}^{2}}-\frac{a\left(n-1\right)}{Td_{tt}^{2}}Z'_{t-1}A^{-1}Z_{t-1}$$

$$=\frac{2a^{2}-4ad_{tt}+4d_{tt}^{2}}{\left(T-L\right)d_{tt}^{3}}-\frac{-nad_{tt}+\left(2n+1\right)d_{tt}^{2}}{\left(T-L\right)d_{tt}^{3}}+4\frac{ad_{tt}-a^{2}}{Td_{tt}^{3}}Z'_{t-1}A^{-1}Z_{t-1}-\frac{\left(n-1\right)ad_{tt}}{Td_{tt}^{3}}Z'_{t-1}A^{-1}Z_{t-1}$$

$$=\frac{2a^{2}-4ad_{tt}+4d_{tt}^{2}+nad_{tt}-\left(2n+1\right)d_{tt}^{2}}{\left(T-L\right)d_{tt}^{3}}+\frac{4ad_{tt}-4a^{2}-\left(n-1\right)ad_{tt}}{Td_{tt}^{3}}Z'_{t-1}A^{-1}Z_{t-1}$$

$$=\frac{2a^{2}+\left(-4a+na\right)d_{tt}+\left[4-\left(2n+1\right)\right]d_{tt}^{2}}{\left(T-L\right)d_{tt}^{3}}+\frac{\left[4a-\left(n-1\right)a\right]d_{tt}-4a^{2}}{Td_{tt}^{3}}Z'_{t-1}A^{-1}Z_{t-1}$$

$$=\frac{2a^{2}+\left(n-4\right)ad_{tt}+\left(3-2n\right)d_{tt}^{2}}{\left(T-L\right)d_{tt}^{3}}+\frac{-4a^{2}-\left(n-5\right)ad_{tt}}{Td_{tt}^{3}}Z'_{t-1}A^{-1}Z_{t-1}$$

$$E\left[\left(\hat{a}-a\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]\cong aE\left[\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right]$$

$$+E\left[\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\left(c_{tt}\mathbf{1}-2b_t\mu_t\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right]$$

$$=\frac{2ab_t^{2}+2a^{2}c_{tt}-4ab_t^{2}}{T-L}=\frac{2a\left(ac_{tt}-b_t^{2}\right)}{T-L}=\frac{2ad_{tt}}{T-L}$$

$$E\left[\left(\hat{b}_k-b_k\right)\left(\hat{d}_{tt}-d_{tt}\right)\right]\cong aE\left[\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\mu_t'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_t\right]$$

$$-E\left[\mathbf{1}'V^{-1}\left(\hat{V}-V\right)V^{-1}\mu_k\left(-c_{tt}\mathbf{1}+2b_t\mu_t\right)'V^{-1}\left(\hat{V}-V\right)V^{-1}\mathbf{1}\right]$$

$$+2E\left[\left(\hat{\mu}_k-\mu_k\right)'V^{-1}\mathbf{1}\left(a\mu_t-b_t\mathbf{1}\right)'V^{-1}\left(\hat{\mu}_t-\mu_t\right)\right]$$

$$=\frac{2ab_t\left(c_{kt}-1\right)-\left[-2ab_kc_{tt}+2b_kb_t^{2}+2b_ta\left(c_{kt}-1\right)\right]}{T-L}+2\left[\frac{ab_t-b_ta}{T}Z'_{k-1}A^{-1}Z_{t-1}\right]$$

$$=\frac{2ab_kc_{tt}-2b_kb_t^{2}}{T-L}=\frac{2b_k\left(ac_{tt}-b_t^{2}\right)}{T-L}=\frac{2b_kd_{tt}}{T-L}$$

$E\left[\left(\hat{b}_t - b_t\right)\left(\hat{d}_{tt} - d_{tt}\right)\right] \cong \frac{2b_t d_{tt}}{T-L}$ then follows directly

$$
\begin{aligned}
E\left[\left(\hat{c}_{kk} - c_{kk}\right)\left(\hat{d}_{tt} - d_{tt}\right)\right] &\cong aE\left[\mu_k' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_k\mu_t' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t\right] \\
&\quad - E\left[\mu_k' V^{-1}\left(\hat{V} - V\right)V^{-1}\mu_k\left(-c_{tt}\mathbf{1} + 2b_t\mu_t\right)' V^{-1}\left(\hat{V} - V\right)V^{-1}\mathbf{1}\right] \\
&\quad + 4E\left[\left(\hat{\mu}_k - \mu_k\right)' V^{-1}\mu_k\left(a\mu_t - b_t\mathbf{1}\right)' V^{-1}\left(\hat{\mu}_t - \mu_t\right)\right] \qquad \text{(A.139)} \\
&= \frac{2a\left(c_{kt} - 1\right)^2 + 2c_{tt}b_k^2 - 2b_t\left(c_{kt} - 1\right)b_k - 2b_t b_k\left(c_{kt} - 1\right)}{T-L} + \frac{4\left[a\left(c_{kt} - 1\right) - b_k b_t\right]}{T} Z_{k-1}' A^{-1} Z_{t-1} \\
&= \frac{2a\left(c_{kt} - 1\right)^2 - 4b_k b_t\left(c_{kt} - 1\right) + 2b_k^2 c_{tt}}{T-L} + \frac{4\left(d_{kt} - a\right)}{T} Z_{k-1}' A^{-1} Z_{t-1}
\end{aligned}
$$

$$
\begin{aligned}
E\left[\left(\hat{c}_{tt} - c_{tt}\right)\left(\hat{d}_{tt} - d_{tt}\right)\right] &\cong \frac{2a\left(c_{tt} - 1\right)^2 - 4b_t^2\left(c_{tt} - 1\right) + 2b_t^2 c_{tt}}{T-L} + \frac{4\left(d_{tt} - a\right)}{T} Z_{t-1}' A^{-1} Z_{t-1} \\
&= \frac{2a\left(c_{tt} - 1\right)^2 - 2b_t^2\left(c_{tt} - 2\right)}{T-L} + \frac{4\left(d_{tt} - a\right)}{T} Z_{t-1}' A^{-1} Z_{t-1}
\end{aligned}
$$

completing the proof. □

Proof of Corollary I:

We begin by noting that the portfolio return at time $t$ is $R_{pt} = w'R_t$, where $R_t = \mu_t + \varepsilon_t = \delta' Z_{t-1} + \varepsilon_t$. The estimated squared Sharpe Ratio is based on the estimated portfolio mean and the estimated portfolio variance. The conditional mean of $R_{p,t}$ is $E\left(R_{p,t} \mid Z_{t-1}\right) = w'\mu_t$, and the conditional variance of $R_{p,t}$ is $\sigma_{p,t}^2 \equiv Var\left(R_{p,t} \mid Z_{t-1}\right) = w'Vw$ using the conditional covariance matrix $V$ of $R_t$ given $Z_{t-1}$. The unconditional variance $\sigma_p^2$ of the portfolio $R_p$ may then be expressed as the expected conditional variance plus the variance of the conditional portfolio means $w'\mu_t$. Given a zero-beta rate $\varphi$, the maximized squared-Sharpe-Ratio

$$S_\varphi^2 = (\bar{\mu} - \varphi\mathbf{1})' U^{-1} (\bar{\mu} - \varphi\mathbf{1}), \tag{A.140}$$

is constructed from the weights

$$w_\varphi = \frac{U^{-1}(\bar{\mu} - \varphi\mathbf{1})}{\mathbf{1}'U^{-1}(\bar{\mu} - \varphi\mathbf{1})} \tag{A.141}$$

for which the portfolio mean is $\mu_\varphi = \frac{\bar{\mu}'U^{-1}(\bar{\mu} - \varphi\mathbf{1})}{\mathbf{1}'U^{-1}(\bar{\mu} - \varphi\mathbf{1})}$ and the portfolio variance is $\sigma_\varphi^2 = \frac{(\bar{\mu} - \varphi\mathbf{1})' U^{-1} (\bar{\mu} - \varphi\mathbf{1})}{\left[\mathbf{1}'U^{-1}(\bar{\mu} - \varphi\mathbf{1})\right]^2} = \frac{S_\varphi^2}{\left[\mathbf{1}'U^{-1}(\bar{\mu} - \varphi\mathbf{1})\right]^2}$ with $S_\varphi^2 = \frac{(\mu_\varphi - \varphi)^2}{\sigma_\varphi^2}$ , allowing us to write $w_\varphi = \frac{\sigma_\varphi^2 U^{-1}(\bar{\mu} - \varphi\mathbf{1})}{\mu_\varphi - \varphi}$. We begin with $\hat{U}^{-1} = \left[\hat{V} + \frac{1}{T}\sum_{t=1}^{T}(\hat{\mu}_t - \hat{\bar{\mu}})(\hat{\mu}_t - \hat{\bar{\mu}})'\right]^{-1}$, which may be written as $\hat{U}^{-1} = \left[\hat{V} + \frac{1}{T}\sum_{t=1}^{T}(\hat{\mu}_t - \bar{\mu})(\hat{\mu}_t - \bar{\mu})'\right]^{-1} + O_p(1/T)$ because $(\hat{\bar{\mu}} - \bar{\mu}) = O_p(1/\sqrt{T})$ , so we have $\frac{1}{T}\sum_{t=1}^{T}(\hat{\mu}_t - \hat{\bar{\mu}})(\hat{\mu}_t - \hat{\bar{\mu}})' = \frac{1}{T}\sum_{t=1}^{T}(\hat{\mu}_t - \bar{\mu})(\hat{\mu}_t - \bar{\mu})' + O_p(1/T)$. We expand as follows:

$$\begin{aligned}
\hat{U}^{-1} &= \left[\hat{V} + \frac{1}{T}\sum_{t=1}^{T}(\hat{\mu}_t - \bar{\mu})(\hat{\mu}_t - \bar{\mu})'\right]^{-1} + O_p(1/T) \\
&= U^{-1} - U^{-1}\left[\hat{V} + \frac{1}{T}\sum_{t=1}^{T}(\hat{\mu}_t - \bar{\mu})(\hat{\mu}_t - \bar{\mu})' - U\right]U^{-1} + O_p(1/T) \\
&= U^{-1} - U^{-1}\left\{\hat{V} - V + \frac{1}{T}\sum_{t=1}^{T}\left[(\hat{\mu}_t - \mu_t)(\mu_t - \bar{\mu})' + (\mu_t - \bar{\mu})(\hat{\mu}_t - \mu_t)'\right]\right\}U^{-1} + O_p(1/T)
\end{aligned} \tag{A.142}$$

Next, we expand the estimated squared Sharpe Ratio:

$$\begin{aligned}
\hat{S}_\varphi^2 &= \left(\hat{\bar{\mu}}-\varphi\mathbf{1}\right)'\hat{U}^{-1}\left(\hat{\bar{\mu}}-\varphi\mathbf{1}\right)\\
&= S_\varphi^2 + 2\left(\bar{\mu}-\varphi\mathbf{1}\right)'U^{-1}\left(\hat{\bar{\mu}}-\bar{\mu}\right)\\
&\quad -\left(\bar{\mu}-\varphi\mathbf{1}\right)'U^{-1}\left\{\hat{V}-V+\frac{1}{T}\sum_{t=1}^{T}\left[\left(\hat{\mu}_t-\mu_t\right)\left(\mu_t-\bar{\mu}\right)'+\left(\mu_t-\bar{\mu}\right)\left(\hat{\mu}_t-\mu_t\right)'\right]\right\}U^{-1}\left(\bar{\mu}-\varphi\mathbf{1}\right)\\
&\quad + O_p\left(1/T\right)\\
&= S_\varphi^2 + \frac{2}{T}\sum_{t=1}^{T}\left[1-\left(\mu_t-\bar{\mu}\right)'U^{-1}\left(\bar{\mu}-\varphi\mathbf{1}\right)\right]\left(\bar{\mu}-\varphi\mathbf{1}\right)'U^{-1}\left(\hat{\mu}_t-\mu_t\right)\\
&\quad -\left(\bar{\mu}-\varphi\mathbf{1}\right)'U^{-1}\left(\hat{V}-V\right)U^{-1}\left(\bar{\mu}-\varphi\mathbf{1}\right)+O_p\left(1/T\right)
\end{aligned} \quad ,\text{(A.143)}$$

from which we see that estimates of the canonical matrices are

$$\begin{aligned}
C &= A^{-1}\frac{2}{T}\sum_{t=1}^{T}Z_{t-1}\left[1-\left(\mu_t-\bar{\mu}\right)'U^{-1}\left(\bar{\mu}-\varphi\mathbf{1}\right)\right]\left(\bar{\mu}-\varphi\mathbf{1}\right)'U^{-1}\\
D &= -U^{-1}\left(\bar{\mu}-\varphi\mathbf{1}\right)\left(\bar{\mu}-\varphi\mathbf{1}\right)'U^{-1}
\end{aligned} \quad \text{(A.144)}$$

Using the fact that $w=\dfrac{\sigma_\varphi^2 U^{-1}\left(\bar{\mu}-\varphi\mathbf{1}\right)}{\mu_\varphi-\varphi}$ along with $w'\bar{\mu}=\mu_\varphi$, we may express these as follows:

$$\begin{aligned}
C &= \frac{2\left(\mu_\varphi-\varphi\right)}{T\sigma_\varphi^2}A^{-1}\sum_{t=1}^{T}Z_{t-1}\left[1-\frac{\mu_\varphi-\varphi}{\sigma_\varphi^2}\left(\mu_t'w-\mu_\varphi\right)\right]w'\\
D &= -\frac{\left(\mu_\varphi-\varphi\right)^2}{\sigma_\varphi^4}ww' = -\frac{S_\varphi^2}{\sigma_\varphi^2}ww'
\end{aligned} \quad \text{(A.145)}$$

To see that the canonical matrices have the same functional form, for the squared Sharpe Ratio, in the case of a given set of fixed weights $w$ and for weights that are estimated so as to maximize the squared Sharpe Ratio, we note that the estimated portfolio mean may be written as

$$\hat{\mu}_p = \mu_p + \frac{1}{T}\sum_{t=1}^{T}w'\left(\hat{\mu}_t-\mu_t\right) \quad \text{(A.146)}$$

where the portfolio mean is $\mu_p \equiv \frac{1}{T}\sum_{t=1}^{T}w'\mu_t$ .

The portfolio variance may be written as the expected conditional variance plus the variance of the conditional mean:

$$\sigma_p^2 = w'Vw + \frac{1}{T}\sum_{t=1}^{T}\left(w'\mu_t\right)^2 - \left(\frac{1}{T}\sum_{t=1}^{T} w'\mu_t\right)^2 \tag{A.147}$$

and may be estimated as

$$\begin{aligned}
\hat{\sigma}_p^2 &= w'\hat{V}w + \frac{1}{T}\sum_{t=1}^{T}\left(w'\hat{\mu}_t\right)^2 - \left(\frac{1}{T}\sum_{t=1}^{T} w'\hat{\mu}_t\right)^2 \\
&= \sigma_p^2 + \frac{2}{T}\sum_{t=1}^{T}\left(w'\mu_t\right)w'\left(\hat{\mu}_t - \mu_t\right) - \left(\frac{2}{T}\sum_{t=1}^{T} w'\mu_t\right)\left(\frac{1}{T}\sum_{t=1}^{T} w'\left(\hat{\mu}_t - \mu_t\right)\right) \\
&\quad + w'\left(\hat{V} - V\right)w + O_p\left(1/T\right) \\
&= \sigma_p^2 + \frac{2}{T}\sum_{t=1}^{T}\left(w'\mu_t - \mu_p\right)w'\left(\hat{\mu}_t - \mu_t\right) + w'\left(\hat{V} - V\right)w + O_p\left(1/T\right)
\end{aligned} \tag{A.148}$$

The estimated squared Sharpe Ratio, given the zero-beta rate $\varphi$, may then be expanded as follows:

$$\begin{aligned}
\hat{S}_p^2 &= \frac{\left(\hat{\mu}_p - \varphi\right)^2}{\hat{\sigma}_p^2} = \frac{\left[\left(\mu_p - \varphi\right) + \left(\hat{\mu}_p - \mu_p\right)\right]^2}{\sigma_p^2 + \left(\hat{\sigma}_p^2 - \sigma_p^2\right)} = \frac{\left(\mu_p - \varphi\right)^2 + 2\left(\mu_p - \varphi\right)\left(\hat{\mu}_p - \mu_p\right)}{\sigma_p^2\left[1 + \left(\hat{\sigma}_p^2 - \sigma_p^2\right)/\sigma_p^2\right]} + O_p\left(1/T\right) \\
&= \frac{\left[\left(\mu_p - \varphi\right)^2 + 2\left(\mu_p - \varphi\right)\left(\hat{\mu}_p - \mu_p\right)\right]\left[1 - \left(\hat{\sigma}_p^2 - \sigma_p^2\right)/\sigma_p^2\right]}{\sigma_p^2} + O_p\left(1/T\right) \\
&= \frac{\left(\mu_p - \varphi\right)^2}{\sigma_p^2} + \frac{2\left(\mu_p - \varphi\right)}{\sigma_p^2}\left(\hat{\mu}_p - \mu_p\right) - \frac{\left(\mu_p - \varphi\right)^2}{\sigma_p^4}\left(\hat{\sigma}_p^2 - \sigma_p^2\right) + O_p\left(1/T\right)
\end{aligned} \tag{A.149}$$

Substituting from (A.113) and (A.115) into (A.116) we find

$$
\begin{aligned}
\hat{S}_p^2 &= \frac{\left(\mu_p-\varphi\right)^2}{\sigma_p^2}+\frac{2\left(\mu_p-\varphi\right)}{\sigma_p^2}\left(\hat{\mu}_p-\mu_p\right)-\frac{\left(\mu_p-\varphi\right)^2}{\sigma_p^4}\left(\hat{\sigma}_p^2-\sigma_p^2\right)+O_p\left(1/T\right) \\
&= S_p^2+\frac{2\left(\mu_p-\varphi\right)}{T\sigma_p^2}\sum_{t=1}^T w'\left(\hat{\mu}_t-\mu_t\right)-\frac{2\left(\mu_p-\varphi\right)^2}{T\sigma_p^4}\sum_{t=1}^T\left(w'\mu_t-\mu_p\right)w'\left(\hat{\mu}_t-\mu_t\right) \\
&\quad -\frac{\left(\mu_p-\varphi\right)^2}{\sigma_p^4}w'\left(\hat{V}-V\right)w+O_p\left(1/T\right) \\
&= S_p^2+\frac{2\left(\mu_p-\varphi\right)}{T\sigma_p^2}\sum_{t=1}^T\left[1-\frac{\mu_p-\varphi}{\sigma_p^2}\left(\mu_t'w-\mu_p\right)\right]w'\left(\hat{\mu}_t-\mu_t\right) \\
&\quad -\frac{\left(\mu_p-\varphi\right)^2}{\sigma_p^4}w'\left(\hat{V}-V\right)w+O_p\left(1/T\right)
\end{aligned}
\tag{A.150}
$$

for which we find estimates of the canonical matrices

$$
\begin{aligned}
C &= \frac{2\left(\mu_p-\varphi\right)A^{-1}}{T\sigma_p^2}\sum_{t=1}^T Z_{t-1}\left[1-\frac{\mu_p-\varphi}{\sigma_p^2}\left(\mu_t'w-\mu_p\right)\right]w' \\
D &= -\frac{\left(\mu_p-\varphi\right)^2}{\sigma_p^4}ww'
\end{aligned}
\tag{A.151}
$$

which have the same functional form as in the case when the weights are estimated to maximize the squared Sharpe Ratio, completing the proof. □

<u>Proof of Corollary II:</u>

To obtain expansions for $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{\alpha}_3$, we begin with $\hat{\Lambda}_t$ to find

$$
\begin{aligned}
\hat{\Lambda}_t &= \left(\hat{\mu}_t\hat{\mu}_t'+\hat{V}\right)^{-1}=\left[\mu_t\mu_t'+V+\mu_t\left(\hat{\mu}_t-\mu_t\right)'+\left(\hat{\mu}_t-\mu_t\right)\mu_t'+\left(\hat{V}-V\right)\right]^{-1}+O_p\left(1/T\right) \\
&= \Lambda_t\left\{I+\left[\mu_t\left(\hat{\mu}_t-\mu_t\right)'+\left(\hat{\mu}_t-\mu_t\right)\mu_t'+\left(\hat{V}-V\right)\right]\Lambda_t\right\}^{-1}+O_p\left(1/T\right) \\
&= \Lambda_t\left\{I-\left[\mu_t\left(\hat{\mu}_t-\mu_t\right)'+\left(\hat{\mu}_t-\mu_t\right)\mu_t'+\left(\hat{V}-V\right)\right]\Lambda_t\right\}+O_p\left(1/T\right) \\
&= \Lambda_t-\Lambda_t\mu_t\left(\hat{\mu}_t-\mu_t\right)'\Lambda_t-\Lambda_t\left(\hat{\mu}_t-\mu_t\right)\mu_t'\Lambda_t-\Lambda_t\left(\hat{V}-V\right)\Lambda_t+O_p\left(1/T\right)
\end{aligned}
\tag{A.152}
$$

and continue, expanding

$$
\begin{aligned}
\mathbf{1}'\hat{\Lambda}_t\mathbf{1} &= \mathbf{1}'\Lambda_t\mathbf{1} - \mathbf{1}'\Lambda_t\mu_t\left(\hat{\mu}_t-\mu_t\right)'\Lambda_t\mathbf{1} - \mathbf{1}'\Lambda_t\left(\hat{\mu}_t-\mu_t\right)\mu_t'\Lambda_t\mathbf{1} - \mathbf{1}'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1} + O_p\left(1/T\right) \\
&= \mathbf{1}'\Lambda_t\mathbf{1} - 2\mathbf{1}'\Lambda_t\mu_t\mathbf{1}'\Lambda_t\left(\hat{\mu}_t-\mu_t\right) - \mathbf{1}'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1} + O_p\left(1/T\right)
\end{aligned}
\tag{A.153}
$$

and

$$
\frac{1}{\mathbf{1}'\hat{\Lambda}_t\mathbf{1}} = \frac{1}{\mathbf{1}'\Lambda_t\mathbf{1}} + \frac{2\mathbf{1}'\Lambda_t\mu_t\mathbf{1}'\Lambda_t\left(\hat{\mu}_t-\mu_t\right)}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} + \frac{\mathbf{1}'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1}}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} + O_p\left(1/T\right) \tag{A.154}
$$

Next, we expand $\hat{\alpha}_1$ as follows

$$
\begin{aligned}
\hat{\alpha}_1 &= \frac{1}{T}\sum_{t=1}^{T}\frac{1}{\mathbf{1}'\hat{\Lambda}_t\mathbf{1}} \\
&= \alpha_1 + \frac{1}{T}\sum_{t=1}^{T}\frac{2\mathbf{1}'\Lambda_t\mu_t\mathbf{1}'\Lambda_t\left(\hat{\mu}_t-\mu_t\right)}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} + \frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{1}'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1}}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} + O_p\left(1/T\right)
\end{aligned}
\tag{A.155}
$$

To expand $\hat{\alpha}_2$, we will need

$$
\begin{aligned}
\mathbf{1}'\hat{\Lambda}_t\hat{\mu}_t &= \mathbf{1}'\Lambda_t\mu_t + \mathbf{1}'\left(\hat{\Lambda}_t-\Lambda_t\right)\mu_t + \mathbf{1}'\Lambda_t\left(\hat{\mu}_t-\mu_t\right) + O_p\left(1/T\right) \\
&= \mathbf{1}'\Lambda_t\mu_t + \mathbf{1}'\left[-\Lambda_t\mu_t\left(\hat{\mu}_t-\mu_t\right)'\Lambda_t - \Lambda_t\left(\hat{\mu}_t-\mu_t\right)\mu_t'\Lambda_t - \Lambda_t\left(\hat{V}-V\right)\Lambda_t\right]\mu_t \\
&\quad + \mathbf{1}'\Lambda_t\left(\hat{\mu}_t-\mu_t\right) + O_p\left(1/T\right) \\
&= \mathbf{1}'\Lambda_t\mu_t + \left(\mathbf{1}' - \mathbf{1}'\Lambda_t\mu_t\mu_t' - \mu_t'\Lambda_t\mu_t\mathbf{1}'\right)\Lambda_t\left(\hat{\mu}_t-\mu_t\right) - \mathbf{1}'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mu_t + O_p\left(1/T\right)
\end{aligned}
\tag{A.156}
$$

and

$$\begin{aligned}
\frac{\mathbf{1}'\hat{\Lambda}_t\hat{\mu}_t}{\mathbf{1}'\hat{\Lambda}_t\mathbf{1}} &= \frac{\mathbf{1}'\Lambda_t\mu_t}{\mathbf{1}'\Lambda_t\mathbf{1}} + \frac{1}{\mathbf{1}'\Lambda_t\mathbf{1}}\left[\left(\mathbf{1}' - \mathbf{1}'\Lambda_t\mu_t\mu_t' - \mu_t'\Lambda_t\mu_t\mathbf{1}'\right)\Lambda_t\left(\hat{\mu}_t - \mu_t\right) - \mathbf{1}'\Lambda_t\left(\hat{V} - V\right)\Lambda_t\mu_t\right] \\
&\quad + \left(\mathbf{1}'\Lambda_t\mu_t\right)\left(\frac{2\mathbf{1}'\Lambda_t\mu_t\mathbf{1}'\Lambda_t\left(\hat{\mu}_t - \mu_t\right)}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} + \frac{\mathbf{1}'\Lambda_t\left(\hat{V} - V\right)\Lambda_t\mathbf{1}}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\right) + O_p\left(1/T\right) \\
&= \frac{\mathbf{1}'\Lambda_t\mu_t}{\mathbf{1}'\Lambda_t\mathbf{1}} + \frac{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)\left(\mathbf{1}' - \mathbf{1}'\Lambda_t\mu_t\mu_t' - \mu_t'\Lambda_t\mu_t\mathbf{1}'\right) + 2\left(\mathbf{1}'\Lambda_t\mu_t\right)^2\mathbf{1}'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\Lambda_t\left(\hat{\mu}_t - \mu_t\right) \\
&\quad + \frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}' - \left(\mathbf{1}'\Lambda_t\mathbf{1}\right)\mu_t'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\Lambda_t\left(\hat{V} - V\right)\Lambda_t\mathbf{1} + O_p\left(1/T\right)
\end{aligned} \tag{A.157}$$

from which we find

$$\begin{aligned}
\hat{\alpha}_2 &= \frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{1}'\hat{\Lambda}_t\hat{\mu}_t}{\mathbf{1}'\hat{\Lambda}_t\mathbf{1}} \\
&= \alpha_2 + \frac{1}{T}\sum_{t=1}^{T}\frac{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)\left(\mathbf{1}' - \mathbf{1}'\Lambda_t\mu_t\mu_t' - \mu_t'\Lambda_t\mu_t\mathbf{1}'\right) + 2\left(\mathbf{1}'\Lambda_t\mu_t\right)^2\mathbf{1}'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\Lambda_t\left(\hat{\mu}_t - \mu_t\right) \\
&\quad + \frac{1}{T}\sum_{t=1}^{T}\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}' - \left(\mathbf{1}'\Lambda_t\mathbf{1}\right)\mu_t'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\Lambda_t\left(\hat{V} - V\right)\Lambda_t\mathbf{1} + O_p\left(1/T\right) \\
&= \alpha_2 + \frac{1}{T}\sum_{t=1}^{T}\left(\frac{\mathbf{1}' - \left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}' - \left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'}{\mathbf{1}'\Lambda_t\mathbf{1}} + \frac{2\left(\mathbf{1}'\Lambda_t\mu_t\right)^2\mathbf{1}'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\right)\Lambda_t\left(\hat{\mu}_t - \mu_t\right) \\
&\quad + \frac{1}{T}\sum_{t=1}^{T}\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}' - \left(\mathbf{1}'\Lambda_t\mathbf{1}\right)\mu_t'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\Lambda_t\left(\hat{V} - V\right)\Lambda_t\mathbf{1} + O_p\left(1/T\right)
\end{aligned} \tag{A.158}$$

To expand $\hat{\alpha}_3$, we will also need

$$\begin{aligned}
\hat{\mu}_t'\hat{\Lambda}_t\hat{\mu}_t &= \mu_t'\Lambda_t\mu_t + 2\mu_t'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) + \mu_t'\left(\hat{\Lambda}_t - \Lambda_t\right)\mu_t + O_p\left(1/T\right) \\
&= \mu_t'\Lambda_t\mu_t + 2\mu_t'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) \\
&\qquad + \mu_t'\left[-\Lambda_t\mu_t\left(\hat{\mu}_t - \mu_t\right)'\Lambda_t - \Lambda_t\left(\hat{\mu}_t - \mu_t\right)\mu_t'\Lambda_t - \Lambda_t\left(\hat{V} - V\right)\Lambda_t\right]\mu_t + O_p\left(1/T\right) \\
&= \mu_t'\Lambda_t\mu_t + 2\left(1 - \mu_t'\Lambda_t\mu_t\right)\mu_t'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) - \mu_t'\Lambda_t\left(\hat{V} - V\right)\Lambda_t\mu_t + O_p\left(1/T\right)
\end{aligned} \tag{A.159}$$

as well as

$$\begin{aligned}
\left(\mathbf{1}'\hat{\Lambda}_t\hat{\mu}_t\right)^2 &= \left(\mathbf{1}'\Lambda_t\mu_t\right)^2 + 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\left(\hat{\Lambda}_t - \Lambda_t\right)\mu_t + 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) + O_p\left(1/T\right) \\
&= \left(\mathbf{1}'\Lambda_t\mu_t\right)^2 + 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\left\{-\Lambda_t\mu_t\left(\hat{\mu}_t - \mu_t\right)'\Lambda_t - \Lambda_t\left(\hat{\mu}_t - \mu_t\right)\mu_t'\Lambda_t - \Lambda_t\left(\hat{V} - V\right)\Lambda_t\right\}\mu_t \\
&\quad + 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) + O_p\left(1/T\right) \\
&= \left(\mathbf{1}'\Lambda_t\mu_t\right)^2 - 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\mu_t\left(\hat{\mu}_t - \mu_t\right)'\Lambda_t\mu_t - 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{\mu}_t - \mu_t\right)\mu_t'\Lambda_t\mu_t \\
&\quad - 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{V} - V\right)\Lambda_t\mu_t + 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) + O_p\left(1/T\right) \\
&= \left(\mathbf{1}'\Lambda_t\mu_t\right)^2 - 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) - 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) \\
&\quad + 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) - 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{V} - V\right)\Lambda_t\mu_t + O_p\left(1/T\right) \\
&= \left(\mathbf{1}'\Lambda_t\mu_t\right)^2 + \left[-2\left(\mathbf{1}'\Lambda_t\mu_t\right)\left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t' - 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}' + 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\right]\Lambda_t\left(\hat{\mu}_t - \mu_t\right) \\
&\quad - 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{V} - V\right)\Lambda_t\mu_t + O_p\left(1/T\right) \\
&= \left(\mathbf{1}'\Lambda_t\mu_t\right)^2 + 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\left[\mathbf{1}' - \left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t' - \left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}'\right]\Lambda_t\left(\hat{\mu}_t - \mu_t\right) \\
&\quad - 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{V} - V\right)\Lambda_t\mu_t + O_p\left(1/T\right)
\end{aligned} \tag{A.160}$$

and

$$\frac{\left(\mathbf{1}'\hat{\Lambda}_t\hat{\mu}_t\right)^2}{\mathbf{1}'\hat{\Lambda}_t\mathbf{1}} = \frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^2}{\mathbf{1}'\Lambda_t\mathbf{1}} + \left(\mathbf{1}'\Lambda_t\mu_t\right)^2\left(\frac{2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{\mu}_t-\mu_t\right)}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} + \frac{\mathbf{1}'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1}}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\right)$$

$$+\frac{1}{\mathbf{1}'\Lambda_t\mathbf{1}}\left[2\left(\mathbf{1}'\Lambda_t\mu_t\right)\left[\mathbf{1}'-\left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'-\left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}'\right]\Lambda_t\left(\hat{\mu}_t-\mu_t\right)-2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mu_t\right]+O_p\left(1/T\right)$$

$$=\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^2}{\mathbf{1}'\Lambda_t\mathbf{1}} + \frac{2\left(\mathbf{1}'\Lambda_t\mu_t\right)^3\mathbf{1}'\Lambda_t\left(\hat{\mu}_t-\mu_t\right)}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} + \frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^2\mathbf{1}'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1}}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}$$

$$+\frac{2\left(\mathbf{1}'\Lambda_t\mu_t\right)\left[\mathbf{1}'-\left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'-\left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}'\right]\Lambda_t\left(\hat{\mu}_t-\mu_t\right)}{\mathbf{1}'\Lambda_t\mathbf{1}} - \frac{2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mu_t}{\mathbf{1}'\Lambda_t\mathbf{1}}+O_p\left(1/T\right)$$

$$=\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^2}{\mathbf{1}'\Lambda_t\mathbf{1}} + \frac{2\left(\mathbf{1}'\Lambda_t\mu_t\right)^3\mathbf{1}'\Lambda_t\left(\hat{\mu}_t-\mu_t\right)}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} + \frac{2\left(\mathbf{1}'\Lambda_t\mu_t\right)\left[\mathbf{1}'-\left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'-\left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}'\right]\Lambda_t\left(\hat{\mu}_t-\mu_t\right)}{\mathbf{1}'\Lambda_t\mathbf{1}}$$

$$+\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^2\mathbf{1}'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1}}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} - \frac{2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mu_t}{\mathbf{1}'\Lambda_t\mathbf{1}}+O_p\left(1/T\right)$$

$$=\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^2}{\mathbf{1}'\Lambda_t\mathbf{1}} + 2\left(\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^3\mathbf{1}'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} + \frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)\left[\mathbf{1}'-\left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'-\left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}'\right]}{\mathbf{1}'\Lambda_t\mathbf{1}}\right)\Lambda_t\left(\hat{\mu}_t-\mu_t\right)$$

$$+\left(\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^2\mathbf{1}'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} - \frac{2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'}{\mathbf{1}'\Lambda_t\mathbf{1}}\right)\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1}+O_p\left(1/T\right) \quad \text{(A.161)}$$

$$=\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^2}{\mathbf{1}'\Lambda_t\mathbf{1}} + 2\left(\mathbf{1}'\Lambda_t\mu_t\right)\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^2\mathbf{1}'+\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)\left[\mathbf{1}'-\left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'-\left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}'\right]}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\Lambda_t\left(\hat{\mu}_t-\mu_t\right)$$

$$+\left(\mathbf{1}'\Lambda_t\mu_t\right)\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'-2\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)\mu_t'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1}+O_p\left(1/T\right)$$

to find

$$\hat{\alpha}_3 = \frac{1}{T}\sum_{t=1}^{T}\hat{\mu}_t'\left(\hat{\Lambda}_t - \frac{\hat{\Lambda}_t\mathbf{1}\mathbf{1}'\hat{\Lambda}_t}{\mathbf{1}'\hat{\Lambda}_t\mathbf{1}}\right)\hat{\mu}_t = \frac{1}{T}\sum_{t=1}^{T}\left(\hat{\mu}_t'\hat{\Lambda}_t\hat{\mu}_t - \frac{\hat{\mu}_t'\hat{\Lambda}_t\mathbf{1}\mathbf{1}'\hat{\Lambda}_t\hat{\mu}_t}{\mathbf{1}'\hat{\Lambda}_t\mathbf{1}}\right)$$

$$=\alpha_3+\frac{2}{T}\sum_{t=1}^{T}\left(1-\mu_t'\Lambda_t\mu_t\right)\mu_t'\Lambda_t\left(\hat{\mu}_t-\mu_t\right)-\frac{1}{T}\sum_{t=1}^{T}\mu_t'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mu_t$$

$$-\frac{2}{T}\sum_{t=1}^{T}\left(\mathbf{1}'\Lambda_t\mu_t\right)\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^2\mathbf{1}'+\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)\left[\mathbf{1}'-\left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}'-\left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'\right]}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\Lambda_t\left(\hat{\mu}_t-\mu_t\right) \quad \text{(A.162)}$$

$$-\frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{1}'\Lambda_t\mu_t\right)\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'-2\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)\mu_t'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1}+O_p\left(1/T\right)$$

$$=\alpha_3+\frac{2}{T}\sum_{t=1}^{T}\left(\left(1-\mu_t'\Lambda_t\mu_t\right)\mu_t'-\left(\mathbf{1}'\Lambda_t\mu_t\right)\frac{\mathbf{1}'-\left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}'-\left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'}{\mathbf{1}'\Lambda_t\mathbf{1}}-\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^3\mathbf{1}'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\right)\Lambda_t\left(\hat{\mu}_t-\mu_t\right)$$

$$-\frac{1}{T}\sum_{t=1}^{T}\left(\mu_t'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mu_t+\left(\mathbf{1}'\Lambda_t\mu_t\right)\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'-2\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)\mu_t'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1}\right)+O_p\left(1/T\right)$$

Next, we use commutativity of matrix multiplication within the trace operator to write

$$\begin{aligned}\hat{\alpha}_3 = \alpha_3 &+ \frac{2}{T}\sum_{t=1}^{T}\left( (1-\mu_t'\Lambda_t\mu_t)\mu_t' - (\mathbf{1}'\Lambda_t\mu_t)\frac{\mathbf{1}' - (\mu_t'\Lambda_t\mu_t)\mathbf{1}' - (\mathbf{1}'\Lambda_t\mu_t)\mu_t'}{\mathbf{1}'\Lambda_t\mathbf{1}} - \frac{(\mathbf{1}'\Lambda_t\mu_t)^3\mathbf{1}'}{(\mathbf{1}'\Lambda_t\mathbf{1})^2}\right)\Lambda_t(\hat{\mu}_t-\mu_t) \\ &-tr\left\{\left[\frac{1}{T}\sum_{t=1}^{T}\left(\Lambda_t\mu_t\mu_t' + (\mathbf{1}'\Lambda_t\mu_t)\Lambda_t\mathbf{1}\frac{(\mathbf{1}'\Lambda_t\mu_t)\mathbf{1}' - 2(\mathbf{1}'\Lambda_t\mathbf{1})\mu_t'}{(\mathbf{1}'\Lambda_t\mathbf{1})^2}\right)\Lambda_t\right]\left(\hat{V}-V\right)\right\} + O_p(1/T)\end{aligned} \tag{A.163}$$

To summarize the expansions of the portfolio coefficients:

$$\begin{aligned}\hat{\alpha}_1 &= \alpha_1 + \frac{1}{T}\sum_{t=1}^{T}\frac{2\mathbf{1}'\Lambda_t\mu_t\mathbf{1}'\Lambda_t(\hat{\mu}_t-\mu_t)}{(\mathbf{1}'\Lambda_t\mathbf{1})^2} + \frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{1}'\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1}}{(\mathbf{1}'\Lambda_t\mathbf{1})^2} + O_p(1/T) \\ &= \alpha_1 + \sum_{t=1}^{T}C_{\alpha_1,t}'(\hat{\mu}_t-\mu_t) + tr\left[D_{\alpha_1}\left(\hat{V}-V\right)\right] + O_p(1/T)\end{aligned} \tag{A.164}$$

$$\begin{aligned}\hat{\alpha}_2 = \alpha_2 &+ \frac{1}{T}\sum_{t=1}^{T}\left(\frac{\mathbf{1}' - (\mu_t'\Lambda_t\mu_t)\mathbf{1}' - (\mathbf{1}'\Lambda_t\mu_t)\mu_t'}{\mathbf{1}'\Lambda_t\mathbf{1}} + \frac{2(\mathbf{1}'\Lambda_t\mu_t)^2\mathbf{1}'}{(\mathbf{1}'\Lambda_t\mathbf{1})^2}\right)\Lambda_t(\hat{\mu}_t-\mu_t) \\ &+ \frac{1}{T}\sum_{t=1}^{T}\frac{(\mathbf{1}'\Lambda_t\mu_t)\mathbf{1}' - (\mathbf{1}'\Lambda_t\mathbf{1})\mu_t'}{(\mathbf{1}'\Lambda_t\mathbf{1})^2}\Lambda_t\left(\hat{V}-V\right)\Lambda_t\mathbf{1} + O_p(1/T) \\ = \alpha_2 &+ \sum_{t=1}^{T}C_{\alpha_2,t}'(\hat{\mu}_t-\mu_t) + tr\left[D_{\alpha_2}\left(\hat{V}-V\right)\right] + O_p(1/T)\end{aligned} \tag{A.165}$$

$$\begin{aligned}\hat{\alpha}_3 = \alpha_3 &+ \frac{2}{T}\sum_{t=1}^{T}\left( (1-\mu_t'\Lambda_t\mu_t)\mu_t' - (\mathbf{1}'\Lambda_t\mu_t)\frac{\mathbf{1}' - (\mu_t'\Lambda_t\mu_t)\mathbf{1}' - (\mathbf{1}'\Lambda_t\mu_t)\mu_t'}{\mathbf{1}'\Lambda_t\mathbf{1}} - \frac{(\mathbf{1}'\Lambda_t\mu_t)^3\mathbf{1}'}{(\mathbf{1}'\Lambda_t\mathbf{1})^2}\right)\Lambda_t(\hat{\mu}_t-\mu_t) \\ &-tr\left\{\left[\frac{1}{T}\sum_{t=1}^{T}\left(\Lambda_t\mu_t\mu_t' + (\mathbf{1}'\Lambda_t\mu_t)\Lambda_t\mathbf{1}\frac{(\mathbf{1}'\Lambda_t\mu_t)\mathbf{1}' - 2(\mathbf{1}'\Lambda_t\mathbf{1})\mu_t'}{(\mathbf{1}'\Lambda_t\mathbf{1})^2}\right)\right]\Lambda_t\left(\hat{V}-V\right)\right\} + O_p(1/T) \\ = \alpha_3 &+ \sum_{t=1}^{T}C_{\alpha_3,t}'(\hat{\mu}_t-\mu_t) + tr\left[D_{\alpha_3}\left(\hat{V}-V\right)\right] + O_p(1/T)\end{aligned} \tag{A.166}$$

The estimates of the terms for the Theorem I are given by

$$\mu^F \equiv \frac{1}{T}\sum_{t=1}^{T}\mu_t^F \tag{A.167}$$

$$\gamma_1 \equiv \frac{1}{T}\sum_{t=1}^{T}\frac{1}{\mathbf{1}'\Lambda_t\mathbf{1}} \tag{A.168}$$

$$\gamma_{\mu} \equiv \frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{1}'\Lambda_t\mu_t}{\mathbf{1}'\Lambda_t\mathbf{1}} \tag{A.169}$$

$$\gamma_{F} \equiv \frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{1}'\Lambda_t E\left(R_tF_t \mid Z_{t-1}\right)}{\mathbf{1}'\Lambda_t\mathbf{1}} = \frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{1}'\Lambda_t V_F + \mathbf{1}'\Lambda_t\mu_t\mu_t^F}{\mathbf{1}'\Lambda_t\mathbf{1}} \tag{A.170}$$

$$\gamma_{\mu\mu} \equiv \frac{1}{T}\sum_{t=1}^{T}\mu_t'\Omega_t\mu_t \tag{A.171}$$

$$\gamma_{\mu F} \equiv \frac{1}{T}\sum_{t=1}^{T}\mu_t'\Omega_t E\left(R_tF_t \mid Z_{t-1}\right) = \frac{1}{T}\sum_{t=1}^{T}\left(\mu_t'\Omega_t V_F + \mu_t'\Omega_t\mu_t\mu_t^F\right) \tag{A.172}$$

$$\begin{aligned}\gamma_{FF} &\equiv \frac{1}{T}\sum_{t=1}^{T}E\left(R_t'F_t \mid Z_{t-1}\right)\Omega_t E\left(R_tF_t \mid Z_{t-1}\right) \\ &= \frac{1}{T}\sum_{t=1}^{T}\left[V_F'\Omega_t V_F + 2\mu_t'\Omega_t V_F\mu_t^F + \mu_t'\Omega_t\mu_t\left(\mu_t^F\right)^2\right]\end{aligned} \tag{A.173}$$

$$\begin{aligned}C_{\alpha_1,t}' &= \frac{2\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t}{T\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} \\ C_{\alpha_1} &= \frac{2A^{-1}}{T}\sum_{t=1}^{T}Z_{t-1}\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}'\Lambda_t}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2} \\ D_{\alpha_1} &= \frac{1}{T}\sum_{t=1}^{T}\frac{\Lambda_t\mathbf{1}\mathbf{1}'\Lambda_t}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\end{aligned} \tag{A.174}$$

$$\begin{aligned}C_{\alpha_2,t}' &= \frac{1}{T}\left(\frac{\mathbf{1}' - \left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}' - \left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'}{\mathbf{1}'\Lambda_t\mathbf{1}} + \frac{2\left(\mathbf{1}'\Lambda_t\mu_t\right)^2\mathbf{1}'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\right)\Lambda_t \\ C_{\alpha_2} &= \frac{A^{-1}}{T}\sum_{t=1}^{T}Z_{t-1}\left(\frac{\mathbf{1}' - \left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}' - \left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'}{\mathbf{1}'\Lambda_t\mathbf{1}} + \frac{2\left(\mathbf{1}'\Lambda_t\mu_t\right)^2\mathbf{1}'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\right)\Lambda_t \\ D_{\alpha_2} &= \frac{1}{T}\sum_{t=1}^{T}\Lambda_t\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}\mathbf{1}' - \left(\mathbf{1}'\Lambda_t\mathbf{1}\right)\mathbf{1}\mu_t'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\Lambda_t\end{aligned} \tag{A.175}$$

$$\begin{aligned}C_{\alpha_3,t}' &= \frac{2}{T}\left(\left(1-\mu_t'\Lambda_t\mu_t\right)\mu_t' - \left(\mathbf{1}'\Lambda_t\mu_t\right)\frac{\mathbf{1}' - \left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}' - \left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'}{\mathbf{1}'\Lambda_t\mathbf{1}} - \frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^3\mathbf{1}'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\right)\Lambda_t \\ C_{\alpha_3} &= \frac{2A^{-1}}{T}\sum_{t=1}^{T}Z_{t-1}\left(\left(1-\mu_t'\Lambda_t\mu_t\right)\mu_t' - \left(\mathbf{1}'\Lambda_t\mu_t\right)\frac{\mathbf{1}' - \left(\mu_t'\Lambda_t\mu_t\right)\mathbf{1}' - \left(\mathbf{1}'\Lambda_t\mu_t\right)\mu_t'}{\mathbf{1}'\Lambda_t\mathbf{1}} - \frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)^3\mathbf{1}'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\right)\Lambda_t \\ D_{\alpha_3} &= -\frac{1}{T}\sum_{t=1}^{T}\Lambda_t\left(\mu_t\mu_t' + \left(\mathbf{1}'\Lambda_t\mu_t\right)\frac{\left(\mathbf{1}'\Lambda_t\mu_t\right)\mathbf{1}\mathbf{1}' - 2\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)\mathbf{1}\mu_t'}{\left(\mathbf{1}'\Lambda_t\mathbf{1}\right)^2}\right)\Lambda_t\end{aligned} \tag{A.176}$$

The estimated squared Sharpe Ratio is

$$\hat{S}_{\varphi}^{2}=\frac{\hat{\alpha}_{2}^{2}+\hat{\alpha}_{1}\hat{\alpha}_{3}-2\varphi\hat{\alpha}_{2}+\varphi^{2}\left(1-\hat{\alpha}_{3}\right)}{\hat{\alpha}_{1}\left(1-\hat{\alpha}_{3}\right)-\hat{\alpha}_{2}^{2}}. \tag{A.177}$$

This estimate may be found by plugging in consistent estimates for the $\alpha$'s or by computing the estimated portfolio weights as a function of $Z$, applying these to the asset returns and computing the unconditional mean and variance of the resulting portfolio. In simulations not reported here, we find that either approach produces similar results. Equation (A.43) follows from maximizing

$$\hat{S}^{2}=\frac{\left(\hat{\mu}_{p}-\varphi\right)^{2}}{\hat{\sigma}_{p}^{2}}=\frac{\left(\hat{\mu}_{p}-\varphi\right)^{2}}{\left(\hat{\alpha}_{1}+\frac{\hat{\alpha}_{2}^{2}}{\hat{\alpha}_{3}}\right)-\frac{2\hat{\alpha}_{2}}{\hat{\alpha}_{3}}\hat{\mu}_{p}+\frac{1-\hat{\alpha}_{3}}{\hat{\alpha}_{3}}\hat{\mu}_{p}^{2}} \tag{A.178}$$

with respect to the mean $\hat{\mu}_{p}$. We expand (A.43) to obtain

$$
\begin{aligned}
\hat{S}_\varphi^2 = S_\varphi^2 &+ \frac{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]\left[2\alpha_2(\hat{\alpha}_2-\alpha_2)+\alpha_1(\hat{\alpha}_3-\alpha_3)+(\hat{\alpha}_1-\alpha_1)\alpha_3-2\varphi(\hat{\alpha}_2-\alpha_2)-\varphi^2(\hat{\alpha}_3-\alpha_3)\right]}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2} \\
&- \frac{\left[\alpha_2^2+\alpha_1\alpha_3-2\varphi\alpha_2+\varphi^2(1-\alpha_3)\right]\left[-\alpha_1(\hat{\alpha}_3-\alpha_3)+(\hat{\alpha}_1-\alpha_1)(1-\alpha_3)-2\alpha_2(\hat{\alpha}_2-\alpha_2)\right]}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2}+O_p(1/T) \\
= S_\varphi^2 &+ \frac{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]\alpha_3(\hat{\alpha}_1-\alpha_1)}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2} \\
&+ \frac{2\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right](\alpha_2-\varphi)(\hat{\alpha}_2-\alpha_2)}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2} \\
&+ \frac{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right](\alpha_1-\varphi^2)(\hat{\alpha}_3-\alpha_3)}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2} \\
&- \frac{\left[\alpha_2^2+\alpha_1\alpha_3-2\varphi\alpha_2+\varphi^2(1-\alpha_3)\right](1-\alpha_3)(\hat{\alpha}_1-\alpha_1)}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2} \\
&+ \frac{2\left[\alpha_2^2+\alpha_1\alpha_3-2\varphi\alpha_2+\varphi^2(1-\alpha_3)\right]\alpha_2(\hat{\alpha}_2-\alpha_2)}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2} \\
&+ \frac{\left[\alpha_2^2+\alpha_1\alpha_3-2\varphi\alpha_2+\varphi^2(1-\alpha_3)\right]\alpha_1(\hat{\alpha}_3-\alpha_3)}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2}+O_p(1/T)
\end{aligned}
\tag{A.179}
$$

and then gather terms to find

$$
\begin{aligned}
\hat{S}_\varphi^2 = S_\varphi^2 &+ \frac{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]\alpha_3-\left[\alpha_2^2+\alpha_1\alpha_3-2\varphi\alpha_2+\varphi^2(1-\alpha_3)\right](1-\alpha_3)}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2}(\hat{\alpha}_1-\alpha_1) \\
&+ \frac{2\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right](\alpha_2-\varphi)+2\left[\alpha_2^2+\alpha_1\alpha_3-2\varphi\alpha_2+\varphi^2(1-\alpha_3)\right]\alpha_2}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2}(\hat{\alpha}_2-\alpha_2) \\
&+ \frac{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right](\alpha_1-\varphi^2)+\left[\alpha_2^2+\alpha_1\alpha_3-2\varphi\alpha_2+\varphi^2(1-\alpha_3)\right]\alpha_1}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2}(\hat{\alpha}_3-\alpha_3)+O_p(1/T)
\end{aligned}
\tag{A.180}
$$

and simplify to obtain

$$
\begin{aligned}
\hat{S}_\varphi^2 = S_\varphi^2 &- \frac{\left[\alpha_2-\varphi(1-\alpha_3)\right]^2}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2}(\hat{\alpha}_1-\alpha_1)+\frac{2(\alpha_1-\varphi\alpha_2)\left[\alpha_2-\varphi(1-\alpha_3)\right]}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2}(\hat{\alpha}_2-\alpha_2) \\
&+ \frac{(\alpha_1-\varphi\alpha_2)^2}{\left[\alpha_1(1-\alpha_3)-\alpha_2^2\right]^2}(\hat{\alpha}_3-\alpha_3)+O_p(1/T)
\end{aligned}
\tag{A.181}
$$

so that the asymptotic variance of $\hat{S}_{\varphi}^{2}$ will follow from the Theorem I together with the following choices for canonical matrices *C* and *D*:

$$C = \frac{-\left[\alpha_2 - \varphi\left(1-\alpha_3\right)\right]^2 C_{\alpha_1} + 2\left(\alpha_1 - \varphi\alpha_2\right)\left[\alpha_2 - \varphi\left(1-\alpha_3\right)\right]C_{\alpha_2} + \left(\alpha_1 - \varphi\alpha_2\right)^2 C_{\alpha_3}}{\left[\alpha_1\left(1-\alpha_3\right)-\alpha_2^2\right]^2}$$
$$D = \frac{-\left[\alpha_2 - \varphi\left(1-\alpha_3\right)\right]^2 D_{\alpha_1} + 2\left(\alpha_1 - \varphi\alpha_2\right)\left[\alpha_2 - \varphi\left(1-\alpha_3\right)\right]D_{\alpha_2} + \left(\alpha_1 - \varphi\alpha_2\right)^2 D_{\alpha_3}}{\left[\alpha_1\left(1-\alpha_3\right)-\alpha_2^2\right]^2} \quad \text{(A.182)}$$

completing the proof. □

Proof of Corollary III:

Our model for the joint distribution of the asset returns and the factor will nest the original specification, $R_t = \mu_t + \varepsilon_t = \delta' Z_{t-1} + \varepsilon_t$ with constant conditional covariance matrix $V$ of $R_t$ given the lagged instruments $Z_{t-1}$, within the extended specification $R_t^* = \mu_t^* + \varepsilon_t^* = \delta^{*\prime} Z_{t-1} + \varepsilon_t^*$ by appending *F* to the list of asset returns. Specifically, we define the extended $N \times 1$ vectors $R_t^* \equiv \left(R_t', F_t\right)' = \left(R_t^1, \ldots, R_t^n, F_t\right)'$, $\mu_t^* \equiv \left(\mu_t', \mu_t^F\right)' = \left(\mu_t^1, \ldots, \mu_t^n, \mu_t^F\right)'$, and $\varepsilon_t^* \equiv \left(\varepsilon_t', \varepsilon_t^F\right)' = \left(\varepsilon_t^1, \ldots, \varepsilon_t^n, \varepsilon_t^F\right)'$. We define the extended $k \times \left(n+1\right)$ matrix

$$\delta^* \equiv \left[\delta \quad \delta_F\right] \quad \text{(A.183)}$$

where $\delta$ is the same $L \times N$ matrix of regression coefficients as before, and $\delta_F$ contains the *L* regression coefficients for *F* given *Z*. We also define the extended $\left(N+1\right)\times\left(N+1\right)$ conditional covariance matrix

$$V^* \equiv \begin{bmatrix} V & V_F \\ V_F' & \sigma_{F|Z}^2 \end{bmatrix} \quad \text{(A.184)}$$

where $V$ is the same $N \times N$ conditional covariance matrix as before, $V_F \equiv Cov(R_t F_t \mid Z_{t-1})$ is $N \times 1$, and $\sigma^2_{F|Z} \equiv Var(F_t \mid Z_{t-1})$ is a scalar. We assume that the unobserved $\varepsilon^*_t$ are independent and identically distributed in this extended model, with mean zero and covariance matrix $V^*$. We note that

$$E(R_t F_t \mid Z_{t-1}) = Cov(R_t F_t \mid Z_{t-1}) + \mu_t \mu^F_t = V_F + \mu_t \mu^F_t \tag{A.185}$$

Using Ferson, Siegel, and Xu (2006) Equation (6) and Corollary to Proposition 2, the maximal correlation portfolio is

$$w'_t \equiv \frac{\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t \mathbf{1}} - \left[\lambda_1 \mu'_t + \lambda_2 E(F_t R'_t \mid Z_{t-1})\right]\left(\Lambda_t - \frac{\Lambda_t \mathbf{1}\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t \mathbf{1}}\right) \tag{A.186}$$

where

$$\Lambda_t \equiv \left(\mu_t \mu'_t + V\right)^{-1} \tag{A.187}$$

$$\lambda_1 \equiv \frac{-\gamma_1\left(\mu^F - \gamma_{\mu F}\right) + \gamma_\mu \gamma_F}{\gamma_\mu\left(\mu^F - \gamma_{\mu F}\right) + \gamma_F\left(\gamma_{\mu\mu} - 1\right)} \tag{A.188}$$

$$\lambda_2 \equiv \frac{-\gamma_1\left(\gamma_{\mu\mu} - 1\right) - \gamma^2_\mu}{\gamma_\mu\left(\mu^F - \gamma_{\mu F}\right) + \gamma_F\left(\gamma_{\mu\mu} - 1\right)} \tag{A.189}$$

Consistent estimates of the model parameters are:

$$\mu^F \equiv \frac{1}{T}\sum_{t=1}^{T} \mu^F_t \tag{A.190}$$

$$\gamma_1 \equiv \frac{1}{T}\sum_{t=1}^{T} \frac{1}{\mathbf{1}'\Lambda_t \mathbf{1}} \tag{A.191}$$

$$\gamma_\mu \equiv \frac{1}{T}\sum_{t=1}^{T} \frac{\mathbf{1}'\Lambda_t \mu_t}{\mathbf{1}'\Lambda_t \mathbf{1}} \tag{A.192}$$

$$\gamma_F \equiv \frac{1}{T}\sum_{t=1}^{T} \frac{\mathbf{1}'\Lambda_t E(R_t F_t \mid Z_{t-1})}{\mathbf{1}'\Lambda_t \mathbf{1}} = \frac{1}{T}\sum_{t=1}^{T} \frac{\mathbf{1}'\Lambda_t V_F + \mathbf{1}'\Lambda_t \mu_t \mu^F_t}{\mathbf{1}'\Lambda_t \mathbf{1}} \tag{A.193}$$

$$\Omega_t \equiv \Lambda_t - \frac{\Lambda_t \mathbf{1}\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t \mathbf{1}} \tag{A.194}$$

$$\gamma_{\mu\mu} \equiv \frac{1}{T}\sum_{t=1}^{T} \mu_t'\Omega_t\mu_t \tag{A.195}$$

$$\gamma_{\mu F} \equiv \frac{1}{T}\sum_{t=1}^{T} \mu_t'\Omega_t E\left(R_t F_t \mid Z_{t-1}\right) = \frac{1}{T}\sum_{t=1}^{T}\left(\mu_t'\Omega_t V_F + \mu_t'\Omega_t\mu_t\mu_t^F\right) \tag{A.196}$$

$$\begin{aligned}\gamma_{FF} &\equiv \frac{1}{T}\sum_{t=1}^{T} E\left(R_t'F_t \mid Z_{t-1}\right)\Omega_t E\left(R_t F_t \mid Z_{t-1}\right) \\ &= \frac{1}{T}\sum_{t=1}^{T}\left[V_F'\Omega_t V_F + 2\mu_t'\Omega_t V_F \mu_t^F + \mu_t'\Omega_t\mu_t\left(\mu_t^F\right)^2\right]\end{aligned} \tag{A.197}$$

Canonical matrices for the squared Sharpe Ratio with respect to the zero-beta rate $\varphi$, $S_p^2 = \left(\mu_p - \varphi\right)^2 / \sigma_p^2$, will initially be developed separately for $\mu_p$ and for $\sigma_p^2$, and then combined. For portfolio returns $R_{p,t} = w_t'R_t$, the portfolio mean return is $\mu_p \equiv \frac{1}{T}\sum_{t=1}^{T} w_t'\mu_t$, and the conditional variance of $R_{p,t}$ is $\sigma_{p,t}^2 \equiv Var\left(R_{p,t} \mid Z_{t-1}\right) = w_t'Vw_t$ using the conditional covariance matrix $V$ of $R_t$ given $Z_{t-1}$. The variance $\sigma_p^2$ of the portfolio $R_p$ may then be expressed as the expected conditional variance plus the variance of the conditional portfolio means $w'\mu_t$, then developed as follows:

$$\begin{aligned}\sigma_p^2 &\equiv \frac{1}{T}\sum_{t=1}^{T}\left[\sigma_{p,t}^2 + \left(w_t'\mu_t\right)^2\right] - \left(\frac{1}{T}\sum_{t=1}^{T} w_t'\mu_t\right)^2 \\ &= \frac{1}{T}\sum_{t=1}^{T}\left[w_t'Vw_t + \left(w_t'\mu_t\right)^2\right] - \mu_p^2 \\ &= \frac{1}{T}\sum_{t=1}^{T} w_t'\left(V + \mu_t\mu_t'\right)w_t - \mu_p^2 \\ &= \frac{1}{T}\sum_{t=1}^{T} w_t'\Lambda_t^{-1}w_t - \mu_p^2\end{aligned} \tag{A.198}$$

For the weights as specified in (A.153) the portfolio mean is

$$\mu_p = \frac{1}{T}\sum_{t=1}^{T} w_t'\mu_t = \gamma_\mu - \lambda_1\gamma_{\mu\mu} - \lambda_2\gamma_{\mu F} \tag{A.199}$$

and the portfolio variance from (A.165) is

$$\sigma_p^2 = \frac{1}{T}\sum_{t=1}^{T} w_t'\Lambda_t^{-1} w_t - \mu_p^2 \tag{A.200}$$

which we will simplify using the fact that

$$w_t'\Lambda_t^{-1} w_t = \frac{1}{\mathbf{1}'\Lambda_t\mathbf{1}} + \left[\lambda_1\mu_t' + \lambda_2 E\left(F_t R_t' \mid Z_{t-1}\right)\right]\left(\Lambda_t - \frac{\Lambda_t\mathbf{1}\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t\mathbf{1}}\right)\left[\lambda_1\mu_t + \lambda_2 E\left(F_t R_t \mid Z_{t-1}\right)\right] \tag{A.201}$$

because

$$\mathbf{1}'\left(\Lambda_t - \frac{\Lambda_t\mathbf{1}\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t\mathbf{1}}\right) = 0 \tag{A.202}$$

and

$$\left(\Lambda_t - \frac{\Lambda_t\mathbf{1}\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t\mathbf{1}}\right)\Lambda_t^{-1}\left(\Lambda_t - \frac{\Lambda_t\mathbf{1}\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t\mathbf{1}}\right) = \left(\Lambda_t - \frac{\Lambda_t\mathbf{1}\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t\mathbf{1}}\right) \tag{A.203}$$

to find that

$$\sigma_p^2 = \gamma_1 + \lambda_1^2\gamma_{\mu\mu} + 2\lambda_1\lambda_2\gamma_{\mu F} + \lambda_2^2\gamma_{FF} - \mu_p^2 \tag{A.204}$$

To facilitate the expansions that follow, we define the following scalars:

$$\theta_t \equiv \frac{1}{\mathbf{1}'\Lambda_t\mathbf{1}} \tag{A.205}$$

$$\theta_{\mu,t} \equiv \mathbf{1}'\Lambda_t\mu_t \tag{A.206}$$

$$\theta_{\mu\mu,t} \equiv \mu_t'\Lambda_t\mu_t \tag{A.207}$$

$$\theta_{F,t} \equiv \mathbf{1}'\Lambda_t V_F \tag{A.208}$$

$$\theta_{FF,t} \equiv V_F' \Lambda_t V_F \tag{A.209}$$

$$\theta_{\mu F,t} \equiv \mu_t' \Lambda_t V_F \tag{A.210}$$

$$\theta_{\mu\Omega\mu,t} \equiv \mu_t' \Omega_t \mu_t = \mu_t' \left( \Lambda_t - \frac{\Lambda_t \mathbf{1}\mathbf{1}' \Lambda_t}{\mathbf{1}' \Lambda_t \mathbf{1}} \right) \mu_t = \theta_{\mu\mu,t} - \theta_t \theta_{\mu,t}^2 \tag{A.211}$$

$$\theta_{\mu\Omega F,t} \equiv \mu_t' \Omega_t V_F = \mu_t' \left( \Lambda_t - \frac{\Lambda_t \mathbf{1}\mathbf{1}' \Lambda_t}{\mathbf{1}' \Lambda_t \mathbf{1}} \right) V_F = \theta_{\mu F,t} - \theta_t \theta_{\mu,t} \theta_{F,t} \tag{A.212}$$

$$\theta_{F\Omega F,t} \equiv V_F' \Omega_t V_F = V_F' \left( \Lambda_t - \frac{\Lambda_t \mathbf{1}\mathbf{1}' \Lambda_t}{\mathbf{1}' \Lambda_t \mathbf{1}} \right) V_F = \theta_{FF,t} - \theta_t \theta_{F,t}^2 \tag{A.213}$$

We may then write the consistent estimators as:

$$\gamma_1 = \frac{1}{T} \sum_{t=1}^{T} \theta_t \tag{A.214}$$

$$\gamma_\mu = \frac{1}{T} \sum_{t=1}^{T} \theta_t \theta_{\mu,t} \tag{A.215}$$

$$\gamma_F = \frac{1}{T} \sum_{t=1}^{T} \theta_t \left( \theta_{F,t} + \theta_{\mu,t} \mu_t^F \right) \tag{A.216}$$

$$\gamma_{\mu\mu} = \frac{1}{T} \sum_{t=1}^{T} \theta_{\mu\Omega\mu,t} \tag{A.217}$$

$$\gamma_{\mu F} = \frac{1}{T} \sum_{t=1}^{T} \left( \mu_t' \Omega_t V_F + \mu_t' \Omega_t \mu_t \mu_t^F \right) = \frac{1}{T} \sum_{t=1}^{T} \left( \theta_{\mu\Omega F,t} + \theta_{\mu\Omega\mu,t} \mu_t^F \right) \tag{A.218}$$

$$\begin{aligned} \gamma_{FF} &= \frac{1}{T} \sum_{t=1}^{T} \left[ V_F' \Omega_t V_F + 2\mu_t' \Omega_t V_F \mu_t^F + \mu_t' \Omega_t \mu_t \left( \mu_t^F \right)^2 \right] \\ &= \frac{1}{T} \sum_{t=1}^{T} \left[ \theta_{F\Omega F,t} + 2\theta_{\mu\Omega F,t} \mu_t^F + \theta_{\mu\Omega\mu,t} \left( \mu_t^F \right)^2 \right] \end{aligned} \tag{A.219}$$

We begin by expanding

$$\hat{\theta}_t = \frac{1}{\mathbf{1}' \hat{\Lambda} \mathbf{1}} = \theta_t + 2\theta_t \theta_{\mu,t} \mathbf{1}' \Lambda_t \left( \hat{\mu}_t - \mu_t \right) + tr\left[ \theta_t \Lambda_t \mathbf{1}\mathbf{1}' \Lambda_t \left( \hat{V} - V \right) \right] + O_p\left( 1/T \right) \tag{A.220}$$

which follows from (A.27). We continue by using (A.25) to find

$$\begin{aligned}\hat{\theta}_{\mu,t} &= \mathbf{1}'\hat{\Lambda}_t\hat{\mu}_t = \theta_{\mu,t} + \mathbf{1}'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) + \mathbf{1}'\left(\hat{\Lambda}_t - \Lambda_t\right)\mu_t + O_p\left(1/T\right) \\ &= \theta_{\mu,t} + \mathbf{1}'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) \\ &\quad + \mathbf{1}'\left[-\Lambda_t\mu_t\left(\hat{\mu}_t - \mu_t\right)'\Lambda_t - \Lambda_t\left(\hat{\mu}_t - \mu_t\right)\mu_t'\Lambda_t - \Lambda_t\left(\hat{V} - V\right)\Lambda_t\right]\mu_t + O_p\left(1/T\right) \\ &= \theta_{\mu,t} + \left(\mathbf{1}' - \theta_{\mu\mu,t}\mathbf{1}' - \theta_{\mu,t}\mu_t'\right)\Lambda_t\left(\hat{\mu}_t - \mu_t\right) + tr\left[-\Lambda_t\mu_t\mathbf{1}'\Lambda_t\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)\end{aligned} \tag{A.221}$$

$$\begin{aligned}\hat{\theta}_{\mu\mu,t} &= \hat{\mu}_t'\hat{\Lambda}_t\hat{\mu}_t = \theta_{\mu\mu,t} + 2\mu_t'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) \\ &\quad + \mu_t'\left[-\Lambda_t\mu_t\left(\hat{\mu}_t - \mu_t\right)'\Lambda_t - \Lambda_t\left(\hat{\mu}_t - \mu_t\right)\mu_t'\Lambda_t - \Lambda_t\left(\hat{V} - V\right)\Lambda_t\right]\mu_t + O_p\left(1/T\right) \\ &= \theta_{\mu\mu,t} + \left(2\mu_t' - 2\theta_{\mu\mu,t}\mu_t'\right)\Lambda_t\left(\hat{\mu}_t - \mu_t\right) + tr\left[-\Lambda_t\mu_t\mu_t'\Lambda_t\left(\hat{V} - V\right)\right] + O_p\left(1/T\right)\end{aligned} \tag{A.222}$$

$$\begin{aligned}\hat{\theta}_{F,t} &= \mathbf{1}'\hat{\Lambda}_t\hat{V}_F = \theta_{F,t} + \mathbf{1}'\left(\hat{\Lambda}_t - \Lambda_t\right)V_F + \mathbf{1}'\Lambda_t\left(\hat{V}_F - V_F\right) + O_p\left(1/T\right) \\ &= \theta_{F,t} + \mathbf{1}'\left[-\Lambda_t\mu_t\left(\hat{\mu}_t - \mu_t\right)'\Lambda_t - \Lambda_t\left(\hat{\mu}_t - \mu_t\right)\mu_t'\Lambda_t - \Lambda_t\left(\hat{V} - V\right)\Lambda_t\right]V_F \\ &\quad + \mathbf{1}'\Lambda_t\left(\hat{V}_F - V_F\right) + O_p\left(1/T\right) \\ &= \theta_{F,t} + \left(-\theta_{\mu,t}V_F' - \theta_{\mu F,t}\mathbf{1}'\right)\Lambda_t\left(\hat{\mu}_t - \mu_t\right) + tr\left[-\Lambda_t V_F\mathbf{1}'\Lambda_t\left(\hat{V} - V\right)\right] \\ &\quad + \mathbf{1}'\Lambda_t\left(\hat{V}_F - V_F\right) + O_p\left(1/T\right)\end{aligned} \tag{A.223}$$

$$\begin{aligned}\hat{\theta}_{FF,t} &= \hat{V}_F'\hat{\Lambda}_t\hat{V}_F = \theta_{FF,t} + 2V_F'\Lambda_t\left(\hat{V}_F - V_F\right) + V_F'\left(\hat{\Lambda}_t - \Lambda_t\right)V_F + O_p\left(1/T\right) \\ &= \theta_{FF,t} + 2V_F'\Lambda_t\left(\hat{V}_F - V_F\right) \\ &\quad + V_F'\left[-\Lambda_t\mu_t\left(\hat{\mu}_t - \mu_t\right)'\Lambda_t - \Lambda_t\left(\hat{\mu}_t - \mu_t\right)\mu_t'\Lambda_t - \Lambda_t\left(\hat{V} - V\right)\Lambda_t\right]V_F + O_p\left(1/T\right) \\ &= \theta_{FF,t} - 2\theta_{\mu F,t}V_F'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) + tr\left[-\Lambda_t V_F V_F'\Lambda_t\left(\hat{V} - V\right)\right] \\ &\quad + 2V_F'\Lambda_t\left(\hat{V}_F - V_F\right) + O_p\left(1/T\right)\end{aligned} \tag{A.224}$$

$$\begin{aligned}\hat{\theta}_{\mu F,t} &= \hat{\mu}_t'\hat{\Lambda}_t\hat{V}_F = \theta_{\mu F,t} + \mu_t'\left(\hat{\Lambda}_t - \Lambda_t\right)V_F + V_F'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) + \mu_t'\Lambda_t\left(\hat{V}_F - V_F\right) + O_p\left(1/T\right) \\ &= \theta_{\mu F,t} + \mu_t'\left[-\Lambda_t\mu_t\left(\hat{\mu}_t - \mu_t\right)'\Lambda_t - \Lambda_t\left(\hat{\mu}_t - \mu_t\right)\mu_t'\Lambda_t - \Lambda_t\left(\hat{V} - V\right)\Lambda_t\right]V_F \\ &\quad + V_F'\Lambda_t\left(\hat{\mu}_t - \mu_t\right) + \mu_t'\Lambda_t\left(\hat{V}_F - V_F\right) + O_p\left(1/T\right) \\ &= \theta_{\mu F,t} + \left(-\theta_{\mu F,t}\mu_t' + V_F' - \theta_{\mu\mu,t}V_F'\right)\Lambda_t\left(\hat{\mu}_t - \mu_t\right) + tr\left[-\Lambda_t V_F \mu_t'\Lambda_t\left(\hat{V} - V\right)\right] \\ &\quad + \mu_t'\Lambda_t\left(\hat{V}_F - V_F\right) + O_p\left(1/T\right)\end{aligned} \tag{A.225}$$

To define the matrices to be used with the Theorem I applied to the extended specification, the following vector and matrix operators will be helpful. The first operator simply appends an additional scalar to a column vector, increasing its dimension from $n$ to $n+1$:

$$\mathcal{L}\left[\left(a_1,\ldots,a_n\right)',b\right] = \left(a_1,\ldots,a_n,b\right)' \tag{A.226}$$

while the second operator enlarges a given $n \times n$ matrix *M*, along with a $1 \times n$ row vector *X*, to size $\left(n+1\right)\times\left(n+1\right)$ by placing *X* below *M* and inserting a column of $n+1$ zeros at the right , as follows:

$$\mathcal{M}\left(M,X\right) \equiv \begin{bmatrix} M & 0 \\ X & 0 \end{bmatrix} \tag{A.227}$$

which will allow us to write, for example,

$$\begin{aligned} tr\left[\mathcal{M}\left(M,X\right)\left(\hat{V}^* - V^*\right)\right] &= tr\left(\begin{bmatrix} M & 0 \\ X & 0 \end{bmatrix}\begin{bmatrix} \hat{V} - V & \hat{V}_F - V_F \\ \hat{V}_F' - \hat{V}_F & \hat{\sigma}^2_{F|Z} - \sigma^2_{F|Z} \end{bmatrix}\right) \\ &= tr\begin{bmatrix} M\left(\hat{V} - V\right) & M\left(\hat{V}_F - V_F\right) \\ X\left(\hat{V} - V\right) & X\left(\hat{V}_F - V_F\right) \end{bmatrix} \\ &= tr\left[M\left(\hat{V} - V\right)\right] + X\left(\hat{V}_F - V_F\right) \end{aligned} \tag{A.228}$$

We are now ready to specify the matrices for the Theorem I, which we will define in sequence. We begin by defining the matrices for the expansion of $\hat{\mu}^F = \mu^F + \frac{1}{T}\sum_{t=1}^{T}\left(\hat{\mu}_t^F - \mu_t^F\right)$, which has no $\hat{V}^* - V^*$ term, as follows:

$$C^F \equiv A^{-1}\sum_{t=1}^{T} Z_{t-1}\left(C_t^F\right)' \quad D^F \equiv \mathcal{M}\left(0,0\right) \tag{A.229}$$

where $C_t^F \equiv \mathcal{L}\left(0,1/T\right)$.

We next define $C'_{\theta t}$ and $D_{\theta t}$ (flattening the subscript levels for readability) for $\theta_t \equiv \dfrac{1}{\mathbf{1}'\Lambda_t\mathbf{1}}$, for which we append zeros (because there are no factor terms in $\theta_t$) to find

$$C_{\theta t} \equiv \mathcal{L}\left(2\theta_t\theta_{\mu,t}\mathbf{1}'\Lambda_t / T,0\right) \quad D_{\theta t} \equiv \mathcal{M}\left(\theta_t\Lambda_t\mathbf{1}\mathbf{1}'\Lambda_t,0\right) \tag{A.230}$$

Proceeding similarly, we find

$$C_{\theta\mu t} \equiv \mathcal{L}\left[\left(\mathbf{1}' - \theta_{\mu\mu,t}\mathbf{1}' - \theta_{\mu,t}\mu'_t\right)\Lambda_t / T,0\right] \quad D_{\theta\mu t} \equiv \mathcal{M}\left(-\Lambda_t\mu_t\mathbf{1}'\Lambda_t,0\right) \tag{A.231}$$

$$C_{\theta\mu\mu t} \equiv \mathcal{L}\left[\left(2\mu'_t - 2\theta_{\mu\mu,t}\mu'_t\right)\Lambda_t / T,0\right] \quad D_{\theta\mu\mu t} \equiv \mathcal{M}\left(-\Lambda_t\mu_t\mu'_t\Lambda_t,0\right) \tag{A.232}$$

For $\hat{\theta}_{F,t}$, we use (A.94) to create the extended *D* matrix as follows:

$$C_{\theta Ft} \equiv \mathcal{L}\left[\left(-\theta_{\mu,t}V'_F - \theta_{\mu F,t}\mathbf{1}'\right)\Lambda_t / T,0\right] \quad D_{\theta Ft} \equiv \mathcal{M}\left(-\Lambda_t V_F\mathbf{1}'\Lambda_t,\mathbf{1}'\Lambda_t\right) \tag{A.233}$$

and similarly we find:

$$C_{\theta FFt} \equiv \mathcal{L}\left(-2\theta_{\mu F,t}V'_F\Lambda_t / T,0\right) \quad D_{\theta FFt} \equiv \mathcal{M}\left(-\Lambda_t V_F V'_F\Lambda_t,2V'_F\Lambda_t\right) \tag{A.234}$$

$$C_{\theta\mu Ft} \equiv \mathcal{L}\left[\left(-\theta_{\mu F,t}\mu'_t + V'_F - \theta_{\mu\mu,t}V'_F\right)\Lambda_t / T,0\right] \quad D_{\theta\mu Ft} \equiv \mathcal{M}\left(-\Lambda_t V_F\mu'_t\Lambda_t,\mu'_t\Lambda_t\right) \tag{A.235}$$

Expanding and using linearity of the Theorem I matrices, we also find

$$C_{\theta\mu\Omega\mu t} = C_{\theta\mu\mu t} - \theta_{\mu,t}^2 C_{\theta t} - 2\theta_t\theta_{\mu,t}C_{\theta\mu t} \quad D_{\theta\mu\Omega\mu t} = D_{\theta\mu\mu t} - \theta_{\mu,t}^2 D_{\theta t} - 2\theta_t\theta_{\mu,t}D_{\theta\mu t} \tag{A.236}$$

$$\begin{aligned} C_{\theta\mu\Omega Ft} &= C_{\theta\mu Ft} - \theta_{\mu,t}\theta_{F,t}C_{\theta t} - \theta_t\theta_{F,t}C_{\theta\mu t} - \theta_t\theta_{\mu,t}C_{\theta Ft} \\ D_{\theta\mu\Omega Ft} &= D_{\theta\mu Ft} - \theta_{\mu,t}\theta_{F,t}D_{\theta t} - \theta_t\theta_{F,t}D_{\theta\mu t} - \theta_t\theta_{\mu,t}D_{\theta Ft} \end{aligned} \tag{A.237}$$

$$C_{\theta F\Omega Ft} = C_{\theta FFt} - \theta_{F,t}^2 C_{\theta t} - 2\theta_t \theta_{F,t} C_{\theta Ft}$$
$$D_{\theta F\Omega Ft} = D_{\theta FFt} - \theta_{F,t}^2 D_{\theta t} - 2\theta_t \theta_{F,t} D_{\theta Ft} \tag{A.238}$$

From (A.181) for $\gamma_1$ along with (A.187) we may define

$$C_{\gamma 1} = A^{-1}\sum\nolimits_{t=1}^{T} Z_{t-1} C'_{\theta t} \quad D_{\gamma 1} = \frac{1}{T}\sum\nolimits_{t=1}^{T} D_{\theta t} \tag{A.239}$$

and similarly,

$$C_{\gamma\mu} = A^{-1}\sum\nolimits_{t=1}^{T} Z_{t-1}\left(\theta_t C_{\theta\mu t} + \theta_{\mu,t} C_{\theta t}\right)' \quad D_{\gamma\mu} = \frac{1}{T}\sum\nolimits_{t=1}^{T}\left(\theta_t D_{\theta\mu t} + \theta_{\mu,t} D_{\theta t}\right) \tag{A.240}$$

For $\gamma_F$ and other expansions involving $\mu_t^F$ we will need to use the Theorem I matrices $C_t^F$ and $D_t^F$ for $\hat{\mu}_F^t$ as defined by (A.196)

$$C_{\gamma F} = A^{-1}\sum\nolimits_{t=1}^{T} Z_{t-1}\left(\theta_t C_{\theta Ft} + \theta_{F,t} C_{\theta t} + \theta_{\mu,t}\mu_t^F C_{\theta t} + \theta_t \mu_t^F C_{\theta\mu t} + \theta_t \theta_{\mu,t} C_t^F\right)'$$
$$D_{\gamma F} = \frac{1}{T}\sum\nolimits_{t=1}^{T}\left(\theta_t D_{\theta Ft} + \theta_{F,t} D_{\theta t} + \theta_{\mu,t}\mu_t^F D_{\theta t} + \theta_t \mu_t^F D_{\theta\mu t} + \theta_t \theta_{\mu,t} D_t^F\right) \tag{A.241}$$

$$C_{\gamma\mu\mu} = A^{-1}\sum\nolimits_{t=1}^{T} Z_{t-1} C'_{\theta\mu\Omega\mu t} \quad D_{\gamma\mu\mu} = \frac{1}{T}\sum\nolimits_{t=1}^{T} D_{\theta\mu\Omega\mu t} \tag{A.242}$$

$$C_{\gamma\mu F} = A^{-1}\sum\nolimits_{t=1}^{T} Z_{t-1}\left(C_{\theta\mu\Omega Ft} + \mu_t^F C_{\theta\mu\Omega\mu t} + \theta_{\mu\Omega\mu,t} C_t^F\right)'$$
$$D_{\gamma\mu F} = \frac{1}{T}\sum\nolimits_{t=1}^{T}\left(D_{\theta\mu\Omega Ft} + \mu_t^F D_{\theta\mu\Omega\mu t} + \theta_{\mu\Omega\mu,t} D_t^F\right) \tag{A.243}$$

$$C_{\gamma FF} = A^{-1}\sum\nolimits_{t=1}^{T} Z_{t-1}\left[C_{\theta F\Omega Ft} + 2\mu_t^F C_{\theta\mu\Omega Ft} + 2\theta_{\mu\Omega F,t} C_t^F + \left(\mu_t^F\right)^2 C_{\theta\mu\Omega\mu t} + 2\theta_{\mu\Omega\mu,t}\mu_t^F C_t^F\right]'$$
$$D_{\gamma FF} = \frac{1}{T}\sum\nolimits_{t=1}^{T}\left[D_{\theta F\Omega Ft} + 2\mu_t^F D_{\theta\mu\Omega Ft} + 2\theta_{\mu\Omega F,t} D_t^F + \left(\mu_t^F\right)^2 D_{\theta\mu\Omega\mu t} + 2\theta_{\mu\Omega\mu,t}\mu_t^F D_t^F\right] \tag{A.244}$$

We are now ready to expand $\lambda_1$, $\lambda_2$, $\mu_p$, and $\sigma_p^2$:

$$
\begin{aligned}
C_{\lambda 1} &\equiv \frac{-\left(\mu^F-\gamma_{\mu F}\right)C_{\gamma 1}-\gamma_1\left(C^F-C_{\gamma\mu F}\right)+\gamma_F C_{\gamma\mu}+\gamma_\mu C_{\gamma F}}{\gamma_\mu\left(\mu^F-\gamma_{\mu F}\right)+\gamma_F\left(\gamma_{\mu\mu}-1\right)} \\
&\quad -\frac{\left[-\gamma_1\left(\mu^F-\gamma_{\mu F}\right)+\gamma_\mu\gamma_F\right]\left[\left(\mu^F-\gamma_{\mu F}\right)C_{\gamma\mu}+\gamma_\mu\left(C^F-C_{\gamma\mu F}\right)+\left(\gamma_{\mu\mu}-1\right)C_{\gamma F}+\gamma_F C_{\gamma\mu\mu}\right]}{\left[\gamma_\mu\left(\mu^F-\gamma_{\mu F}\right)+\gamma_F\left(\gamma_{\mu\mu}-1\right)\right]^2} \\
D_{\lambda 1} &\equiv \frac{-\left(\mu^F-\gamma_{\mu F}\right)D_{\gamma 1}-\gamma_1\left(D^F-D_{\gamma\mu F}\right)+\gamma_F D_{\gamma\mu}+\gamma_\mu D_{\gamma F}}{\gamma_\mu\left(\mu^F-\gamma_{\mu F}\right)+\gamma_F\left(\gamma_{\mu\mu}-1\right)} \\
&\quad -\frac{\left[-\gamma_1\left(\mu^F-\gamma_{\mu F}\right)+\gamma_\mu\gamma_F\right]\left[\left(\mu^F-\gamma_{\mu F}\right)D_{\gamma\mu}+\gamma_\mu\left(D^F-D_{\gamma\mu F}\right)+\left(\gamma_{\mu\mu}-1\right)D_{\gamma F}+\gamma_F D_{\gamma\mu\mu}\right]}{\left[\gamma_\mu\left(\mu^F-\gamma_{\mu F}\right)+\gamma_F\left(\gamma_{\mu\mu}-1\right)\right]^2}
\end{aligned}
\tag{A.245}
$$

$$
\begin{aligned}
C_{\lambda 2} &\equiv \frac{-\gamma_1 C_{\gamma\mu\mu}-\left(\gamma_{\mu\mu}-1\right)C_{\gamma 1}-2\gamma_\mu C_{\gamma\mu}}{\gamma_\mu\left(\mu^F-\gamma_{\mu F}\right)+\gamma_F\left(\gamma_{\mu\mu}-1\right)} \\
&\quad -\frac{\left[-\gamma_1\left(\gamma_{\mu\mu}-1\right)-\gamma_\mu^2\right]\left[\left(\mu^F-\gamma_{\mu F}\right)C_{\gamma\mu}+\gamma_\mu\left(C^F-C_{\gamma\mu F}\right)+\left(\gamma_{\mu\mu}-1\right)C_{\gamma F}+\gamma_F C_{\gamma\mu\mu}\right]}{\left[\gamma_\mu\left(\mu^F-\gamma_{\mu F}\right)+\gamma_F\left(\gamma_{\mu\mu}-1\right)\right]^2} \\
D_{\lambda 2} &\equiv \frac{-\gamma_1 D_{\gamma\mu\mu}-\left(\gamma_{\mu\mu}-1\right)D_{\gamma 1}-2\gamma_\mu D_{\gamma\mu}}{\gamma_\mu\left(\mu^F-\gamma_{\mu F}\right)+\gamma_F\left(\gamma_{\mu\mu}-1\right)} \\
&\quad -\frac{\left[-\gamma_1\left(\gamma_{\mu\mu}-1\right)-\gamma_\mu^2\right]\left[\left(\mu^F-\gamma_{\mu F}\right)D_{\gamma\mu}+\gamma_\mu\left(D^F-D_{\gamma\mu F}\right)+\left(\gamma_{\mu\mu}-1\right)D_{\gamma F}+\gamma_F D_{\gamma\mu\mu}\right]}{\left[\gamma_\mu\left(\mu^F-\gamma_{\mu F}\right)+\gamma_F\left(\gamma_{\mu\mu}-1\right)\right]^2}
\end{aligned}
\tag{A.246}
$$

From (A.166) and (A.171) we find

$$
\begin{aligned}
C_{\mu p} &\equiv C_{\gamma\mu}-\gamma_{\mu\mu}C_{\lambda 1}-\lambda_1 C_{\gamma\mu\mu}-\gamma_{\mu F}C_{\lambda 2}-\lambda_2 C_{\gamma\mu F} \\
D_{\mu p} &\equiv D_{\gamma\mu}-\gamma_{\mu\mu}D_{\lambda 1}-\lambda_1 D_{\gamma\mu\mu}-\gamma_{\mu F}D_{\lambda 2}-\lambda_2 D_{\gamma\mu F}
\end{aligned}
\tag{A.247}
$$

$$
\begin{aligned}
C_{\sigma p 2} &\equiv C_{\gamma 1}+2\gamma_{\mu\mu}\lambda_1 C_{\lambda 1}+\lambda_1^2 C_{\gamma\mu\mu}+2\lambda_2\gamma_{\mu F}C_{\lambda 1}+2\lambda_1\gamma_{\mu F}C_{\lambda 2} \\
&\quad +2\lambda_1\lambda_2 C_{\gamma\mu F}+2\gamma_{FF}\lambda_2 C_{\lambda 2}+\lambda_2^2 C_{\gamma FF}-2\mu_p C_{\mu p} \\
D_{\sigma p 2} &\equiv D_{\gamma 1}+2\gamma_{\mu\mu}\lambda_1 D_{\lambda 1}+\lambda_1^2 D_{\gamma\mu\mu}+2\lambda_2\gamma_{\mu F}D_{\lambda 1}+2\lambda_1\gamma_{\mu F}D_{\lambda 2} \\
&\quad +2\lambda_1\lambda_2 D_{\gamma\mu F}+2\gamma_{FF}\lambda_2 D_{\lambda 2}+\lambda_2^2 D_{\gamma FF}-2\mu_p D_{\mu p}
\end{aligned}
\tag{A.248}
$$

and, finally, we find the canonical matrices for the squared Sharpe Ratio $S_p^2=\left(\mu_p-\varphi\right)^2/\sigma_p^2$ to be

$$C \equiv \frac{2\sigma_p^2\left(\mu_p-\varphi\right)C_{\mu p}-\left(\mu_p-\varphi\right)^2 C_{\sigma p2}}{\sigma_p^4}$$

$$D \equiv \frac{2\sigma_p^2\left(\mu_p-\varphi\right)D_{\mu p}-\left(\mu_p-\varphi\right)^2 D_{\sigma p2}}{\sigma_p^4} \qquad \text{(A.249)}$$

completing the proof. □

Proof of Corollary IIII:

The optimum zero-beta rate is $\phi = \frac{b-b^*}{c-c^*}$. Expanding this equation we obtain:

$$\hat{\phi}-\phi = \frac{\partial\phi}{\partial b}(\hat{b}-b) + \frac{\partial\phi}{\partial b^*}(\hat{b}^*-b^*) + \frac{\partial\phi}{\partial c}(\hat{c}-c) + \frac{\partial\phi}{\partial c^*}(\hat{c}^*-c^*) + O(\tfrac{1}{T}), \qquad \text{(A.250)}$$

where

$$\frac{\partial\phi}{\partial b} = \frac{1}{c-c^*},$$

$$\frac{\partial\phi}{\partial b^*} = -\frac{1}{c-c^*},$$

$$\frac{\partial\phi}{\partial c} = -\frac{b-b^*}{(c-c^*)^2},$$

$$\frac{\partial\phi}{\partial c^*} = \frac{b-b^*}{(c-c^*)^2}.$$

Now, $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{a}^*$, $\hat{b}^*$ and $\hat{c}^*$ can be similarly expanded with canonical matrices $C_a$, $D_a$, $C_b$, $D_b$, $C_c$, $D_c$, $C_{a^*}$, $D_{a^*}$, $C_{b^*}$, $D_{b^*}$, $C_{c^*}$, and $D_{c^*}$. Since $\phi$ is a continuous differentiable function of $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{a}^*$, $\hat{b}^*$ and $\hat{c}^*$, $\hat{\phi}$ can be written in the same format. To determine the canonical matrices, based on the expansion of $\hat{\phi}-\phi$, we obtain

$$C_\phi = \frac{1}{c-c^*}C_b - \frac{1}{c-c^*}C_{b^*} - \frac{b-b^*}{(c-c^*)^2}C_c + \frac{b-b^*}{(c-c^*)^2}C_{c^*}.$$

$$D_\phi = \frac{1}{c-c^*}D_b - \frac{1}{c-c^*}D_{b^*} - \frac{b-b^*}{(c-c^*)^2}D_c + \frac{b-b^*}{(c-c^*)^2}D_{c^*}. \qquad \text{(A.251)}$$

Plug the optimum zero-beta rate to the difference between squared Sharpe ratios (Barillas and Shanken (2017)),

$$BS = (a-a^*) - 2(b-b^*)\phi + (c-c^*)\phi^2. \qquad \text{(A.252)}$$

Expand $BS$,

$$\widehat{BS} - BS = (\hat{a} - a) - (\hat{a}^* - a^*) - 2\phi(\hat{b} - b) + 2\phi(\hat{b}^* - b^*) - 2(b - b^*)(\hat{\phi} - \phi)$$

$$+\phi^2(\hat{c} - c) - \phi^2(\hat{c}^* - c^*) + 2\phi(c - c^*)(\hat{\phi} - \phi) + O(\tfrac{1}{T}) . \quad \text{(A.253)}$$

BS can be can be expanded in the same way as $\hat{\theta}$ in the Theorem I, and the canonical matrices for BS test are:

$$C_{BS} = C_a - C_{a^*} - 2\phi(C_b - C_{b^*}) - 2(b - b^*)C_\phi + \phi^2(C_c - C_{c^*}) + 2\phi(c - c^*)C_\phi ,$$

$$D_{BS} = D_a - D_{a^*} - 2\phi(D_b - D_{b^*}) - 2(b - b^*)D_\phi + \phi^2(D_c - D_{c^*}) + 2\phi(c - c^*)D_\phi . \quad \text{(A.254)}$$

The Theorem I applies to this test using $C_{BS}$ and $D_{BS}$. QED.

## 2. Asymptotic Results with Time-varying covariances (Preliminary)

We present some analysis for how to extend the Theorem I to the case with conditional heteroscedasticity The estimates for $C$ and $D$ for the Squared Sharpe Ratio with Constant Conditional Covariance are $C_t' = \frac{2}{T}\mu_t' V^{-1}$ and $D = -\frac{1}{T}\sum_{t=1}^{T} V^{-1}\mu_t\mu_t' V^{-1} = -\frac{1}{T}V^{-1}\left(\sum_{t=1}^{T}\mu_t\mu_t'\right)V^{-1}$

Theorem I with Time-Varying Conditional Covariance: Consider a scalar estimator of the form $\hat{\theta} \cong \theta + \sum_{t=1}^{T} C_t'(\hat{\mu}_t - \mu_t) + \sum_{t=1}^{T} tr\left[D_t\left(\hat{V}_t - V_t\right)\right]$ where $C_1,\ldots,C_T$ are $N\times 1$ vectors, and the $D_t$ are $N \times N$ matrices with $D$ denoting their sum. Define the $L \times N$ matrix $C \equiv A^{-1}\sum_{t=1}^{T} Z_{t-1}C_t'$ , where $A \equiv \frac{1}{T}\sum_{t=1}^{T} Z_{t-1}Z_{t-1}'$ , and let $\tau_t$ denote the time-$t$ $N \times N$ matrix of residuals obtained by regressing $\hat{V}_t - \hat{V}$ on $Z_{t-1}$. The asymptotic variance of $\hat{\theta}$ may be estimated as:

$$AVAR\left(\hat{\theta}\right) = \frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{i=1}^{T}\left(v_{it} - \bar{v}_{t}\right)^2\right] \quad \text{(A.255)}$$

with $v_{it} \equiv Z'_{t-1}\hat{C}\hat{\varepsilon}_i + \hat{\varepsilon}'_i\hat{D}\hat{\varepsilon}_i + T\, tr\left(\hat{D}_t\hat{\tau}_i\right)$ and $\overline{v}_t \equiv \sum_{i=1}^{T} v_{it} / T$, where $\hat{C}$, $\hat{D}_t$, and $\hat{D}$ are consistent estimates for the canonical matrices $C$, $D_t$, and $D$ respectively.

Proof: We begin with an identity:

$$\hat{\mu}'_t\hat{V}^{-1}\hat{\mu}_t = \left[\mu_t + (\hat{\mu}_t - \mu_t)\right]'\left[V + \left(\hat{V} - V\right)\right]^{-1}\left[\mu_t + (\hat{\mu}_t - \mu_t)\right] \tag{A.256}$$

and continue with a first-order approximation:

$$\begin{aligned}
\hat{\mu}'_t\hat{V}^{-1}\hat{\mu}_t &\cong 2\mu'_tV^{-1}(\hat{\mu}_t - \mu_t) + \mu'_t\left[V + \left(\hat{V} - V\right)\right]^{-1}\mu_t \\
&= 2\mu'_tV^{-1}(\hat{\mu}_t - \mu_t) + \mu'_tV^{-1}\left[I + \left(\hat{V} - V\right)V^{-1}\right]^{-1}\mu_t \\
&\cong 2\mu'_tV^{-1}(\hat{\mu}_t - \mu_t) + \mu'_tV^{-1}\left[I - \left(\hat{V} - V\right)V^{-1}\right]\mu_t \\
&= 2\mu'_tV^{-1}(\hat{\mu}_t - \mu_t) + \mu'_t\left[V^{-1} - V^{-1}\left(\hat{V} - V\right)V^{-1}\right]\mu_t \\
&= \mu_tV^{-1}\mu_t + 2\mu'_tV^{-1}(\hat{\mu}_t - \mu_t) - \mu'_tV^{-1}\left(\hat{V} - V\right)V^{-1}\mu_t \\
&= \mu_tV^{-1}\mu_t + 2\mu'_tV^{-1}(\hat{\mu}_t - \mu_t) - tr\left[V^{-1}\mu_t\mu'_tV^{-1}\left(\hat{V} - V\right)\right]
\end{aligned} \tag{A.257}$$

The estimated average Sharpe Ratio may therefore be written as

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}\hat{\mu}'_t\hat{V}^{-1}\hat{\mu}_t \\
&\quad\cong \frac{1}{T}\sum_{t=1}^{T}\mu_tV^{-1}\mu_t + \frac{1}{T}\sum_{t=1}^{T}2\mu'_tV^{-1}(\hat{\mu}_t - \mu_t) - \frac{1}{T}\sum_{t=1}^{T}tr\left[V^{-1}\mu_t\mu'_tV^{-1}\left(\hat{V} - V\right)\right]
\end{aligned} \tag{A.258}$$

which is in the form of the Theorem I

$$\hat{\theta} \cong \theta + \sum_{t=1}^{T}C'_t(\hat{\mu}_t - \mu_t) + tr\left[D\left(\hat{V} - V\right)\right] \tag{A.259}$$

with $C'_t = \frac{2}{T}\mu'_tV^{-1}$ and $D = -\frac{1}{T}\sum_{t=1}^{T}V^{-1}\mu_t\mu'_tV^{-1} = -\frac{1}{T}V^{-1}\left(\sum_{t=1}^{T}\mu_t\mu'_t\right)V^{-1}$

Recall that

$$\hat{\theta} \cong \theta + \sum_{t=1}^{T} C_t'\left(\hat{\mu}_t - \mu_t\right) + \sum_{t=1}^{T} tr\left[D_t\left(\hat{V}_t - V_t\right)\right] \tag{A.260}$$

We begin with the middle term on the right, and proceed as in the constant conditional variance case, to find:

$$\sum_{t=1}^{T} C_t'\left(\hat{\mu}_t - \mu_t\right) = \sum_{t=1}^{T} C_t'\left(\frac{1}{T}\sum_{i=1}^{T} \varepsilon_i Z_{i-1}'\right) A^{-1} Z_{t-1} \tag{A.261}$$

Using the $L \times N$ matrix (without subscript) $C = A^{-1}\sum_{t=1}^{T} Z_{t-1}C_t'$, and using matrix commutativity within the trace operator, this becomes

$$\begin{aligned} \sum_{t=1}^{T} C_t'\left(\hat{\mu}_t - \mu_t\right) &= tr\left[\sum_{t=1}^{T} C_t'\left(\frac{1}{T}\sum_{i=1}^{T} \varepsilon_i Z_{i-1}'\right) A^{-1} Z_{t-1}\right] \\ &= tr\left[\sum_{t=1}^{T}\left(\frac{1}{T}\sum_{i=1}^{T} \varepsilon_i Z_{i-1}'\right) A^{-1} Z_{t-1} C_t'\right] \\ &= tr\left[\left(\frac{1}{T}\sum_{i=1}^{T} \varepsilon_i Z_{i-1}'\right) A^{-1} \sum_{t=1}^{T} Z_{t-1} C_t'\right] = \frac{1}{T} tr\left(\sum_{i=1}^{T} \varepsilon_i Z_{i-1}' C\right) \\ &= \frac{1}{T}\sum_{t=1}^{T} Z_{t-1}' C \varepsilon_t \end{aligned} \tag{A.262}$$

Because conditional constants will not contribute to the conditional variance of $\hat{\theta}$, we may work with $\sum_{t=1}^{T} tr\left(D_t \hat{V}_t\right)$ in place of the last term on the right for the conditional variance of $\hat{\theta}$. We will use the following decomposition:

$$\hat{V}_t = \hat{V} + \left(\hat{V}_t - \hat{V}\right) \cong \frac{1}{T}\sum_{t=1}^{T} \varepsilon_t \varepsilon_t' + \left(\hat{V}_t - \hat{V}\right) \tag{A.263}$$

We now regress $\hat{V}_t - \hat{V}$ on $Z$ (i.e., one bivariate regression for each matrix entry) and assume that the matrix residuals at time *t*, denoted $\tau_t$, are independent over time but may be correlated with the $\varepsilon_t$. Note that if this regression is perfect (i.e., residuals are zero) then the term $\hat{V}_t - \hat{V}$ becomes a known function of the $Z$ and does not contribute to the conditional variance; hence it is natural to concentrate on the residuals $\tau_t$ while accounting for the conditional variance contributed by $\hat{V}_t - \hat{V}$. Up to a conditional constant, for the purpose of computing the conditional variance, we may therefore represent $\hat{V}_t = \hat{V} + \left(\hat{V}_t - \hat{V}\right)$ using $\left(\sum_{i=1}^{T} \varepsilon_i \varepsilon_i'\right) / T + \tau_t$. This implies that we may, for this purpose, represent $\sum_{t=1}^{T} tr\left(D_t \hat{V}_t\right)$ using

$$\begin{aligned} \sum_{i=1}^{T} tr\left[D_i\left(\frac{1}{T}\sum_{t=1}^{T} \varepsilon_t \varepsilon_t' + \tau_i\right)\right] &= \sum_{i=1}^{T} tr\left(D_i \frac{1}{T}\sum_{t=1}^{T} \varepsilon_t \varepsilon_t'\right) + \sum_{t=1}^{T} tr\left(D_t \tau_t\right) \\ &= \frac{1}{T}\sum_{i=1}^{T}\left(\sum_{t=1}^{T} \varepsilon_t' D_i \varepsilon_t\right) + \sum_{t=1}^{T} tr\left(D_t \tau_t\right) = \frac{1}{T}\sum_{t=1}^{T} \varepsilon_t' D \varepsilon_t + \sum_{t=1}^{T} tr\left(D_t \tau_t\right) \end{aligned} \tag{A.264}$$

where $D \equiv \sum_{t=1}^{T} D_t$.

Putting these two terms for $\hat{\theta}$ together, for the purpose of finding its conditional variance, we may work with

$$\frac{1}{T}\sum_{t=1}^{T} Z_{t-1}' C \varepsilon_t + \frac{1}{T}\sum_{t=1}^{T} \varepsilon_t' D \varepsilon_t + \sum_{t=1}^{T} tr\left(D_t \tau_t\right) \tag{A.265}$$

It now follows, by conditional independence of $\left(\varepsilon_t, \tau_t\right)$ over time, that the conditional asymptotic variance of $\hat{\theta}$ is

$$AVAR\left(\hat{\theta}\right) = \frac{1}{T}\sum_{t=1}^{T} Var\left[Z_{t-1}' C \varepsilon_t + \varepsilon_t' D \varepsilon_t + T\ tr\left(D_t \tau_t\right) \mid Z_{0,} ..., Z_{T-1}\right] \tag{A.266}$$

If we define $v_{it} \equiv Z_{t-1}' \hat{C} \hat{\varepsilon}_i + \hat{\varepsilon}_i' \hat{D} \hat{\varepsilon}_i + T\ tr\left(\hat{D}_t \hat{\tau}_i\right)$ and $\overline{v}_t \equiv \sum_{i=1}^{T} v_{it} / T$

then we may write for the unconditional variance,

$$AVAR\left(\hat{\theta}\right)=\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{i=1}^{T}\left(v_{it}-\bar{v}_{t}\right)^{2}\right] \tag{A.267}$$

completing the proof. □

## 3. Equivalence of Two Formulations

We wish to show equivalence of two formulas for conditional portfolio weights in the presence of a time-varying conditionally riskless asset with return $R_f = R_f(Z)$. One formula is from Theorem 3 using Equation (14) on page 977 of FS2001 for the case of no riskless asset (where we instead include the time-varying conditionally riskless asset $R_f$ as risky asset $n+1$). This formula gives the portfolio weights of the original *n* risky assets followed by the weight in the conditionally riskless asset as the $(n+1)$ vector:

$$x' = \frac{\mathbf{1}'_{n+1}\Lambda}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}} + \frac{\mu_p - \alpha_2}{\alpha_3}\mu'_{n+1}\left(\Lambda - \frac{\Lambda\mathbf{1}_{n+1}\mathbf{1}'_{n+1}\Lambda}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}}\right) \tag{A.268}$$

where $\mu_p$ is the target unconditional portfolio mean, $\mu_{n+1} = \mu_{n+1}(Z)$ is the $n+1$ vector of conditional means whose last entry is $R_f$, and $V_{n+1}$ is the conditional covariance matrix of the risky assets whose final row and column are zeros,

$$\Lambda = \Lambda(Z) = \left(\mu_{n+1}\mu'_{n+1} + V_{n+1}\right)^{-1} \qquad \alpha_2 = E\left(\frac{\mathbf{1}'_{n+1}\Lambda\mu_{n+1}}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}}\right)$$

and $$\alpha_3 = E\left[\mu'_{n+1}\left(\Lambda - \frac{\Lambda\mathbf{1}_{n+1}\mathbf{1}'_{n+1}\Lambda}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}}\right)\mu_{n+1}\right] \tag{A.269}$$

We assume that $R_f \neq 0$ almost surely so that the inverse $\Lambda = \Lambda(Z) = \left(\mu_{n+1}\mu'_{n+1} + V_{n+1}\right)^{-1}$ exists.

The other formula is from Equation (35.18) of Ferson and Siegel (2015) which models the initial $n$ risky assets separately from the conditionally riskless asset $R_f$ and gives the portfolio weights of the original $n$ risky assets as the $n$ vector:

$$y' = \left[(c+1)\mu_p + b - R_f\right]\left(\mu_n - R_f\mathbf{1}_n\right)' Q \qquad \text{(A.270)}$$

where $\mu_n$ is the vector of means for the $n$ risky assets (excluding the conditionally riskless asset) and the amount invested in the conditionally riskless asset is $1 - y'\mathbf{1}_n$ to preserve the portfolio constraint,

$$Q = \left\{\left(\mu_n - R_f\mathbf{1}_n\right)\left(\mu_n - R_f\mathbf{1}_n\right)' + V_n\right\}^{-1}$$

$$b = \frac{E\left[R_f\left(\mu_n - R_f\mathbf{1}_n\right)' Q\left(\mu_n - R_f\mathbf{1}_n\right)\right] - E\left(R_f\right)}{E\left[\left(\mu_n - R_f\mathbf{1}_n\right)' Q\left(\mu_n - R_f\mathbf{1}_n\right)\right]}$$

and

$$c = \frac{1 - E\left[\left(\mu_n - R_f\mathbf{1}_n\right)' Q\left(\mu_n - R_f\mathbf{1}_n\right)\right]}{E\left[\left(\mu_n - R_f\mathbf{1}_n\right)' Q\left(\mu_n - R_f\mathbf{1}_n\right)\right]} \qquad \text{(A.271)}$$

Please note that, with this notation, $\mu_n$ denotes the vector of the first $n$ entries of $\mu_{n+1}$ and similarly for $V_n$ and $V_{n+1}$. That is, if we partition into the first $n$ and the last 1, we have

$$\mu_{n+1} = \begin{bmatrix} \mu_n \\ R_f \end{bmatrix}$$

$$V_{n+1} = \begin{bmatrix} V_n & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{(A.272)}$$

<u>Proposition</u>: The vector of the first *n* entries of *x* is equal to *y*. That is, $x'_{1\ldots n} = y'$. That is, the two methods produce the same weights.

<u>Proof</u>: It is sufficient to show that $x'_{1\ldots n} = y'$ because the final weight of $x_{n+1}$ is the amount invested in the conditionally riskless asset and the entries of $x_{n+1}$ sum to 1 by construction, while the portfolio defined by *y* is financed using this same investment in the conditionally riskless asset, thereby also achieving the portfolio constraint. We now expand $\Lambda$ using Lemma 2 while noting that the final entry of $\mu_{n+1}$ is $R_f$

$$\begin{aligned}\Lambda &= \left(V_{n+1} + \mu_{n+1}\mu'_{n+1}\right)^{-1} = \left(\begin{bmatrix} V_n & 0 \\ 0 & 0 \end{bmatrix} + \mu_{n+1}\mu'_{n+1}\right)^{-1} \\ &= \begin{bmatrix} V_n^{-1} & -V_n^{-1}\mu_n / R_f \\ -\mu'_n V_n^{-1} / R_f & \left(1 + \mu'_n V_n^{-1}\mu_n\right) / R_f^2 \end{bmatrix}\end{aligned} \tag{A.273}$$

To find $x' = \dfrac{\mathbf{1}'_{n+1}\Lambda}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}} + \dfrac{\mu_p - \alpha_2}{\alpha_3}\mu'_{n+1}\left(\Lambda - \dfrac{\Lambda\mathbf{1}_{n+1}\mathbf{1}'_{n+1}\Lambda}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}}\right)$ using the expanded $\Lambda$, we will need each of the following:

$$\begin{aligned}\mathbf{1}'_{n+1}\Lambda &= \mathbf{1}'_{n+1}\begin{bmatrix} V_n^{-1} & -V_n^{-1}\mu_n / R_f \\ -\mu'_n V_n^{-1} / R_f & \left(1 + \mu'_n V_n^{-1}\mu_n\right) / R_f^2 \end{bmatrix} \\ &= \left[\mathbf{1}'_n V_n^{-1} - \mu'_n V_n^{-1} / R_f \quad -\mathbf{1}'_n V_n^{-1}\mu_n / R_f + \left(1 + \mu'_n V_n^{-1}\mu_n\right) / R_f^2\right]\end{aligned}$$

$$\begin{aligned}\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1} &= \left[\mathbf{1}'_n V_n^{-1} - \mu'_n V_n^{-1} / R_f \quad -\mathbf{1}'_n V_n^{-1}\mu_n / R_f + \left(1 + \mu'_n V_n^{-1}\mu_n\right) / R_f^2\right]\mathbf{1}_{n+1} \\ &= \mathbf{1}'_n V_n^{-1}\mathbf{1}_n - \mu'_n V_n^{-1}\mathbf{1}_n / R_f - \mathbf{1}'_n V_n^{-1}\mu_n / R_f + \left(1 + \mu'_n V_n^{-1}\mu_n\right) / R_f^2 \\ &= \frac{R_f^2\mathbf{1}'_n V_n^{-1}\mathbf{1}_n - R_f\mu'_n V_n^{-1}\mathbf{1}_n - R_f\mathbf{1}'_n V_n^{-1}\mu_n + \left(1 + \mu'_n V_n^{-1}\mu_n\right)}{R_f^2} \\ &= \frac{1 + \left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}{R_f^2}\end{aligned}$$

$$\mu'_{n+1}\Lambda = \mu'_{n+1}\begin{bmatrix} V_n^{-1} & -V_n^{-1}\mu_n / R_f \\ -\mu'_n V_n^{-1} / R_f & \left(1+\mu'_n V_n^{-1}\mu_n\right) / R_f^2 \end{bmatrix}$$
$$= \begin{bmatrix} \mu'_n V_n^{-1} - R_f \mu'_n V_n^{-1} / R_f & -\mu'_n V_n^{-1}\mu_n / R_f + R_f\left(1+\mu'_n V_n^{-1}\mu_n\right) / R_f^2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \dfrac{1}{R_f} \end{bmatrix}$$

$$\mu'_{n+1}\Lambda\mathbf{1}_{n+1} = \begin{bmatrix} 0 & 1/R_f \end{bmatrix}' \mathbf{1}_{n+1} = 1/R_f$$

$$\mu'_{n+1}\Lambda\mu_{n+1} = \mu'_{n+1}\begin{bmatrix} V_n^{-1} & -V_n^{-1}\mu_n / R_f \\ -\mu'_n V_n^{-1} / R_f & \left(1+\mu'_n V_n^{-1}\mu_n\right) / R_f^2 \end{bmatrix}\mu_{n+1}$$
$$= \begin{bmatrix} \mu'_n V_n^{-1} - R_f \mu'_n V_n^{-1} / R_f & -\mu'_n V_n^{-1}\mu_n / R_f + R_f\left(1+\mu'_n V_n^{-1}\mu_n\right) / R_f^2 \end{bmatrix}\mu_{n+1} \qquad \text{(A.274)}$$
$$= \begin{bmatrix} 0 & 1/R_f \end{bmatrix}\mu_{n+1} = 1$$

Using these in the formula for $x'$ we find

$$x' = \frac{\mathbf{1}'_{n+1}\Lambda}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}} + \frac{\mu_p - \alpha_2}{\alpha_3}\mu'_{n+1}\left(\Lambda - \frac{\Lambda\mathbf{1}_{n+1}\mathbf{1}'_{n+1}\Lambda}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}}\right)$$
$$= \frac{\mathbf{1}'_{n+1}\Lambda}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}} + \frac{\mu_p - \alpha_2}{\alpha_3}\mu'_{n+1}\Lambda - \frac{\mu_p - \alpha_2}{\alpha_3}\frac{\mu'_{n+1}\Lambda\mathbf{1}_{n+1}\mathbf{1}'_{n+1}\Lambda}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}}$$
$$= \left(1 - \frac{\mu_p - \alpha_2}{\alpha_3}\mu'_{n+1}\Lambda\mathbf{1}_{n+1}\right)\frac{\mathbf{1}'_{n+1}\Lambda}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}} + \frac{\mu_p - \alpha_2}{\alpha_3}\mu'_{n+1}\Lambda$$
$$= \left(1 - \frac{\mu_p - \alpha_2}{\alpha_3 R_f}\right)\frac{\begin{bmatrix} \mathbf{1}'_n V_n^{-1} - \mu'_n V_n^{-1} / R_f & -\mathbf{1}'_n V_n^{-1}\mu_n / R_f + \left(1+\mu'_n V_n^{-1}\mu_n\right) / R_f^2 \end{bmatrix}}{1 + \left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)} R_f^2 + \frac{\mu_p - \alpha_2}{\alpha_3}\begin{bmatrix} 0 & 1/R_f \end{bmatrix}$$
$$= \left(1 - \frac{\mu_p - \alpha_2}{\alpha_3 R_f}\right)\frac{\begin{bmatrix} \mathbf{1}'_n V_n^{-1} R_f^2 - \mu'_n V_n^{-1} R_f & -\mathbf{1}'_n V_n^{-1}\mu_n R_f + \left(1+\mu'_n V_n^{-1}\mu_n\right) \end{bmatrix}}{1 + \left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)} + \frac{\mu_p - \alpha_2}{\alpha_3}\begin{bmatrix} 0 & 1/R_f \end{bmatrix}$$

(A.275)

Because we know that $x'\mathbf{1}_{n+1} = 1$ from direct calculation, we may look at the first $n$ coordinates $x_{1\ldots n}$ of $x$, knowing that this position will be financed using a position in the conditionally risk-free asset. We find

$$\begin{aligned} x'_{1\ldots n} &= \left(1 - \frac{\mu_p - \alpha_2}{\alpha_3 R_f}\right) \frac{\mathbf{1}'_n V_n^{-1} R_f^2 - \mu'_n V_n^{-1} R_f}{1 + \left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \left(\mu_n - R_f \mathbf{1}_n\right)} \\ &= \left(\frac{\mu_p - \alpha_2}{\alpha_3} - R_f\right) \frac{\left(\mu - R_f \mathbf{1}_n\right)'_n V_n^{-1}}{1 + \left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \left(\mu_n - R_f \mathbf{1}_n\right)} \end{aligned} \tag{A.276}$$

Now use Lemma 4 to expand

$$\begin{aligned} Q &= \left\{V_n + \left(\mu_n - R_f \mathbf{1}_n\right)\left(\mu_n - R_f \mathbf{1}_n\right)'\right\}^{-1} \\ &= V_n^{-1} - \frac{V_n^{-1}\left(\mu_n - R_f \mathbf{1}_n\right)\left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1}}{1 + \left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \left(\mu_n - R_f \mathbf{1}_n\right)} \end{aligned} \tag{A.277}$$

from which we find

$$\begin{aligned} y' &= \left[(c+1)\mu_p + b - R_f\right]\left(\mu_n - R_f \mathbf{1}_n\right)' Q \\ &= \left[(c+1)\mu_p + b - R_f\right]\left(\mu_n - R_f \mathbf{1}_n\right)' \left(V_n^{-1} - \frac{V_n^{-1}\left(\mu_n - R_f \mathbf{1}_n\right)\left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1}}{1 + \left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \left(\mu_n - R_f \mathbf{1}_n\right)}\right) \\ &= \left[(c+1)\mu_p + b - R_f\right]\left(\left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} - \frac{\left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f \mathbf{1}_n\right)\left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1}}{1 + \left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \left(\mu_n - R_f \mathbf{1}_n\right)}\right) \\ &= \left[(c+1)\mu_p + b - R_f\right]\left(1 - \frac{\left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f \mathbf{1}_n\right)}{1 + \left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \left(\mu_n - R_f \mathbf{1}_n\right)}\right)\left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \\ &= \left[(c+1)\mu_p + b - R_f\right] \frac{\left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1}}{1 + \left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \left(\mu_n - R_f \mathbf{1}_n\right)} \end{aligned}$$

(A.278)

Comparing $x'_{1\ldots n}$ to $y'$, we see that they will be equal provided we show that

$$\frac{\mu_p - \alpha_2}{\alpha_3} - R_f = (c+1)\mu_p + b - R_f \tag{A.279}$$

or, more simply, that

$$\frac{\mu_p - \alpha_2}{\alpha_3} = (c+1)\mu_p + b \tag{A.280}$$

We next show that the multiples of $\mu_p$ are identical, that is: $\alpha_3 = 1/(1+c)$. We find

$$\begin{aligned}\alpha_3 &= E\left[\mu'_{n+1}\left(\Lambda - \frac{\Lambda \mathbf{1}_{n+1}\mathbf{1}'_{n+1}\Lambda}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}}\right)\mu_{n+1}\right] = E\left(\mu'_{n+1}\Lambda\mu_{n+1} - \frac{\mu'_{n+1}\Lambda\mathbf{1}_{n+1}\mathbf{1}'_{n+1}\Lambda\mu_{n+1}}{\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}}\right)\\ &= E\left(1 - \frac{1}{R_f^2\mathbf{1}'_{n+1}\Lambda\mathbf{1}_{n+1}}\right) = E\left(1 - \frac{R_f^2}{R_f^2\left[1 + \left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)\right]}\right)\\ &= E\left(1 - \frac{1}{1 + \left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}\right)\end{aligned} \tag{A.281}$$

and

$$\frac{1}{1+c} = \frac{1}{1 + \dfrac{1 - E\left[\left(\mu_n - R_f\mathbf{1}_n\right)' Q\left(\mu_n - R_f\mathbf{1}_n\right)\right]}{E\left[\left(\mu_n - R_f\mathbf{1}_n\right)' Q\left(\mu_n - R_f\mathbf{1}_n\right)\right]}} = E\left[\left(\mu_n - R_f\mathbf{1}_n\right)' Q\left(\mu_n - R_f\mathbf{1}_n\right)\right] \tag{A.282}$$

From the derivation of $y'$, we know that

$$\left(\mu_n - R_f \mathbf{1}_n\right)' Q = \frac{\left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1}}{1 + \left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \left(\mu_n - R_f \mathbf{1}_n\right)} \tag{A.283}$$

which implies that

$$\begin{aligned}\frac{1}{1+c} &= E\left[\left(\mu_n - R_f \mathbf{1}_n\right)' Q \left(\mu_n - R_f \mathbf{1}_n\right)\right] = E\left[\frac{\left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1}}{1 + \left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \left(\mu_n - R_f \mathbf{1}_n\right)} \left(\mu_n - R_f \mathbf{1}_n\right)\right] \\ &= E\left[\frac{1 + \left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \left(\mu_n - R_f \mathbf{1}_n\right) - 1}{1 + \left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \left(\mu_n - R_f \mathbf{1}_n\right)}\right] = E\left(1 - \frac{1}{1 + \left(\mu_n - R_f \mathbf{1}_n\right)' V_n^{-1} \left(\mu_n - R_f \mathbf{1}_n\right)}\right) = \alpha_3\end{aligned} \tag{A.284}$$

showing that the coefficients multiplying $\mu_p$ are identical. It remains only to show that the remaining terms are equal, that is, that $b = -\alpha_2 / \alpha_3$. For this we need

$$b = \frac{E\left[R_f\left(\mu_n - R_f\mathbf{1}_n\right)' Q\left(\mu_n - R_f\mathbf{1}_n\right)\right] - E\left(R_f\right)}{E\left[\left(\mu_n - R_f\mathbf{1}_n\right)' Q\left(\mu_n - R_f\mathbf{1}_n\right)\right]}$$

$$= \frac{E\left[R_f \frac{\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}}{1+\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}\left(\mu_n - R_f\mathbf{1}_n\right)\right] - E\left(R_f\right)}{E\left[\frac{\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}}{1+\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}\left(\mu_n - R_f\mathbf{1}_n\right)\right]}$$

$$= \frac{E\left[R_f \frac{1+\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right) - 1}{1+\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}\right] - E\left(R_f\right)}{E\left[\frac{\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}{1+\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}\right]}$$

$$= -\frac{E\left[\frac{R_f}{1+\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}\right]}{E\left[\frac{\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}{1+\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}\right]}$$

$$= -\frac{E\left[\frac{R_f}{1+\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}\right]}{E\left[1 - \frac{1}{1+\left(\mu_n - R_f\mathbf{1}_n\right)' V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}\right]}$$

$$\alpha_2 = E\left(\frac{\mathbf{1}_{n+1}'\Lambda\mu_{n+1}}{\mathbf{1}_{n+1}'\Lambda\mathbf{1}_{n+1}}\right) = E\left(\frac{1/R_f}{\frac{1+\left(\mu_n - R_f\mathbf{1}_n\right)'V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}{R_f^2}}\right)$$
$$= E\left(\frac{R_f}{1+\left(\mu_n - R_f\mathbf{1}_n\right)'V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}\right) \tag{A.285}$$

Putting these together with the $\alpha_3$ derived earlier, we find that

$$-\frac{\alpha_2}{\alpha_3} = -\frac{E\left(\frac{R_f}{1+\left(\mu_n - R_f\mathbf{1}_n\right)'V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}\right)}{E\left(1-\frac{1}{1+\left(\mu_n - R_f\mathbf{1}_n\right)'V_n^{-1}\left(\mu_n - R_f\mathbf{1}_n\right)}\right)} = b \tag{A.286}$$

completing the proof. □

<u>Lemma 1</u>: Let $u$ be an $n+1$ vector with its last entry $u_{n+1} \neq 0$, and let $u_{1\ldots n} = \left(u_1,\ldots,u_n\right)'$. Then

$$\left(\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} + uu'\right)^{-1} = \begin{bmatrix} I_n & -u_{1\ldots n}/u_{n+1} \\ -u_{1\ldots n}'/u_{n+1} & \left(1+\left\|u_{1\ldots n}\right\|^2\right)/u_{n+1}^2 \end{bmatrix} \tag{A.287}$$

<u>Proof</u>:

$$\left(\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} + uu'\right)\begin{bmatrix} I_n & -u_{1\ldots n} / u_{n+1} \\ -u'_{1\ldots n} / u_{n+1} & \left(1+\|u_{1\ldots n}\|^2\right) / u_{n+1}^2 \end{bmatrix}$$
$$= \left(\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{1\ldots n}u'_{1\ldots n} & u_{1\ldots n}u_{n+1} \\ u'_{1\ldots n}u_{n+1} & u_{n+1}^2 \end{bmatrix}\right)\begin{bmatrix} I_n & -u_{1\ldots n} / u_{n+1} \\ -u'_{1\ldots n} / u_{n+1} & \left(1+\|u_{1\ldots n}\|^2\right) / u_{n+1}^2 \end{bmatrix} \quad \text{(A.288)}$$
$$= \begin{bmatrix} I_n + u_{1\ldots n}u'_{1\ldots n} & u_{1\ldots n}u_{n+1} \\ u'_{1\ldots n}u_{n+1} & u_{n+1}^2 \end{bmatrix}\begin{bmatrix} I_n & -u_{1\ldots n} / u_{n+1} \\ -u'_{1\ldots n} / u_{n+1} & \left(1+\|u_{1\ldots n}\|^2\right) / u_{n+1}^2 \end{bmatrix}$$

The top left entry of this matrix product is

$$\left(I_n + u_{1\ldots n}u'_{1\ldots n}\right)I_n + u_{1\ldots n}u_{n+1}\left(-u'_{1\ldots n} / u_{n+1}\right)$$
$$= I_n + u_{1\ldots n}u'_{1\ldots n} - u_{1\ldots n}u'_{1\ldots n} = I_n \quad \text{(A.289)}$$

The top right entry is

$$\left(I_n + u_{1\ldots n}u'_{1\ldots n}\right)\left(-u_{1\ldots n} / u_{n+1}\right) + u_{1\ldots n}u_{n+1}\left(1+\|u_{1\ldots n}\|^2\right) / u_{n+1}^2$$
$$= I_n\left(-u_{1\ldots n} / u_{n+1}\right) + u_{1\ldots n}u'_{1\ldots n}\left(-u_{1\ldots n} / u_{n+1}\right) + u_{1\ldots n}u_{n+1} / u_{n+1}^2 + u_{1\ldots n}u_{n+1}\|u_{1\ldots n}\|^2 / u_{n+1}^2 \quad \text{(A.290)}$$
$$= -u_{1\ldots n} / u_{n+1} - u_{1\ldots n}\|u_{1\ldots n}\|^2 / u_{n+1} + u_{1\ldots n} / u_{n+1} + u_{1\ldots n}\|u_{1\ldots n}\|^2 / u_{n+1} = 0_n$$

The bottom left entry is

$$u'_{1\ldots n}u_{n+1}I_n + u_{n+1}^2\left(-u'_{1\ldots n} / u_{n+1}\right) = u'_{1\ldots n}u_{n+1} - u_{n+1}u'_{1\ldots n} = 0'_n \quad \text{(A.291)}$$

Finally, the bottom right entry is

$$u'_{1\ldots n}u_{n+1}\left(-u_{1\ldots n} / u_{n+1}\right) + u_{n+1}^2\left(1+\|u_{1\ldots n}\|^2\right) / u_{n+1}^2 = -u'_{1\ldots n}u_{1\ldots n} + u_{n+1}^2 / u_{n+1}^2 + u_{n+1}^2\|u_{1\ldots n}\|^2 / u_{n+1}^2$$
$$= -\|u_{1\ldots n}\|^2 + 1 + \|u_{1\ldots n}\|^2 = 1 \quad \text{(A.292)}$$

Therefore

$$\left(\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} + uu'\right)\begin{bmatrix} I_n & -u_{1\ldots n} / u_{n+1} \\ -u'_{1\ldots n} / u_{n+1} & \left(1+\|u_{1\ldots n}\|^2\right) / u_{n+1}^2 \end{bmatrix} = \begin{bmatrix} I_n & 0_n \\ 0'_n & 1 \end{bmatrix} = I_{n+1} \quad \text{(A.293)}$$

completing the proof. □

<u>Lemma 2</u>: Let $u$ be an $n+1$ vector with its last entry $u_{n+1} \neq 0$, and let $u_{1\ldots n} = (u_1,\ldots,u_n)'$. Let $V_n$ be an $n \times n$ symmetric positive definite matrix. Then

$$\left(\begin{bmatrix} V_n & 0 \\ 0 & 0 \end{bmatrix} + uu'\right)^{-1} = \begin{bmatrix} V_n^{-1} & -V_n^{-1}u_{1\ldots n} / u_{n+1} \\ -u'_{1\ldots n}V_n^{-1} / u_{n+1} & \left[1 + u'_{1\ldots n}V_n^{-1}u_{1\ldots n}\right] / u_{n+1}^2 \end{bmatrix} \tag{A.294}$$

<u>Proof</u>: Let $V_n = AA'$ be the Cholesky decomposition. Lemma 1 implies that

$$\left(\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix} u \left(\begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix} u\right)'\right)^{-1} = \begin{bmatrix} I_n & -A^{-1}u_{1\ldots n} / u_{n+1} \\ -\left(A^{-1}u_{1\ldots n}\right)' / u_{n+1} & \left(1 + \left\|A^{-1}u_{1\ldots n}\right\|^2\right) / u_{n+1}^2 \end{bmatrix} \tag{A.295}$$

Pre- and post- multiplying the left-hand side, we find

$$\begin{aligned} &\begin{bmatrix} \left(A^{-1}\right)' & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix} u \left(\begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix} u\right)'\right)^{-1} \begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix} \\ &= \left(\begin{bmatrix} AA' & 0 \\ 0 & 0 \end{bmatrix} + uu'\right)^{-1} = \left(\begin{bmatrix} V_n & 0 \\ 0 & 0 \end{bmatrix} + uu'\right)^{-1} \end{aligned} \tag{A.296}$$

Multiplying similarly for the right-hand side, we find

$$\begin{bmatrix} \left(A^{-1}\right)' & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_n & -A^{-1}u_{1\ldots n} / u_{n+1} \\ -\left(A^{-1}u_{1\ldots n}\right)' / u_{n+1} & \left(1+\left\|A^{-1}u_{1\ldots n}\right\|^2\right) / u_{n+1}^2 \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \left(A^{-1}\right)' A^{-1} & -\left(A^{-1}\right)' A^{-1}u_{1\ldots n} / u_{n+1} \\ -\left(A^{-1}u_{1\ldots n}\right)' A^{-1} / u_{n+1} & \left(1+\left\|A^{-1}u_{1\ldots n}\right\|^2\right) / u_{n+1}^2 \end{bmatrix}$$
$$= \begin{bmatrix} \left(AA'\right)^{-1} & -\left(AA'\right)^{-1} u_{1\ldots n} / u_{n+1} \\ -\left(A^{-1}u_{1\ldots n}\right)' A^{-1} / u_{n+1} & \left[1+u'_{1\ldots n}\left(AA'\right)^{-1} u_{1\ldots n}\right] / u_{n+1}^2 \end{bmatrix}$$
$$= \begin{bmatrix} V_n^{-1} & -V_n^{-1}u_{1\ldots n} / u_{n+1} \\ -u'_{1\ldots n}V_n^{-1} / u_{n+1} & \left[1+u'_{1\ldots n}V_n^{-1}u_{1\ldots n}\right] / u_{n+1}^2 \end{bmatrix} \tag{A.297}$$

Equating these expressions completes the proof. □

<u>Lemma 3</u>: Let $u$ be a vector. Then

$$\left(I+uu'\right)^{-1} = I - \frac{uu'}{1+u'u} \tag{A.298}$$

<u>Proof</u>: We multiply:

$$\left(I+uu'\right)\left(I-\frac{uu'}{1+u'u}\right) = \left(I-\frac{uu'}{1+u'u}\right) + uu'\left(I-\frac{uu'}{1+u'u}\right)$$
$$= I - \frac{uu'}{1+u'u} + uu' - \frac{u\left(u'u\right)u'}{1+u'u} = I + uu'\left(-\frac{1}{1+u'u} + 1 - \frac{u'u}{1+u'u}\right) = I \tag{A.299}$$

completing the proof. □

<u>Lemma 4</u>: Let $u$ be a vector and let $V$ be a symmetric positive-definite matrix. Then

$$\left(V+uu'\right)^{-1} = V^{-1} - \frac{V^{-1}uu'V^{-1}}{1+u'V^{-1}u} \tag{A.300}$$

<u>Proof</u>: Let $V = AA'$ be the Cholesky decomposition. Then we have:

$$\left(V+uu'\right)^{-1}=\left(AA'+uu'\right)^{-1}=\left(A'\right)^{-1}\left[I+\left(A^{-1}u\right)\left(A^{-1}u\right)'\right]^{-1}A^{-1} \tag{A.301}$$

Applying Lemma 3, we find

$$\begin{aligned}
\left(V+uu'\right)^{-1}&=\left(A'\right)^{-1}\left(I-\frac{\left(A^{-1}u\right)\left(A^{-1}u\right)'}{1+\left(A^{-1}u\right)'\left(A^{-1}u\right)}\right)A^{-1}\\
&=\left(A'\right)^{-1}A^{-1}-\frac{\left(A'\right)^{-1}\left(A^{-1}u\right)\left(A^{-1}u\right)'A^{-1}}{1+\left(A^{-1}u\right)'\left(A^{-1}u\right)}=\left(AA'\right)^{-1}-\frac{\left(A'\right)^{-1}A^{-1}uu'\left(A^{-1}\right)'A^{-1}}{1+u'\left(A^{-1}\right)'\left(A^{-1}\right)u} \qquad \text{(A.302)}\\
&=\left(AA'\right)^{-1}-\frac{\left(AA'\right)^{-1}uu'\left(AA'\right)^{-1}}{1+u'\left(AA'\right)^{-1}u}=V^{-1}-\frac{V^{-1}uu'V^{-1}}{1+u'V^{-1}u}
\end{aligned}$$

completing the proof. □

**4 The Impacts of Dynamic Trading**

We dig into the sources of the Sharpe ratio improvments from dynamic trading of the factors in the FF3, FF5 and Q4 models. The largest squared Sharpe ratio available in each design is $S^2_{ue}(r,f)$, the squared Sharpe ratio attainable by dynamically trading the test assets and a model's factors. The maximum difference between any two squared Sharpe ratios in each design is $[S^2_{ue}(r,f) - S^2_{fix}(f)]$. Two alternative decompositions of this maximum Sharpe ratio difference are shown:

Max $S^2$ Difference = $[S^2_{ue}(r,f) - S^2_{fix}(f)]$

$= [S^2_{fix}(r,f) - S^2_{fix}(f)] + [S^2_{ue}(r,f) - S^2_{fix}(r,f)]$ (A.303)

$= [S^2_{ue}(r,f) - S^2_{ue}(f)] + [S^2_{ue}(f) - S^2_{fix}(f)]$.

The first decomposition shows the sum of the squared Sharpe ratio differences associated with a classical fixed-weight factor model test, plus a measure of how trading dynamicially expands the mean variance boundary of the factors and the test assets combined. The second decomposition shows the squared Sharpe ratio differences associated with a dynamic model test, plus a measure of how trading dynamically increases the squared Sharpe ratio available from the factors alone.

Table A.1 shows that all of the pieces of the decompositions are statistically significant, except in the Q4 model, where the maximum squared Sharpe ratio of the models factors is not significantly improved by dynamic trading, and in the FF5 model with the 25 Investment x productivity portfolios. The fixed weight tests do reject the FF3, FF5 and Q4 models, except in the 25 investment-productivity portfolios, where neither the FF5 nor the Q4 models is rejected.

The point estimates of the first decomposition indicate that the ability of dynamic trading to improve the squared Sharpe ratio of the combined model factors and test assets is the largest component, accounting for 40-55% of the total maximum squared Sharpe ratio. The additional squared Sharpe ratio attributed to dynamic trading is economically significant. Previous studies find economically large benefits to optimally trading with lagged instruments in out of sample analyses (see Abhyankar, Basu and Stremme (2005) and Chiang (2016) for example). This is different from simple predictive regressions, where the variables enter linearly and the out-of sample performance is poor. Here the variables enter nonlinearly through the optimal dynamic strategy. In the first decomposition, a fixed weight portfolio of the factors accounts for 12-37% of the total squared Sharpe ratio, with the largest value holding in the Q4 model, while the contribution associated with the classical fixed factor model test is 23-34% of the total. Thus, tests of the factor models that do not allow for dynamic trading miss a large fraction of the story.

The point estimates of the second decomposition indicate that the dynamic model test accounts for 60-80% of the total squared Sharpe ratio, and thus the quadratic utility attained by a UE portfolio of the factors and test assets combined. None of the models does a good job of capturing the maximum quadratic utility implied by a dynamic strategy. A fixed weight portfolio of the factors accounts for 12-37% of the total squared

Sharpe ratio. The contribution made by expanding the mean variance boundary of the factors with dynamic trading is smaller and varies across the models. This contributes 16% of the squared Sharpe ratio in the FF5 model, 9% in the FF3, but only 2% in the Q4 model.

# Table A.1: Decomposing Sharpe Ratio Improvements

## Panel A: 25 size-value portfolios

| | | [Ue(r,f) - fix(r,f)] | + | [fix(r,f)-fix(f)] | = | [Ue(r,f)-fix(f)] | = | [Ue(r,f) - ue(f)] |
|---|---|---|---|---|---|---|---|---|
| **FF3** | BS test value | 0.18 | | 0.11 | | 0.29 | | 0.26 |
| | *t*-statistics | 7.02 | | 3.38 | | 6.94 | | 6.35 |
| **FF5** | BS test value | 0.20 | | 0.09 | | 0.29 | | 0.23 |
| | *t*-statistics | 7.41 | | 2.82 | | 6.87 | | 5.76 |
| **Q4** | BS test value | 0.19 | | 0.11 | | 0.30 | | 0.28 |
| | *t*-statistics | 6.74 | | 3.07 | | 6.56 | | 6.10 |

## Panel B: 25 investment-productivity portfolios

| | | | | | |
|---|---|---|---|---|---|
| **FF3** | BS test value | 0.15 | 0.06 | 0.20 | 0.17 |
| | *t*-statistics | 6.11 | 2.28 | 5.88 | 5.17 |
| **FF5** | BS test value | 0.16 | 0.01 | 0.17 | 0.11 |
| | *t*-statistics | 6.37 | 0.56 | 5.35 | 3.78 |
| **Q4** | BS test value | 0.14 | 0.02 | 0.16 | 0.14 |
| | *t*-statistics | 5.41 | 0.82 | 4.58 | 4.15 |

**Panel C: 32 size-investment-productivity portfolios**

| | | [Ue(r,f) - fix(r,f)] | + | [fix(r,f)-fix(f)] | = | [Ue(r,f)-fix(f)] | = | [Ue(r,f) - ue(f)] |
|---|---|---|---|---|---|---|---|---|
| **FF3** | BS test value | 0.18 | | 0.17 | | 0.35 | | 0.32 |
| | *t*-statistics | 6.90 | | 4.04 | | 7.06 | | 6.55 |
| **FF5** | BS test value | 0.19 | | 0.12 | | 0.31 | | 0.25 |
| | *t*-statistics | 7.16 | | 3.17 | | 6.66 | | 5.65 |
| **Q4** | BS test value | 0.19 | | 0.09 | | 0.28 | | 0.27 |
| | *t*-statistics | 6.73 | | 2.38 | | 5.98 | | 5.63 |

**Panel D: 49 industry portfolios**

| | | | | | |
|---|---|---|---|---|---|
| **FF3** | BS test value | 0.27 | 0.20 | 0.47 | 0.44 |
| | *t*-statistics | 8.59 | 4.77 | 8.78 | 8.28 |
| **FF5** | BS test value | 0.29 | 0.25 | 0.55 | 0.49 |
| | *t*-statistics | 8.72 | 5.28 | 8.77 | 8.04 |
| **Q4** | BS test value | 0.30 | 0.15 | 0.44 | 0.43 |
| | *t*-statistics | 8.01 | 3.23 | 7.13 | 7.13 |

Squared Sharpe ratio differences and their t-ratios are shown. Ue refers to a portfolio of the indicated assets that trades optimally with the conditioning information. The test asset portfolios are denoted by r, and the factors by f.

## 5. Results for Alternative Lagged Instruments

We show results here using a set of "classical" lagged instruments *Z* including: (i) the lagged value of a one-month Treasury bill yield, (ii) the dividend yield of the market index; (iii) the spread between Moody's Baa and Aaa corporate bond yields; (iv) the spread between ten-year and one-year constant maturity Treasury bond yields. We choose these variables because they have been among the mostly commonly used in the asset pricing literature.

We also examine a set of more "modern" lagged instruments. Goyal Welch and Zafirov (GWZ, 2022) dismiss most of the 46 predictors they examine because their predictive ability either becomes insignificant in data extended to 2020, has different signs in subsamples or has poor step ahead (OOS) predictive ability. GWZ find that among our "classical" predictors only the short term tbill rate survives all of their criteria for a predictor. We pick a set of monthly modern predictors to serve as a robustness check on our empirical findings. We pick monthly variables which have data available for our sample and, even if they fail on some of the three GWZ criteria, manage to satisfy two. GWZ must have been able to roughly replicate the in-sample performance found in the original study, the OOS R-squared cannot be negative, and the extended sample t-ratio must be of the same sign as in the original study, but not necessarily statistically significant. We use the following as our modern predictors:

1. Illiq is from Chen, Eaton, and Paye (2018). This is the log of the number of zero returns, measuring stock market illiquidity. The series has structural break adjustments for tick-size reductions in 1997 and 2001, found by regressing the series on dummy variables equal to 1 after the tick-size reductions, and 0 otherwise, then taking the residuals.

2. New durables orders from Jones and Tuzel (2013) is the ratio of new orders to shipments of durable goods, obtained from the Census Bureau.

3. Technical indicators is from Neely, Rapach, Tu, and Zhou (2014). This is the first principal component of 14 technical indicators, mainly versions of moving price averages, momentum, and dollar trading volume.

4. Average Correlation is from Pollet and Wilson (2010). This is the average correlation between daily stock returns among the 500 largest stocks (by capitalization). The daily pairwise correlations of stock returns are multiplied by the product of both stock's weights relative to total sample market capitalization, then summed.

Tables of results with these alternative instruments follow.

## *6. Non-traded Factors*

When the factors in a model are not traded assets, or the model is fully conditional, mimicking portfolios must be found. With dynamic trading Ferson and Siegel (2009) show that given *m* satisfies the pricing Equation (3), then a portfolio that is maximum correlation to *m* with respect to lagged information *Z*, must be UE with respect to *Z*. Tests compare the squared Sharpe ratio of the maximum correlation portfolio with $S^2_{ue}(r)$. Supressing the time subscripts, we state a definition.

***Definition.*** *A portfolio* $R_P$ *is* ***maximum correlation for a random variable, m, with respect to lagged conditioning information Z****, iff:*

$$\rho^2\left(R_P, m\right) \geq \rho^2\left[w'(Z)R, m\right] \quad \forall w(Z): w'(Z)\mathbf{1} = 1,$$

*where* $\rho^2(.,.)$ *is the squared unconditional correlation coefficient and we restrict to functions* $w(.)$ *for which the correlation exists*.

Ferson, Siegel and Xu (2006) present solutions for maximum correlation portfolio weights with respect to conditioning information. Tests with nontraded factors and fully conditional models depend on the choice of test assets from which the mimicking portfolios are estimated. Tests with dynamic trading depend on the choice of the lagged instruments, and we compare several specifications. In models with non-traded factors and conditioning information, a portfolio with maximum correlation (with respect to *Z*) to the SDF is hypothesized to be UE. Tests compare the squared Sharpe ratio of the mimicking portfolio to the maximum squared Sharpe ratio of the test assets.

Let $R_t^* \equiv \left(R_t', F_t\right)' = \left(R_t^1, ..., R_t^n, F_t\right)'$ , $\mu_t^* \equiv \left(\mu_t', \mu_t^F\right)' = \left(\mu_t^1, ..., \mu_t^n, \mu_t^F\right)'$ , and $\varepsilon_t^* \equiv \left(\varepsilon_t', \varepsilon_t^F\right)' = \left(\varepsilon_t^1, ..., \varepsilon_t^n, \varepsilon_t^F\right)'$. Define the k x (N+1) matrix $\delta^* \equiv \left[\delta \quad \delta_F\right]$ where $\delta_F$ contains the *L* regression coefficients for *F* given *Z*. Define the $\left(N+1\right)\times\left(N+1\right)$ conditional covariance matrix

$$V^* \equiv \begin{bmatrix} V & V_F \\ V_F' & \sigma^2_{F|Z} \end{bmatrix}$$

where $V$ is the same $N \times N$ conditional covariance matrix as before, $V_F \equiv Cov\left(R_t F_t \mid Z_{t-1}\right)$ is $N \times 1$, and $\sigma^2_{F|Z} \equiv Var\left(F_t \mid Z_{t-1}\right)$ is a scalar. We assume that the unobserved $\varepsilon_t^*$ are independent and identically distributed, with mean zero and covariance matrix $V^*$. We note that

$$E\left(R_t F_t \mid Z_{t-1}\right) = Cov\left(R_t F_t \mid Z_{t-1}\right) + \mu_t \mu_t^F = V_F + \mu_t \mu_t^F$$

Using Ferson, Siegel, and Xu (2006) Equation (6) the maximal correlation portfolio weight is

$$w_t' \equiv \frac{\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t\mathbf{1}} - \left[\lambda_1 \mu_t' + \lambda_2 E\left(F_t R_t' \mid Z_{t-1}\right)\right]\left(\Lambda_t - \frac{\Lambda_t \mathbf{1}\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t\mathbf{1}}\right)$$

With $\Lambda_t \equiv \left(\mu_t \mu_t' + V\right)^{-1}$, $\Omega_t \equiv \Lambda_t - \dfrac{\Lambda_t \mathbf{1}\mathbf{1}'\Lambda_t}{\mathbf{1}'\Lambda_t\mathbf{1}}$, $\lambda_1 \equiv \dfrac{-\gamma_1\left(\mu^F - \gamma_{\mu F}\right) + \gamma_\mu \gamma_F}{\gamma_\mu\left(\mu^F - \gamma_{\mu F}\right) + \gamma_F\left(\gamma_{\mu\mu} - 1\right)}$

and $\lambda_2 \equiv \dfrac{-\gamma_1\left(\gamma_{\mu\mu} - 1\right) - \gamma_\mu^2}{\gamma_\mu\left(\mu^F - \gamma_{\mu F}\right) + \gamma_F\left(\gamma_{\mu\mu} - 1\right)}$. The parameters $\mu_F$, $\gamma_1$, $\gamma_\mu$, $\gamma_F$, $\gamma_{\mu F}$, $\gamma_{\mu\mu}$ and their estimates are presented with the proof.

**<u>Corollary</u>**: The asymptotic variance of the estimated squared Sharpe Ratio of the portfolio $R_p$ having maximal correlation with a given scalar factor $F$ with respect to lagged information Z may be obtained using the Theorem I together with canonical matrices

$$C \equiv \frac{2\sigma_p^2\left(\mu_p - \varphi\right)C_{\mu p} - \left(\mu_p - \varphi\right)^2 C_{\sigma p2}}{\sigma_p^4}$$

$$D \equiv \frac{2\sigma_p^2\left(\mu_p - \varphi\right)D_{\mu p} - \left(\mu_p - \varphi\right)^2 D_{\sigma p2}}{\sigma_p^4}$$

where $\varphi$ is the given zero-beta rate, $\mu_p$ and $\sigma_p^2$ are the mean and variance of the maximal correlation portfolio, and expressions for the matrices $C_{\mu p}$, $C_{\sigma p2}$, $D_{\mu p}$, and $D_{\sigma p2}$ are provided with the proof. The Corollary applies to any general factor, $F$. It does not impose the assumption that the factor is the SDF.

Our data for nontraded factors follow Chen, Roll and Ross (1986). Their factors include a monthly growth in industrial production with a one-month lead, a change in expected inflation following Fama and Gibbons (1984), a corresponding unexpected inflation based on the US CPI. They also examine two traded factors: the return difference between Baa corporate bond index and the long term government bond return and the difference between the long term bond and the short term bill return. We compute a real consumption growth using the personal consumption expenditure (PCE in Table 2.3.5U) and price index data (Table 2.3.4U) from Bureau of Economic Analysis. These monthly data start from January of 1959 until December 2020. We also use a non-traded broker-dealer leverage factor, which Adrian, Etula and Muir (2014) propose as a single-factor model, the data graciously provided on Tyler Muir's web site.

In Table 8 we estimate mimicking portfolio weights in each simulation trial to capture the effects of estimation error. The "true" values of the mimicking portfolio squared Sharpe ratios are computed using simulations with 1,000 times as many time series as in the original data. Bias adjustment uses the JK (1980) adjustment. The largest true squared Sharpe ratios for the individual mimicking portfolios are for the leverage and industrial production factors, followed by consumption growth. These range from 1.5 to 2.5% per month. The goal of the dynamic trading is to increase the squared correlation with the mimicked factor. The absolute correlation with the consumption factor increases to 0.28, versus 0.26 with fixed weights. For industrial production it rises from 0.24 to 0.34. For unexpected inflation it rises from 0.29 to 0.36.

Because mimicking portfolios are not formed to maximize their sample Sharpe ratios, they should not be as biased as maximum Sharpe ratio portfolios. The Sharpe ratios of the fixed-weight consumption and industrial production portfolios actually have a small downward bias. Other mimicking portfolios show an upward bias, especially when the true squared Sharpe ratio is close to zero (unexpected inflation).

The JK (1980) adjustment works in the right direction in these cases, because the adjustment shrinks the estimated Sharpe ratio towards zero. No bias adjustment is available for the dynamically trading mimicking portfolios.

Near the bottom of the table we combine the three nontraded Chen, Roll and Ross (1986) mimicking portfolios (CRR3) to find their maxiumum squared Sharpe ratio, which is just over 3.0% with fixed weights and 4.6% with time-varying weights. The mimicking portfolios are formed with fixed weights and combined into portfolios with time-varying weights.

The right hand columns of Table 8 report the empirical standard deviations of the mimicking portfolio squared Sharpe ratios, taken across the 1,000 simulation trials, and the average result from the Theorem I, Corolllary III. The asymptotics do a pretty good job of predicting the simulated standard errors for most of the mimicking portfolios, getting within 10% in 40% of the cases, but they greatly overstate the sampling variability when the true Sharpe ratios are close to zero (unexpected inflation).

## Table 1. Summary Statistics of factors and lagged conditioning information

This table contains summary statistics for our sample of factors from the French data library (February 1959 to December 2020), the Q factors from Hou, Xue and Zhang (2015), monthly consumption growth data (from January 1959 to Dec 2020), and the Broker-dealer leverage factor (from Adrian, Etula and Muir 2014) (from December 1970 to November 2018). The coefficient, $\rho_1$, is the first order autcorrelation (after stochastic detrending in the case of the lagged instruments). Squared SR is the squared Sharpe ratio, where the zero-beta rate is the average Treasury bill rate, equal to 0.39 percent per month. The R-square is obtained by regressing market excess return on conditioning information. Returns, yields and yield spreads are measured as percent per month.

| **Model Factors** | **Mean** | **Std** | **AR(1)** | **Squared SR** |
|---|---|---|---|---|
| Market-risk free | 0.57 | 4.43 | 0.066 | 0.016 |
| SMB | 0.2 | 2.96 | 0.064 | 0.005 |
| HML | 0.26 | 2.81 | 0.179 | 0.008 |
| RMW | 0.23 | 2.09 | 0.149 | 0.012 |
| CMA | 0.24 | 1.92 | 0.121 | 0.015 |
| Momentum | 0.61 | 4.06 | 0.047 | 0.022 |
| Investment | 0.29 | 1.77 | 0.099 | 0.027 |
| Profitability | 0.44 | 2.4 | 0.117 | 0.034 |
| Investment Growth | 0.7 | 1.87 | 0.102 | 0.141 |

| **Non-traded Factors** | | | | |
|---|---|---|---|---|
| Consumption growth | 0.26 | 0.82 | 0.01 | na |
| Broker-dealier Leverage | 0.09 | 6.71 | 0.09 | na |

| **Lagged instruments** | | | | |
|---|---|---|---|---|
| **Old** | **Mean** | **Std** | **AR(1)** | **R-Square (Perc)** |
| Dividend yield | -0.02 | 0.32 | 0.89 | 0.01 |
| Yield spread | 0.00 | 0.27 | 0.90 | 0.02 |
| Term spread | -0.01 | 0.84 | 0.87 | 0.00 |
| Past risk free rate | 0.00 | 0.09 | 0.77 | 0.21 |
| **New** | | | | |
| Illiq | -1.75 | 0.20 | 0.84 | 0.99 |
| New durables orders | 0.00 | 0.04 | 0.69 | 0.28 |
| Technical indicators | -0.08 | 1.47 | 0.91 | 0.55 |
| Average Corrrelation | 0.28 | 0.11 | 0.90 | 0.80 |

## Table 2: Accuracy of Bias Adjustments for Squared Sharpe Ratios

The "true" squared Sharpe ratios are from simulations with a large number (1000*743) of time series observations. The values are stated in percent (multiplied by 100). The Average values across 5,000 simulation trials are shown for five alternative bias adjustment methods. The number of time series observations in the finite samples is 743. The JK uses the results of Jobson and Korkie (1980), and are based on a Non-central F distribution. The four adjustments for dynamic portfolios are the Chi-square, Non-central F, Odd-even and Direct Expansion. The adjustments are applied to the squared Sharpe ratios of fixed weight portfolios in Panel A and to Efficient with respect to Z portfolios in Panel B. The four lagged instruments that comprise the vector Z are described in the text. The N=25 portfolios are the 5x5 size x book/market sorts, the N=49 are industry portfolios and the N=99 combine the first two sets with 25 investment x profitability portfolios from Kenneth French.

**Panel A: Fixed-weight Factor Portfolios**

| | TRUE | JK | % Difference |
|---|---|---|---|
| $S(R_m)$ | 1.64 | 1.65 | -1% |
| $S_{fix}(FF3)$ | 3.27 | 3.33 | -2% |
| $S_{fix}(FF6)$ | 11.56 | 11.89 | -3% |
| $S_{fix}(Q5)$ | 31.06 | 31.05 | 0% |
| $S_{fix}(r)$ N=25 | 16.14 | 16.50 | -2% |
| $S_{fix}(r)$ N=49 | 31.87 | 32.25 | -1% |
| $S_{fix}(r)$ N=99 | 76.87 | 78.35 | -2% |

**Panel B: Efficient-with-Respect to Z Portfolios**

| | TRUE | No-Adj | Chi-Square | Non-Central F | Odd-Even | Direct Expansion |
|---|---|---|---|---|---|---|
| $S_{UE}(FF3)$ | 7.13 | 8.56 | 8.10 | 7.64 | 7.58 | 7.29 |
| $S_{UE}(FF5)$ | 15.85 | 18.61 | 17.77 | 16.93 | 16.76 | 16.20 |
| $S_{UE}(FF6)$ | 20.04 | 23.53 | 22.47 | 21.25 | 20.99 | 20.52 |
| $S_{UE}(Q5)$ | 33.77 | 36.72 | 35.70 | 34.35 | 34.01 | 34.12 |
| $S_{UE}(r)$ N=25 | 39.16 | 54.64 | 49.08 | 41.64 | 39.63 | 39.86 |
| $S_{UE}(r)$ N=49 | 69.42 | 100.85 | 75.71 | 69.01 | 75.46 | 70.58 |
| $S_{UE}(r)$ N=99 | 154.07 | 236.29 | 188.25 | 156.40 | 188.01 | 158.28 |

## Table 3: Accuracy of Asymptotic Standard Deviations

A parametric bootstrap generates 1000 simulation trials, each with 743 observations. Squared Sharpe ratios and squared Sharpe ratio differences are estimated and the asymptotic standard deviations are calculated using the propositions and the Theorem I. The first columns (Empirical) are the standard deviations of the estimates across the 1,000 simulation trials. The second columns (Avg Asymptotic) are the averages of the estimated asymptotic standard deviations. Fix(r) or fix(f) refers to a mean-variance efficient portfolio that ignores the conditioning information and uses fixed weights. UE is efficient with respect to Z. The lagged instruments are the four described in the data section. The average return of a three-month Treasury bill is taken to be the zero beta rate. The N=25 portfolios are the 5x5 size x book/market sorts, the N=49 are industry portfolios and the N=99 combine the first two sets with 25 investment x profitability portfolios from Kenneth French.

**Panel A: Standard Errors for Squared Sharpe Ratio Levels**

| | Empirical (simulated) | Average FSW Asymptotic | Average BKRS Asymptotic | Difference FSW (% empirical) | Difference BKRS (% empirical) |
|---|---|---|---|---|---|
| **$R_m$** | 0.28 | 0.28 | 0.28 | -2% | -1% |
| **$S_{fix}$(FF3)** | 0.45 | 0.40 | 0.41 | -11% | -9% |
| **$S_{fix}$(FF6)** | 0.87 | 0.80 | 0.81 | -9% | -7% |
| **$S_{UE}$(FF3)** | 0.60 | 0.58 | | -3% | |
| **$S_{UE}$(FF6)** | 1.09 | 1.08 | | -2% | |
| | | | | | |
| **$S_{fix}$(r) N=25** | 1.02 | 0.95 | 0.95 | -7% | -7% |
| **$S_{fix}$(r) N=49** | 1.25 | 1.24 | 1.30 | 0% | 4% |
| **$S_{fix}$(r) N=99** | 2.33 | 2.48 | 2.54 | 7% | 9% |
| | | | | | |
| **$S_{UE}$(r) N=25** | 1.49 | 1.38 | | -8% | |
| **$S_{UE}$(r) N=49** | 1.91 | 1.79 | | -6% | |
| **$S_{UE}$(r) N=99** | 3.50 | 3.33 | | -5% | |

**Panel B: Standard Errors for Squared Sharpe Ratio Differences (N=25)**

| | Empirical (simulated) | Average FSW Asymptotic | Average BKRS Asymptotic | Difference FSW (% empirical) | Difference BKRS (% empirical) |
|---|---|---|---|---|---|
| **$S_{fix}$(r) - Rm** | 0.92 | 0.91 | 0.93 | -1% | 1% |
| **$S_{fix}$(r) – $S_{fix}$(FF3)** | 0.93 | 0.89 | 0.91 | -4% | -2% |
| **$S_{fix}$(r) – $S_{fix}$(FF5)** | 0.89 | 0.89 | 0.91 | 1% | 2% |
| | | | | | |
| **$S_{UE}$(r) – Rm** | 1.45 | 1.35 | | -7% | |
| **$S_{UE}$(r) – $S_{UE}$(FF3)** | 1.46 | 1.34 | | -9% | |
| **$S_{UE}$(r) – $S_{UE}$(FF3)** | 1.43 | 1.34 | | -6% | |
| | | | | | |
| **$S_{UE}$(r) – $S_{fix}$(r)** | 1.11 | 1.02 | | -8% | |

**Panel C: Standard Errors for Squared Sharpe Ratio Differences (N=49)**

| | Empirical (simulated) | Average FSW Asymptotic | Average BKRS Asymptotic | Difference FSW (% empirical) | Difference BKRS (% empirical) |
|---|---|---|---|---|---|
| **$S_{fix}$(r) - Rm** | 1.22 | 1.21 | 1.25 | -1% | 3% |
| **$S_{fix}$(r) – $S_{fix}$(FF3)** | 1.29 | 1.25 | 1.29 | -3% | 0% |
| **$S_{fix}$(r) – $S_{fix}$(FF5)** | 1.34 | 1.31 | 1.35 | -2% | 1% |
| | | | | | |
| **$S_{UE}$(r) – Rm** | 1.86 | 1.82 | | -2% | |
| **$S_{UE}$(r) – $S_{UE}$(FF3)** | 1.90 | 1.79 | | -5% | |
| **$S_{UE}$(r) – $S_{UE}$(FF3)** | 1.93 | 1.88 | | -2% | |
| | | | | | |
| **$S_{UE}$(r) – $S_{fix}$(r)** | 1.47 | 1.40 | | -5% | |

**Panel D: Standard Errors for Squared Sharpe Ratio Differences (N=99)**

| | Empirical (simulated) | Average FSW Asymptotic | Average BKRS Asymptotic | Difference FSW (% empirical) | Difference BKRS (% empirical) |
|---|---|---|---|---|---|
| **$S_{fix}$(r) - Rm** | 2.29 | 2.46 | 2.50 | 7% | 9% |
| **$S_{fix}$(r) – $S_{fix}$(FF3)** | 2.31 | 2.45 | 2.49 | 6% | 8% |
| **$S_{fix}$(r) – $S_{fix}$(FF5)** | 2.22 | 2.39 | 2.43 | 8% | 10% |
| | | | | | |
| **$S_{UE}$(r) – Rm** | 3.60 | 3.34 | | -7% | |
| **$S_{UE}$(r) – $S_{UE}$(FF3)** | 3.40 | 3.29 | | -3% | |
| **$S_{UE}$(r) – $S_{UE}$(FF3)** | 3.29 | 3.23 | | -2% | |
| | | | | | |
| **$S_{UE}$(r) – $S_{fix}$(r)** | 2.47 | 2.54 | | 3% | |

**Panel E: Standard Errors for Squared Sharpe Ratio Differences (Factors Alone)**

| | Empirical (simulated) | Average FSW Asymptotic | Average BKRS Asymptotic | Difference FSW (% empirical) | Difference BKRS (% empirical) |
|---|---|---|---|---|---|
| **$S_{fix}$(FF5) – $S_{fix}$(FF3)** | 0.54 | 0.48 | 0.49 | -12% | -10% |
| **$S_{fix}$(FF6) – $S_{fix}$(FF5)** | 0.49 | 0.47 | 0.47 | -4% | -3% |
| **$S_{fix}$(FF6) – $S_{fix}$(Q5)** | 1.08 | 1.01 | 1.02 | -7% | -6% |
| | | | | | |
| **$S_{UE}$(FF5) – $S_{UE}$(FF3)** | 0.65 | 0.67 | | 2% | |
| **$S_{UE}$(FF6) – $S_{UE}$(FF5)** | 0.55 | 0.55 | | -1% | |
| **$S_{UE}$(FF6) – $S_{fix}$(Q5)** | 1.20 | 1.37 | | 14% | |

## Table 4: The Empirical Distributions of t-ratios

A parametric bootstrap generates 1000 simulation trials. Each set of simulated data has 743 observations. Squared Sharpe ratios, S(.), and their differences are estimated, bias adjusted using the second order expansion method, and their asymptotic standard deviations are calculated using the propositions and the Theorem I. Squared t-ratios are formed as the squared adjusted Sharpe ratio or difference less its "true" value, divided by its asymptotic standard error. The true values are from simulations with 743*1000 observations. Fractiles of the empirical distribution from the 1,000 simulation trials are shown. $\chi(1)$ are the values for a Chi distribution with one degree of freedom. Fixed weight portfolios ignore the conditioning information. UE is efficient with respect to Z. The lagged instruments are the four "classical" ones. The average return of a three-month Treasury bill is taken to be the zero beta rate. The N=25 portfolios are the 5x5 size x book/market sorts, the N=49 are industry portfolios and the N=99 combine the first two sets with 25 investment x profitability portfolios from Kenneth French.

**Panel A: Fractile Values of T-ratios for Squared Sharpe Ratio Levels**

| | Fixed Weight Portfolios | | | Dynamic UE Portfolios | | |
|---|---|---|---|---|---|---|
| **Percentile:** | 90% | 95% | 98% | 90% | 95% | 98% |
| $\chi(1)$ | 1.65 | 1.96 | 2.33 | 1.65 | 1.96 | 2.33 |
| **S($R_m$)** | 2.01 | 2.82 | 3.88 | 1.79 | 2.61 | 3.69 |
| **S(FF3)** | 2.01 | 2.39 | 3.07 | 1.72 | 2.04 | 2.53 |
| **S(FF5)** | 1.87 | 2.18 | 2.87 | 1.63 | 2.00 | 2.45 |
| **S(FF6)** | 1.80 | 2.17 | 2.68 | 1.73 | 1.99 | 2.46 |
| **S(Q5)** | 1.67 | 2.00 | 2.28 | 1.28 | 1.55 | 1.86 |
| **S(r) N=25** | 1.73 | 2.08 | 2.55 | 1.76 | 2.18 | 2.49 |
| **S(r) N=49** | 1.57 | 1.84 | 2.28 | 1.72 | 2.06 | 2.42 |
| **S(r) N=99** | 1.59 | 1.96 | 2.40 | 1.82 | 2.17 | 2.55 |

**Panel B: Squared Sharpe Ratio Differences (N=25)**

| | Fixed Weight Portfolios | | | Dynamic UE Portfolios | | |
|---|---|---|---|---|---|---|
| **Percentile:** | 90% | 95% | 98% | 90% | 95% | 98% |
| $\chi(1)$ | 1.65 | 1.96 | 2.33 | 1.65 | 1.96 | 2.33 |
| **S(r) - Rm** | 1.62 | 1.96 | 2.40 | 1.84 | 2.12 | 2.35 |
| **S(r) – S(FF3)** | 1.76 | 2.05 | 2.56 | 1.78 | 2.11 | 2.57 |
| **S(r) – S(FF5)** | 1.63 | 1.99 | 2.43 | 1.73 | 2.12 | 2.38 |
| **$S_{UE}$(r) – $S_{fix}$(r)** | 1.83 | 2.13 | 2.48 | NA | NA | NA |

**Panel C: Squared Sharpe Ratio Differences (N=49)**

| | Fixed Weight Portfolios | | | Dynamic UE Portfolios | | |
|---|---|---|---|---|---|---|
| **Percentile:** | 90% | 95% | 98% | 90% | 95% | 98% |
| $\chi(1)$ | 1.65 | 1.96 | 2.33 | 1.65 | 1.96 | 2.33 |
| **S(r) - Rm** | 1.96 | 2.25 | 2.57 | 1.81 | 2.01 | 2.32 |
| **S(r) – S(FF3)** | 1.71 | 2.00 | 2.34 | 1.71 | 2.05 | 2.28 |
| **S(r) – S(FF5)** | 1.65 | 1.98 | 2.38 | 1.69 | 1.95 | 2.29 |
| **$S_{UE}(r) - S_{fix}(r)$** | 1.83 | 2.13 | 2.47 | NA | NA | NA |

**Panel D: Squared Sharpe Ratio Differences (N=99)**

| | Fixed Weight Portfolios | | | Dynamic UE Portfolios | | |
|---|---|---|---|---|---|---|
| **Percentile:** | 90% | 95% | 98% | 90% | 95% | 98% |
| $\chi(1)$ | 1.65 | 1.96 | 2.33 | 1.65 | 1.96 | 2.33 |
| **S(r) - Rm** | 1.54 | 1.84 | 2.23 | 2.01 | 2.41 | 2.73 |
| **S(r) – S(FF3)** | 1.59 | 1.88 | 2.25 | 1.83 | 2.13 | 2.62 |
| **S(r) – S(FF5)** | 1.57 | 1.91 | 2.25 | 1.79 | 2.13 | 2.56 |
| **$S_{UE}(r) - S_{fix}(r)$** | 1.80 | 2.07 | 2.44 | NA | NA | NA |

**Panel E: Squared Sharpe Ratio Differences for Factors Alone**

| | Fixed Weight Portfolios | | | Dynamic UE Portfolios | | |
|---|---|---|---|---|---|---|
| **Percentile:** | 90% | 95% | 98% | 90% | 95% | 98% |
| $\chi(1)$ | 1.65 | 1.96 | 2.33 | 1.65 | 1.96 | 2.33 |
| **S(FF5) – S(FF3)** | 1.90 | 2.32 | 3.02 | 1.60 | 1.96 | 2.53 |
| **S(FF6) – S(FF5)** | 1.88 | 2.43 | 3.42 | 1.79 | 2.27 | 2.88 |
| **S(FF6) – S(Q5)** | 1.74 | 2.00 | 2.55 | 1.41 | 1.75 | 2.03 |

## Table 5: Squared Sharpe Ratios for Mimicking Portfolios

A parametric bootstrap generates 1000 simulation trials. Each set of simulated data has T=743 observations for consumption growth and T=587 for the leverage risk factor. The squared Sharpe ratios of mimicking portfolios are estimated and shown, Unadjusted and Adjusted using the JK (1980) bias adjustment. The values are stated in percent (multiplied by 100). The true values are from simulations with T*1,000 observations. Their asymptotic standard deviations are calculated using the propositions and the Theorem I. The Average Asymptotic value is taken across the 1,000 simulation trials. The empirical standard error is the standard deviation of the Sharpe ratio point estimates taken across the 1,000 simulation trials. The lagged instruments are the four classical instruments described in the data section. The average return of a three-month Treasury bill is taken to be the zero beta rate. The mimicking portolios are formed from 25 size x value portfolios, with either fixed weights (fix) or efficient portfolio weights with respect to the lagged information (UE).

| | Squared Sharpe Ratio Levels | | | Standard Errors | |
|---|---|---|---|---|---|
| | TRUE | No-Adj | Adj | Average Asymptotic | Empirical (simulated) |
| **Consumption (fixed)** | 1.41 | 1.34 | 1.20 | 0.35 | 0.38 |
| **Consumption (UE)** | 1.40 | 1.65 | 1.51 | 0.51 | 0.49 |
| | | | | | |
| **Leverage (fixed)** | 2.80 | 3.63 | 3.44 | 0.81 | 0.41 |
| **Leverage (UE)** | 2.77 | 3.67 | 3.48 | 0.83 | 0.38 |

## Table 6: Relative Tests of Factor Models

The test statistic is the difference in the bias-adjusted squared Sharpe ratios (not multiplied by 100) for the test assets and the factors versus the factors alone. The t-ratios are in parentheses. The factor model abbreviations and test asset portfolios are described in the text. Monthly Sharpe ratios are computed using the average Treasury bill return of 0.39 percent per month as the zero-beta rate. The dynamic models trade optimally using the four classical lagged instruments in Panel B and using the modern instruments previously described in Panel C. The sample period is January 1967 to December, 2020.

**Panel A: Fixed weight Models**

| | CAPM | FF3 | FF5 | FF6 | Q4 | Q5 |
|---|---|---|---|---|---|---|
| 25 size x value portfolios: | 0.16 | 0.13 | 0.11 | 0.09 | 0.12 | 0.11 |
| | (4.03) | (3.46) | (2.91) | (2.58) | (3.13) | (2.73) |
| 25 Investment x productivity: | 0.06 | 0.05 | -0.01 | -0.01 | 0.00 | 0.01 |
| | (2.35) | (1.89) | (-0.27) | (-0.72) | (0.20) | (0.27) |
| 32 size x value portfolios x prod: | 0.09 | 0.07 | 0.05 | 0.04 | 0.05 | 0.06 |
| | (2.73) | (2.23) | (1.49) | (1.42) | (1.69) | (1.56) |
| 49 Industry: | 0.08 | 0.12 | 0.17 | 0.15 | 0.14 | 0.12 |
| | (2.59) | (3.08) | (3.84) | (3.44) | (2.98) | (2.55) |

**Panel B: Dynamic Models using Classical Instruments**

| | CAPM | FF3 | FF5 | FF6 | Q4 | Q5 |
|---|---|---|---|---|---|---|
| 25 size x value portfolios: | 0.24 | 0.19 | 0.17 | 0.15 | 0.20 | 0.19 |
| | (4.68) | (3.94) | (3.58) | (3.29) | (3.86) | (3.22) |
| 25 Investment x productivity: | 0.11 | 0.10 | 0.03 | 0.02 | 0.06 | 0.07 |
| | (2.95) | (2.53) | (0.84) | (0.64) | (1.72) | (1.58) |
| 32 size x value portfolios x prod: | 0.18 | 0.15 | 0.11 | 0.11 | 0.16 | 0.17 |
| | (3.76) | (3.03) | (2.53) | (2.46) | (3.07) | (2.70) |
| 49 Industry: | 0.20 | 0.21 | 0.26 | 0.24 | 0.26 | 0.24 |
| | (3.89) | (3.68) | (4.19) | (3.93) | (4.03) | (3.71) |

**Panel C: Dynamic Models using Modern Instruments**

| | CAPM | FF3 | FF5 | FF6 | Q4 | Q5 |
|---|---|---|---|---|---|---|
| 25 size x value portfolios: | 0.32 | 0.27 | 0.24 | 0.24 | 0.28 | 0.27 |
| | (5.62) | (4.94) | (4.45) | (4.42) | (4.78) | (4.11) |
| 25 Investment x productivity: | 0.23 | 0.21 | 0.15 | 0.14 | 0.18 | 0.18 |
| | (4.80) | (4.59) | (3.42) | (3.35) | (3.97) | (3.29) |
| 32 size x value portfolios x prod: | 0.21 | 0.18 | 0.15 | 0.15 | 0.19 | 0.19 |
| | (3.94) | (3.35) | (2.83) | (2.89) | (3.30) | (2.96) |
| 49 Industry: | 0.47 | 0.49 | 0.53 | 0.53 | 0.52 | 0.50 |
| | (6.43) | (6.55) | (6.41) | (6.48) | (6.40) | (6.16) |

**Table 7: Relative Tests of Factor Models with an Estimated Zero Beta Rate**

The test statistic is the difference in bias-adjusted squared Sharpe ratios for the test assets and the factors versus the factors alone. The factor model abbreviations and test asset portfolios are described in the text. Monthly Sharpe ratios are computed using the estimated zero beta rate, assuming no risk-free asset exists, and shown as percent per month. The dynamic models trade optimally using the four lagged instruments previously described. The sample period is January 1967 to December, 2020.

**Panel A: Fixed weight Models**

| | CAPM | FF3 | FF5 | FF6 | Q4 | Q5 |
|---|---|---|---|---|---|---|
| 25 size x value portfolios: | 0.05 | 0.07 | 0.06 | 0.05 | 0.06 | 0.08 |
| | (2.00) | (1.20) | (0.85) | (0.63) | (1.08) | (0.82) |
| Zero Beta rate | 0.0155 | 0.0038 | 0.0036 | 0.0035 | 0.0028 | 0.0023 |
| 25 Investment x productivity: | 0.03 | 0.05 | -0.01 | -0.02 | 0.00 | -0.01 |
| | (1.33) | (1.80) | (-0.13) | (-0.29) | (-0.06) | (-0.09) |
| Zero Beta rate | 0.0152 | 0.0052 | 0.0038 | 0.0036 | 0.0043 | 0.0042 |
| 32 size x value portfolios x prod: | 0.03 | 0.04 | 0.02 | 0.02 | 0.03 | 0.04 |
| | (1.11) | (0.68) | (0.28) | (0.23) | (0.56) | (0.43) |
| Zero Beta rate | 0.0126 | 0.0042 | 0.0039 | 0.0037 | 0.0036 | 0.0033 |
| 49 Industry: | 0.08 | 0.08 | 0.08 | 0.09 | 0.08 | 0.08 |
| | (2.67) | (2.44) | (1.96) | (1.99) | (2.39) | (2.05) |
| Zero Beta rate | 0.0083 | 0.0070 | 0.0064 | 0.0060 | 0.0076 | 0.0067 |

**Panel B: Dynamic Models using Classical Instruments**

| | CAPM | FF3 | FF5 | FF6 | Q4 | Q5 |
|---|---|---|---|---|---|---|
| 25 size x value portfolios: | 0.11 | 0.11 | 0.11 | 0.09 | 0.14 | 0.15 |
| | (2.90) | (1.82) | (1.36) | (1.09) | (2.07) | (1.52) |
| Zero Beta rate | 0.0155 | 0.0039 | 0.0037 | 0.0036 | 0.0028 | 0.0023 |
| 25 Investment x productivity: | 0.03 | 0.05 | -0.02 | -0.03 | 0.01 | 0.01 |
| | (0.98) | (1.51) | (-0.40) | (-0.53) | (0.14) | (0.11) |
| Zero Beta rate | 0.0154 | 0.0052 | 0.0038 | 0.0036 | 0.0042 | 0.0042 |
| 32 size x value portfolios x prod: | 0.10 | 0.09 | 0.06 | 0.06 | 0.11 | 0.12 |
| | (2.32) | (1.41) | (0.75) | (0.66) | (1.82) | (1.35) |
| Zero Beta rate | 0.0128 | 0.0042 | 0.0039 | 0.0038 | 0.0036 | 0.0033 |
| 49 Industry: | 0.19 | 0.16 | 0.15 | 0.17 | 0.19 | 0.20 |
| | (3.76) | (3.29) | (3.17) | (3.28) | (3.97) | (3.82) |
| Zero Beta rate | 0.0084 | 0.0071 | 0.0065 | 0.0060 | 0.0076 | 0.0067 |

| Panel C: Dynamic Models using Modern Instruments | | | | | | |
|---|---|---|---|---|---|---|
| | CAPM | FF3 | FF5 | FF6 | Q4 | Q5 |
| 25 size x value portfolios: | 0.12 | 0.12 | 0.10 | 0.09 | 0.14 | 0.16 |
| | (2.83) | (1.65) | (1.03) | (0.85) | (2.01) | (1.57) |
| Zero Beta rate | 0.0162 | 0.0039 | 0.0037 | 0.0036 | 0.0029 | 0.0025 |
| 25 Investment x productivity: | 0.07 | 0.09 | 0.02 | 0.01 | 0.04 | 0.04 |
| | (1.89) | (2.20) | (0.40) | (0.23) | (0.75) | (0.47) |
| Zero Beta rate | 0.0154 | 0.0053 | 0.0038 | 0.0037 | 0.0044 | 0.0044 |
| 32 size x value portfolios x prod: | 0.08 | 0.07 | 0.05 | 0.04 | 0.10 | 0.11 |
| | (1.79) | (1.13) | (0.49) | (0.38) | (1.57) | (1.25) |
| Zero Beta rate | 0.0131 | 0.0042 | 0.0039 | 0.0038 | 0.0037 | 0.0034 |
| 49 Industry: | 0.38 | 0.37 | 0.36 | 0.36 | 0.37 | 0.38 |
| | (5.51) | (5.72) | (5.62) | (5.56) | (6.16) | (6.36) |
| Zero Beta rate | 0.0088 | 0.0072 | 0.0065 | 0.0061 | 0.0080 | 0.0073 |

**Table 8: Direct Factor Model Comparisons**

The test statistic is the difference in bias-adjusted squared Sharpe ratios (not multiplied by 100) for the first model less the second model. The factors are held with fixed weights over time (no instruments) or dynamically traded using either the four classical or the four modern instruments. The t-ratios are the differences divided by the asymptotic standard errors for the difference. The factor model abbreviations and test asset portfolios are described in the text. FF6* replaces the HML factor in FF6 factors version. The sample period is January 1967 to December, 2020.

| | No Instruments | Classical Instruments | Modern Instruments |
|---|---|---|---|
| FF3 – $R_m$ | 0.03 | 0.05 | 0.06 |
| (t-ratio) | (1.73) | (2.45) | (2.74) |
| FF5 – FF3 | 0.06 | 0.08 | 0.08 |
| (t-ratio) | (2.38) | (2.74) | (2.44) |
| FF6 – FF5 | 0.03 | 0.03 | 0.06 |
| (t-ratio) | (1.37) | (1.46) | (2.40) |
| Q5 – Q4 | 0.20 | 0.20 | 0.21 |
| (t-ratio) | (3.75) | (3.80) | (3.81) |
| Q4 – FF5 | 0.08 | 0.03 | 0.06 |
| (t-ratio) | (2.20) | (0.81) | (1.36) |
| Q5 – FF6 | 0.25 | 0.21 | 0.20 |
| (t-ratio) | (4.05) | (3.23) | (3.09) |
| Q5 – FF6* | 0.22 | 0.19 | 0.19 |
| (t-ratio) | (3.32) | (2.86) | (2.81) |
| Q5 – M4 | 0.16 | 0.13 | 0.15 |
| (t-ratio) | (2.66) | (2.16) | (2.32) |

## Table 5: Squared Sharpe Ratios for Mimicking Portfolios

A parametric bootstrap generates 1000 simulation trials. Each set of simulated data has T=743 observations for consumption growth and T=587 for the leverage risk factor and T=743 for the Chen, Roll and Ross (1986) factors. The squared Sharpe ratios of mimicking portfolios are estimated and shown, Unadjusted and Adjusted using the JK (1980) bias adjustment. The values are stated in percent (multiplied by 100). The true values are from simulations with T*1,000 observations. Asymptotic standard deviations are calculated using the propositions and Theorem I. The Average Asymptotic value is the average taken across the 1,000 simulation trials. The empirical standard error is the standard deviation of the Sharpe ratio bias-adjusted point estimates taken across the 1,000 simulation trials. The lagged instruments are described in the data section. The average return of a three-month Treasury bill is taken to be the zero beta rate. The mimicking portolios are formed from 25 size x value portfolios, with either fixed weights (fixed) or time-varying weights (with Z). The bottom rows use fixed-weight mimicking portfolios for the CRR factors, combined with time-varying weights (with Z).

| | Squared Sharpe Ratio Levels | | | Standard Errors | |
|---|---|---|---|---|---|
| | TRUE | No-Adj | Adj | Average Asymptotic | Empirical (simulated) |
| **Consumption (fixed)** | 1.54 | 1.35 | 1.21 | 0.39 | 0.34 |
| **Consumption (with Z)** | 1.60 | 1.69 | 1.55 | 0.49 | 0.53 |
| **Leverage (fixed)** | 2.47 | 2.70 | 2.52 | 0.36 | 0.38 |
| **Leverage (with Z)** | 2.43 | 2.73 | 2.54 | 0.36 | 0.41 |
| **Chen Roll and Ross (CRR) factors** | | | | | |
| **Change Expected Inflation (fixed)** | 0.28 | 0.58 | 0.45 | 0.22 | 0.30 |
| **Change Expected Inflation (with Z)** | 0.27 | 1.12 | 0.99 | 0.42 | 0.45 |
| **Industry Production (fixed)** | 2.05 | 1.86 | 1.72 | 0.41 | 0.38 |
| **Industry Production (with Z)** | 1.98 | 2.09 | 1.94 | 0.47 | 0.58 |
| **Unexpected Inflation (fixed)** | 0.03 | 0.37 | 0.23 | 0.39 | 0.14 |
| **Unexpected Inflation (with Z)** | 0.04 | 0.80 | 0.66 | 1.67 | 0.30 |
| **Three CRR factors (fixed)** | 3.03 | 3.02 | 2.60 | 0.36 | 0.47 |
| **Three CRR factors (with Z)** | 4.61 | 6.13 | 4.34 | 0.50 | 0.57 |