Appendix F. Proofs for the LRR-Vol Model

The stochastic discount factor in equation (9) can be written as:

$$M_{t+1} = \exp\left\{a + b^m z_t^m - \frac{1}{2}\underline{x}_t' \lambda \Sigma_{\varepsilon,\varepsilon} \lambda' \underline{x}_t - \underline{x}_t' \lambda \underline{\varepsilon}_{t+1}\right\}$$

where $b^m = b/\delta^m$, $z_t^m = \delta^m z_t$, $\Sigma_{\varepsilon,\varepsilon}$ is the variance of $\underline{\varepsilon} \equiv \left[\varepsilon^d, \varepsilon^u, \varepsilon^z, \varepsilon^x, \varepsilon^w\right]'$, $\underline{x}_t \equiv [x_t, \sigma_t - \bar{\sigma}]'$, and

$\lambda \equiv \frac{1}{\sigma_d}\begin{bmatrix} \bar{\sigma} & 0 & 0 & 0 & 0 \\ \bar{x} & 0 & 0 & 0 & 0 \end{bmatrix}$ in all our 4 models.

1. Price-Dividend Ratio

Equation (11) says that the price-dividend ratio of the strip with n quarters to maturity at time t can be expressed as:

$$\frac{P_{n,t}^m}{D_t^m} = \exp\left\{A(n) + B_{\underline{x}}(n)\underline{x}_t + B_z(n) z_t^m\right\}$$

where $B_{\underline{x}}(n) \equiv [B_x(n),\ B_\sigma(n)]$. Using the boundary condition that $P_{0,t}^m = D_t^m$, we see that this holds when $n = 0$, with $A(0) = B_z(0) = B_x(0) = B_\sigma(0) = 0$. We proceed by induction on n. We can write the price of the strip with n quarters to maturity as a function of the price of a strip with $n-1$ quarters to maturity:

$$P_{n,t}^m = E_t[M_{t+1} P_{n-1,t+1}^m]$$

Dividing by the market dividend at time t:

$$\frac{P_{n,t}^m}{D_t^m} = E_t\left[M_{t+1}\left(\frac{D_{t+1}^m}{D_t^m}\right)\left(\frac{P_{n-1,t+1}^m}{D_{t+1}^m}\right)\right]$$

Plugging in the expressions for the stochastic discount factor and dividend growth:

$$\begin{aligned}\frac{P^m_{n,t}}{D^m_t} &= E_t\Big[\exp\Big\{\Big(a+b^m z^m_t-\frac{1}{2}\underline{x}_t'\lambda\Sigma_{\varepsilon,\varepsilon}\lambda'\underline{x}_t-\underline{x}_t'\lambda\underline{\varepsilon}_{t+1}\Big)+\left(g^m+z^m_t+\delta^m\varepsilon^d_{t+1}+\varepsilon^u_{t+1}\right)\\ &\qquad +A\left(n-1\right)+B_{\underline{x}}\left(n-1\right)\underline{x}_{t+1}+B_z\left(n-1\right)z^m_{t+1}\Big\}\Big]\end{aligned}$$

Here $\underline{x}$ evolves as a 2-dimensional VAR(1):

$$\underline{x}_{t+1}=\left(I-\Phi_x\right)\underline{\bar{x}}+\Phi_x\underline{x}_t+\underline{\varepsilon}^x_{t+1}$$

where $\underline{\bar{x}}=[\bar{x},0]'$, $\underline{\varepsilon}^x_t\equiv[\varepsilon^x_t,\varepsilon^w_t]'$, and $\Phi_x=\begin{bmatrix}\phi_x & 0\\ 0 & \phi_\sigma\end{bmatrix}$. Factoring out the time-t information, and expanding the $\underline{x}_{t+1}$ and z^m_{t+1} processes:

$$\begin{aligned}\frac{P^m_{n,t}}{D^m_t} &= \exp\Big\{a+b^m z^m_t-\frac{1}{2}\underline{x}_t'\lambda\Sigma_{\varepsilon,\varepsilon}\lambda'\underline{x}_t+g^m+z^m_t+A\left(n-1\right)\Big\}\\ &\quad\times E_t\left[\exp\left\{-\underline{x}_t'\lambda\underline{\varepsilon}_{t+1}+\delta^m\varepsilon^d_{t+1}+\varepsilon^u_{t+1}+B_{\underline{x}}\left(n-1\right)\left[\left(I-\Phi_x\right)\underline{\bar{x}}+\Phi_x\underline{x}_t+\underline{\varepsilon}^x_{t+1}\right]\right.\right.\\ &\qquad\left.\left.+B_z\left(n-1\right)\left[\phi_z z^m_t+\varepsilon^z_{t+1}\right]\right\}\right]\\ &= \exp\Big\{a+b^m z^m_t-\frac{1}{2}\underline{x}_t'\lambda\Sigma_{\varepsilon,\varepsilon}\lambda'\underline{x}_t+g^m+z^m_t+A\left(n-1\right)\\ &\qquad+B_{\underline{x}}\left(n-1\right)\left[\left(I-\Phi_x\right)\underline{\bar{x}}+\Phi_x\underline{x}_t\right]+\phi_z B_z\left(n-1\right)z^m_t\Big\}\\ &\quad\times E_t\left[\exp\left\{\left(\begin{bmatrix}\delta^m, & 1, & B_z\left(n-1\right), & B_{\underline{x}}\left(n-1\right)\end{bmatrix}-\underline{x}_t'\lambda\right)\underline{\varepsilon}_{t+1}\right\}\right]\\ &= \exp\Big\{a+b^m z^m_t-\cancel{\frac{1}{2}\underline{x}_t'\lambda\Sigma_{\varepsilon,\varepsilon}\lambda'\underline{x}_t}+g^m+z^m_t+A\left(n-1\right)\\ &\qquad+B_{\underline{x}}\left(n-1\right)\left[\left(I-\Phi_x\right)\underline{\bar{x}}+\Phi_x\underline{x}_t\right]+\phi_z B_z\left(n-1\right)z^m_t\Big\}\\ &\quad\times\exp\Big\{\cancel{\frac{1}{2}\underline{x}_t'\lambda\Sigma_{\varepsilon,\varepsilon}\lambda'\underline{x}_t}+\frac{1}{2}C^m_{n-1}\Sigma_{\varepsilon,\varepsilon}(C^m_{n-1})'-C^m_{n-1}\Sigma_{\varepsilon,\varepsilon}\lambda'\underline{x}_t\Big\}\end{aligned}$$

where $C^m_{n-1} \equiv [\ \delta^m, \quad 1, \quad B_z(n-1), \quad B_{\underline{x}}(n-1)\ ]$. Collecting constant, z^m_t and $\underline{x}_t$ terms:

$$\frac{P^m_{n,t}}{D^m_t} = \exp\left\{\underbrace{A(n-1) + a + g^m + B_{\underline{x}}(n-1)(I-\Phi_x)\underline{\bar{x}} + \frac{1}{2}C^m_{n-1}\Sigma_{\varepsilon,\varepsilon}(C^m_{n-1})'}_{A(n)}\right.$$
$$\left. + [\underbrace{1 + b^m + \phi_z B_z(n-1)}_{B_z(n)}] z^m_t\} + [\underbrace{B_{\underline{x}}(n-1)\Phi_x - C^m_{n-1}\Sigma_{\varepsilon,\varepsilon}\lambda'}_{B_{\underline{x}}(n)}]\underline{x}_t \right\}$$

Matching coefficients, and plugging in for λ:

$$\begin{aligned}
A(n) &= A(n-1) + a + g^m + (1-\phi_x)\bar{x}B_x(n-1) + \frac{1}{2}C^m_{n-1}\Sigma_{\varepsilon,\varepsilon}(C^m_{n-1})' \\
B_z(n) &= 1 + b^m + \phi_z B_z(n-1) \\
&= \frac{(1+b^m)(1-\phi_z^n)}{1-\phi_z} \\
B_x(n) &= \phi_x B_x(n-1) - \frac{\bar{\sigma}}{\sigma_d}\Sigma_{d,\varepsilon}(C^m_{n-1})' \\
B_\sigma(n) &= \phi_\sigma B_\sigma(n-1) - \frac{\bar{x}}{\sigma_d}\Sigma_{d,\varepsilon}(C^m_{n-1})'
\end{aligned}$$

where $\Sigma_{d,\varepsilon} \equiv E[\varepsilon^d \underline{\varepsilon}']$.

2. Mean and Variance of the Log Return on Market-Dividend Strips

The return from time t to $t+1$ of the strip with a maturity at time $t+n$ is given by:

$$
\begin{aligned}
r^m_{n,t+1} &\equiv \log\left(R^m_{n,t+1}\right) \\
&= \log\left(\frac{P^m_{n-1,t+1}}{P^m_{n,t}}\right) \\
&= \log\left[\left(\frac{P^m_{n-1,t+1}/D^m_{t+1}}{P^m_{n,t}/D^m_t}\right)\left(\frac{D^m_{t+1}}{D^m_t}\right)\right] \\
&= \log\left(\frac{P^m_{n-1,t+1}}{D^m_{t+1}}\right) - \log\left(\frac{P^m_{n,t}}{D^m_t}\right) + \log\left(\frac{D^m_{t+1}}{D^m_t}\right) \\
&= A(n-1) + B_{\underline{x}}(n-1)\,\underline{x}_{t+1} + B_z(n-1)\,z^m_{t+1} - A(n) - B_{\underline{x}}(n)\,\underline{x}_t \\
&\quad -B_z(n)\,z^m_t + \Delta d^m_{t+1} \\
&= A(n-1) + B_{\underline{x}}(n-1)\left[\underline{\bar{x}}\,(I-\Phi_x) + \Phi_x\underline{x}_t\right] + B_z(n-1)\,\phi_z z^m_t - A(n) - B_{\underline{x}}(n)\,\underline{x}_t \\
&\quad -B_z(n)\,z^m_t + g^m + z^m_t + B_{\underline{x}}(n-1)\,\underline{\varepsilon}^x_{t+1} + B_z(n-1)\,\varepsilon^z_{t+1} + \delta^m\varepsilon^d_{t+1} + \varepsilon^u_{t+1} \\
&= E_t[r^m_{n,t+1}] + C^m_{n-1}\underline{\varepsilon}_{t+1}
\end{aligned}
$$

The time-t conditional variance of the return from time t to $t+1$ of the strip with a maturity of n periods is therefore given by:

$$
\begin{aligned}
\sigma^2_t[r^m_{n,t+1}] &= \sigma^2_t\left[E_t[r^m_{n,t+1}] + C^m_{n-1}\underline{\varepsilon}_{t+1}\right] \\
&= C^m_{n-1}\Sigma_{\varepsilon,\varepsilon}(C^m_{n-1})'
\end{aligned}
$$

3. Riskfree Rate

The riskfree rate is given by:

$$
\begin{aligned}
R^f_{t+1} &= \frac{1}{E_t[M_{t+1}]} \\
&= \exp\{-a - bz_t\} = \exp\{-a - b^m z^m_t\}
\end{aligned}
$$

from the conditional log-nomality of M_{t+1}.

4. Log Risk Premium of Market-Dividend Strips

By definition, the log risk premium of a strip is given by:

$$\begin{aligned}
\log\left(E_t\left[\frac{R^m_{n,t+1}}{R^f_t}\right]\right) &= \log\left(E_t\left[\exp\left\{r^m_{n,t+1} - r^f_t\right\}\right]\right) \\
&= E_t[r^m_{n,t+1} - r^f_t] + \frac{1}{2}\sigma^2_t[r^m_{n,t+1}]
\end{aligned}$$

since $r^m_{n,t+1}$ is normally distributed. This expression can be obtained from the Euler equation:

$$\begin{aligned}
1 &= E_t[M_{t+1}R^m_{n,t+1}] \\
&= E_t[\exp\{\log(M_{t+1}) + r^m_{n,t+1}\}] \\
&= E_t\left[\exp\left\{-r^f_t - \frac{1}{2}\underline{x}'_t\lambda\Sigma_{\varepsilon,\varepsilon}\lambda'\underline{x}_t - \underline{x}'_t\lambda\underline{\varepsilon}_{t+1} + E_t[r^m_{n,t+1}] + C^m_{n-1}\underline{\varepsilon}_{t+1}\right\}\right]
\end{aligned}$$

Taking logs:

$$\begin{aligned}
0 &= \log\left(E_t\left[\exp\left\{-r^f_t - \frac{1}{2}\underline{x}'_t\lambda\Sigma_{\varepsilon,\varepsilon}\lambda'\underline{x}_t - \underline{x}'_t\lambda\underline{\varepsilon}_{t+1} + E_t[r^m_{n,t+1}] + C^m_{n-1}\underline{\varepsilon}_{t+1}\right\}\right]\right) \\
&= E_t[r^m_{n,t+1} - r^f_t] - \cancel{\frac{1}{2}\underline{x}'_t\lambda\Sigma_{\varepsilon,\varepsilon}\lambda'}\underline{x}_t + \frac{1}{2}\left(\cancel{\underline{x}_t\lambda\Sigma_{\varepsilon,\varepsilon}\lambda'}\underline{x}_t + \sigma^2_t[r^m_{n,t+1}] - 2C^m_{n-1}\Sigma_{\varepsilon,\varepsilon}\lambda'\underline{x}_t\right)
\end{aligned}$$

Rearranging, the log risk premium is given by equation (12):

$$\begin{aligned}
&E_t[r^m_{n,t+1} - r^f_t] + \frac{1}{2}\sigma^2_t[r^m_{n,t+1}] \\
&\quad = \left(C^m_{n-1}\Sigma_{\varepsilon,\varepsilon}\lambda'\right)\underline{x}_t \\
&\quad = \left(\delta^m\sigma^2_d + \sigma_{d,u} + B_x(n-1)\,\sigma_{d,x} + B_\sigma(n-1)\,\sigma_{d,w} + B_z(n-1)\,\sigma_{d,z}\right) \\
&\qquad \times\left(\frac{\bar{\sigma}}{\sigma_d}x_t + \frac{\bar{x}}{\sigma_d}(\sigma_t - \bar{\sigma})\right)
\end{aligned}$$

Appendix G. Proofs for the CC Model

The representative agent has external habit preferences as in equation (16) and we specify the law of motion for s_t as in equation (17):

$$s_{t+1} = (1-\phi_s)\bar{s} + \phi_s s_t + \lambda(\bar{s})z_t + \lambda(s_t)\sigma_t \varepsilon^d_{t+1}$$

where $\lambda(.)$ is as defined by CC and in footnote 3:

$$\lambda(s_t) = \begin{cases} \frac{1}{\bar{S}}\sqrt{1-2(s_t-\bar{s})} - 1 & s_t \leq s_{\max} \\ 0 & s_t \geq s_{\max} \end{cases}$$

with $\bar{s} \equiv \log(\bar{S})$ and $s_{\max} = \bar{s} + \frac{1}{2}(1-(\bar{S})^2)$. We set $\bar{S} \equiv \bar{\sigma}\sigma_d\sqrt{\frac{\gamma}{1-\phi_s}}$.

1. Riskfree Rate

Using the law of motion for Δc_{t+1} and subtracting s_t from the law of motion for s_{t+1}, the stochastic discount factor in CC is:

$$\begin{aligned} M_{t+1} &= \exp\{\log(\delta) - \gamma(\Delta c_{t+1} + \Delta s_{t+1})\} \\ &= \exp\{\log(\delta) - \gamma(g + z_t + \sigma_t\varepsilon^d_{t+1} + (1-\phi_s)\bar{s} + \phi_s s_t + \lambda(\bar{s})z_t + \lambda(s_t)\sigma_t\varepsilon^d_{t+1} - s_t)\} \\ &= \exp\{\log(\delta) - \gamma g - \gamma(1+\lambda(\bar{s}))z_t + \gamma(1-\phi_s)(s_t-\bar{s}) - \gamma(1+\lambda(s_t))\sigma_t\varepsilon^d_{t+1}\} \end{aligned}$$

The log riskfree rate is then:

$$\begin{aligned} r^f_{t+1} &\equiv \log(R^f_{t+1}) \\ &= -\log(E_t[M_{t+1}]) \\ &= -\log(\delta) + \gamma g + \gamma(1+\lambda(\bar{s}))z_t - \gamma(1-\phi_s)(s_t - \bar{s}) - \frac{\gamma^2}{2} \underbrace{(1+\lambda(s_t))^2}_{=\frac{(1-\phi_s)(1-2(s_t-\bar{s}))}{\gamma \not\sigma_d^2 \bar{\sigma}^2}} \not\sigma_d^2 \sigma_t^2 \\ &= -\log(\delta) + \gamma g + \gamma(1+\lambda(\bar{s}))z_t - \frac{\gamma}{2}(1-\phi_s)\frac{\sigma_t^2}{\bar{\sigma}^2} + \gamma(1-\phi_s)(s_t-\bar{s})\left(\frac{\sigma_t^2}{\bar{\sigma}^2} - 1\right) \end{aligned}$$

2. Properties of the Assumed Habit Process

As in CC, we require habit to be pre-determined at, and near, the steady state for the consumption surplus, $s_t = \bar{s}$:

$$\left[\frac{\partial h_{t+1}}{\partial d_{t+1}}\right]_{s_t=\bar{s}} = 0 \tag{28}$$

$$\text{and } \left[\frac{\partial}{\partial s}\left(\frac{\partial h_{t+1}}{\partial d_{t+1}}\right)\right]_{s_t=\bar{s}} = 0 \tag{29}$$

We first calculate $\frac{\partial h_{t+1}}{\partial d_{t+1}}$ using an expression for h_{t+1} from the definition of s_{t+1}:

$$\begin{aligned} s_{t+1} &\equiv \log\left(\frac{D_{t+1} - H_{t+1}}{D_{t+1}}\right) \\ &= \log\left(1 - e^{h_{t+1} - d_{t+1}}\right) \\ \Rightarrow h_{t+1} &= d_{t+1} + \log\left(1 - e^{s_{t+1}}\right) \end{aligned}$$

Differentiating:

$$\begin{aligned}
\frac{\partial h_{t+1}}{\partial d_{t+1}} &= 1 + \frac{\partial s_{t+1}}{\partial d_{t+1}} \frac{\partial}{\partial s_{t+1}} \left[\log\left(1 - e^{s_{t+1}}\right)\right] \\
&= 1 - \lambda(s_t) \frac{e^{s_{t+1}}}{1 - e^{s_{t+1}}} \\
&= 1 - \frac{\lambda(s_t)}{e^{-s_{t+1}} - 1} \\
&\approx 1 - \frac{\lambda(s_t)}{e^{-s_t} - 1}
\end{aligned}$$

with the approximation since $s_{t+1} \approx s_t$ for small time intervals.

Habit is pre-determined at the steady state $s_t = \bar{s}$:

The first condition, equation (28), is equivalent to:

$$\begin{aligned}
\left[\frac{\partial h_{t+1}}{\partial d_{t+1}}\right]_{s_t = \bar{s}} = 0 \quad &\Leftrightarrow \quad 1 - \frac{\lambda(\bar{s})}{e^{-\bar{s}} - 1} = 0 \\
&\Leftrightarrow \quad \lambda(\bar{s}) = e^{-\bar{s}} - 1
\end{aligned}$$

which holds from the definition of $\lambda(.)$.

Habit is pre-determined near the steady state $s_t = \bar{s}$:

The second condition, equation (29), is equivalent to:

$$\begin{aligned}
0 &= \left[\frac{\partial}{\partial s}\left(\frac{\partial h_{t+1}}{\partial d_{t+1}}\right)\right]_{s_t = \bar{s}} \\
&= \left[\frac{\partial}{\partial s}\left(1 - \frac{\lambda(s_t)}{e^{-s_t} - 1}\right)\right]_{s_t = \bar{s}} \\
&= \left[-\frac{\left(e^{-s_t} - 1\right)\lambda_s(s_t) + e^{-s_t}\lambda(s_t)}{\left(e^{-s_t} - 1\right)^2}\right]_{s_t = \bar{s}} \\
\Leftrightarrow 0 &= \left(e^{-\bar{s}} - 1\right)\lambda_s(\bar{s}) + e^{-\bar{s}}\lambda(\bar{s})
\end{aligned}$$

Rearranging, this is equivalent to:

$$\begin{aligned}\lambda_s(\bar{s}) &= \frac{\frac{1}{\bar{S}}\lambda(\bar{s})}{1-\frac{1}{\bar{S}}} \\ &= -\frac{1}{\bar{S}}\end{aligned}$$

Differentiating the sensitivity function, we verify that this holds:

$$\begin{aligned}\lambda(s_t) &= \frac{1}{\bar{S}}\sqrt{1-2(s_t-\bar{s})}-1 \\ \Rightarrow \lambda_s(s_t) &= -\frac{1}{\bar{S}\sqrt{1-2(s_t-\bar{s})}} \\ \Rightarrow \lambda_s(\bar{s}) &= -\frac{1}{\bar{S}}\end{aligned}$$

So the second condition, equation (29), holds.

3. Relation between External Habit and Past Consumption

From the definition of s_t, we have shown that a first-order log-linear approximation is given by:

$$\begin{aligned}s_t &= \log\left(1-e^{h_t-d_t}\right) \\ &\approx \underbrace{\log\left(1-e^{\overline{h-d}}\right)}_{\equiv \bar{s}} + \left[(h_t-d_t)-\left(\overline{h-d}\right)\right]\left(\frac{-e^{\overline{h-d}}}{1-e^{\overline{h-d}}}\right)\end{aligned}$$

Substituting the log-linear approximations for s_{t+1} and s_t into the law of motion for the s process:

$$\begin{aligned}&\cancel{\bar{s}} + \left[(h_{t+1}-d_{t+1})-\left(\overline{h-d}\right)\right]\left(\frac{-e^{\overline{h-d}}}{1-e^{\overline{h-d}}}\right) \\ \approx\ &\cancel{(1-\phi_s)\bar{s}} + \phi_s\left(\cancel{\bar{s}} + \left[(h_t-d_t)-\left(\overline{h-d}\right)\right]\left(\frac{-e^{\overline{h-d}}}{1-e^{\overline{h-d}}}\right)\right) + \lambda(\bar{s})\left(d_{t+1}-d_t-g\right)\end{aligned}$$

When $s_t \leq s_{\max}$ the sensitivity function is:

$$
\begin{aligned}
\lambda(s_t) &= \frac{1}{\bar{S}}\sqrt{1-2(s_t-\bar{s})}-1 \\
\Rightarrow \lambda(\bar{s}) &= \frac{1}{\bar{S}}-1 = \frac{1}{e^{\bar{s}}}-1 = \frac{1}{1-e^{\overline{h-d}}}-1 = \frac{e^{\overline{h-d}}}{1-e^{\overline{h-d}}}
\end{aligned} \tag{30}
$$

$$
\Rightarrow (h_{t+1} - \cancel{d_{t+1}}) - \left(\overline{h-d}\right) \approx \phi_s \left((h_t - d_t) - \left(\overline{h-d}\right)\right) - (\cancel{d_{t+1}} - d_t - g)
$$

which yields:

$$
h_{t+1} \approx (1-\phi_s)\left(\overline{h-d}\right) + \phi_s h_t + (1-\phi_s) d_t + g \tag{31}
$$

Iterating this recursion back to the start of time gives equation (18):

$$
h_{t+1} \approx \overline{h-d} + (1-\phi_s)\sum_{j=0}^{\infty}(\phi_s)^j d_{t-j} + \frac{g}{1-\phi_s}
$$

since the transversality condition $\lim_{j\to\infty}[\phi_s^j h_{t-j-1}] = 0$ holds. Note that subtracting h_t from both sides of equation (31) gives:

$$
h_{t+1} - h_t \approx g + (1-\phi_s)\left[(d_t - h_t) - \overline{d-h}\right]
$$

Appendix H. z_t^m is a Proxy for the Consumption-Market Dividend Ratio

Lettau and Ludvigson (2005) show that:

$$
d_t - \nu d_t^m - (1-\nu) y_t \approx E_t \sum_{i=1}^{\infty} \rho_w^i \left(\nu \Delta d_{t+i}^m + (1-\nu)\Delta y_{t+i} - \Delta d_{t+i}\right)
$$

where y_t is log labor income at time t, ν is the average share of aggregate wealth from financial assets (as opposed to human capital), and ρ_w is a constant. If there is no labor income, as in our model, then $\nu = 1$ and this gives an expression for the log consumption-market dividend ratio:

$$\begin{aligned}\log\left(\frac{D_t}{D_t^m}\right) = d_t - d_t^m &\approx E_t \sum_{i=1}^{\infty} \rho_w^i \left(\Delta d_{t+i}^m - \Delta d_{t+i}\right) \\ &= \sum_{i=1}^{\infty} \rho_w^i \left(E_t[\Delta d_{t+i}^m] - E_t[\Delta d_{t+i}]\right)\end{aligned}$$

We now show that $E_t[\Delta d_{t+i}^m]$ and $E_t[\Delta d_{t+i}]$ are each affine in z_t^m:

$$\begin{aligned}E_t[\Delta d_{t+i}^m] &= E_t[\delta^m g + z_{t+i-1}^m + \delta^m \varepsilon_{t+i}^d + \varepsilon_{t+i}^u] \\ &= \delta^m g + E_t[z_{t+i-1}^m] \\ &= \delta^m g + \phi_z^{i-1} z_t^m\end{aligned}$$

$$\begin{aligned}E_t[\Delta d_{t+i}] &= E_t[g + z_{t+i-1} + \sigma_{t+i-1}\varepsilon_{t+i}^d] \\ &= g + \frac{1}{\delta^m} E_t[z_{t+i-1}^m] \\ &= g + \frac{\phi_z^{i-1} z_t^m}{\delta^m}\end{aligned}$$

since $E_t[\sigma_{t+i-1}\varepsilon_{t+i}^d] = E_t[E_{t+i-1}[\sigma_{t+i-1}\varepsilon_{t+i}^d]] = E_t[\sigma_{t+i-1}E_{t+i-1}[\varepsilon_{t+i}^d]] = 0$.

Putting this together:

$$\begin{aligned}\log\left(\frac{D_t}{D_t^m}\right) = d_t - d_t^m &\approx \sum_{i=1}^{\infty} \rho_w^i \left(E_t[\Delta d_{t+i}^m] - E_t[\Delta d_{t+i}]\right) \\ &= \sum_{i=1}^{\infty} \rho_w^i \left(\left(\delta^m g + \phi^{i-1} z_t^m\right) - \left(g + \frac{\phi_z^{i-1} z_t^m}{\delta^m}\right)\right) \\ &= (\delta^m - 1) g \sum_{i=1}^{\infty} \rho_w^i + \frac{\delta^m - 1}{\delta^m}\left(\sum_{i=1}^{\infty} \rho_w^i \phi_z^{i-1}\right) z_t^m\end{aligned}$$

which is affine in z_t^m. So z_t^m is a proxy for the log consumption-market dividend ratio.