J. Appl. Prob. **62,** 1–15 (2025)
Printed in England

SUPPLEMENTARY MATERIAL: DISTRIBUTION THEORY FOR DEPENDENT RENEWAL-REWARD PROCESSES AND THEIR FIRST-PASSAGE TIMES USING SADDLEPOINT AND RESIDUE EXPANSIONS

RONALD W. BUTLER,*

Keywords: Compound renewal processes; cumulative processes; double transforms, renewal-reward processes; residue approximations; saddlepoint approximations; shock models.

2010 Mathematics Subject Classification: Primary 60K05
Secondary 60K10

Summary of Skovgaard [18] double-saddlepoint approximations.

These approximations start with the joint CGF for random variables $R(T)$ and T and compute approximations for the conditional survival function $\mathbb{P}\{R(T) \geq x \mid T = n\}$ when T represents discrete time with counting measure on $\mathbb{N}$.

Let $\mathcal{K}(\mathrm{r}, \mathrm{s}) = \ln \mathcal{M}(\mathrm{r}, \mathrm{s})$ be the joint CGF of $\{R(T), T\}$ defined on the open convex region $(\mathrm{r}, \mathrm{s}) \in \mathcal{B} \cap \mathcal{C}$ with $(\mathrm{r}, \mathrm{s}) \leftrightsquigarrow (x, n)$. When $R(T)$ is absolutely continuous, Skovgaard [18, §3] developed an approximation $\mathbb{P}_C$ for the conditional survival function such that

$$\mathbb{P}_C\{R(T) \geq x \mid T = n\} \simeq \mathbb{P}\{R(T) \geq x \mid T = n\} \qquad x > 0. \tag{53}$$

He used the Bleistein [22] method for inversion of transform $\mathcal{M}(\mathrm{r}, \mathrm{s})/\mathrm{r}$ which carefully deals with the fact that this integrand has a simple pole at $\mathrm{r} = 0$.

For the case in which $R(T)$ is integer-valued, [18, §4] provides two continuity corrections $\mathbb{P}_{D1}$ and $\mathbb{P}_{D2}$ for the conditional survival function as in (53). These approximations follow the approach used in Daniels [24, §6] for single-saddlepoint approximations of unconditional survival functions and also use the Bleistein [22] method. We summarise all three formulas below for completeness.

Skovgaard [18] approximation $\mathbb{P}_C$ for $R(T)$ continuous

The Skovgaard double-saddlepoint $\mathbb{P}_C$ approximation is

$$\mathbb{P}_C\{R(T) \geq x \mid T = n\} = 1 - \Phi(\hat{w}) - \phi(\hat{w}) \left(\frac{1}{\hat{w}} - \frac{1}{\hat{u}} \right) \qquad \hat{r} \neq 0, \tag{54}$$

where Φ and ϕ are the standard normal CDF and density functions. Here,

$$\hat{w} = \operatorname{sgn}(\hat{r}) \sqrt{2 \left[\{\mathcal{K}(0, \hat{s}_0) - \hat{s}_0 n\} - \{\mathcal{K}(\hat{r}, \hat{s}) - \hat{r}x - \hat{s}n\} \right]} \tag{55}$$

$$\hat{u} = \hat{r} \sqrt{|\mathcal{K}''(\hat{r}, \hat{s})| \; / \; |\mathcal{K}''_{\mathrm{ss}}(0, \hat{s}_0)|}, \tag{56}$$

---

* Email address: brennerweg@yahoo.com

where $\mathcal{K}''$ is the 2×2 Hessian of $\mathcal{K}$ and $\mathcal{K}''_{\mathrm{ss}} = \partial^2\mathcal{K}/\partial \mathrm{s}^2$. The saddlepoint $(\hat{r}, \hat{s})$ is determined as

$$\begin{aligned} x &= \mathcal{K}'_{\mathrm{r}}(\hat{r}, \hat{s}) \qquad (\hat{r}, \hat{s}) \in \mathcal{B} \cap \mathcal{C} \\ n &= \mathcal{K}'_{\mathrm{s}}(\hat{r}, \hat{s}) \end{aligned} \tag{57}$$

where $\mathcal{K}'_{\mathrm{r}} = \partial\mathcal{K}/\partial \mathrm{r}$. The other saddlepoint $\hat{s}_0$ solves $n = \mathcal{K}'_{\mathrm{s}}(0, \hat{s}_0)$ for $(0, \hat{s}_0) \in \mathcal{B} \cap \mathcal{C}$. The expression (54) is meaningful so long as $n > 0$ and x is in the interior of the convex hull of the support of $R(T)$. When $(\hat{r}, \hat{s}) \in \mathcal{B} \cap \mathcal{C}$ and $(0, \hat{s}_0) \in \mathcal{B} \cap \mathcal{C}$, then the arguments for the $\sqrt{\cdot}$ in (55) and (56) are positive.

Skovgaard [18] approximations $\mathbb{P}_{D1}$ and $\mathbb{P}_{D2}$ for $R(T)$ integer-valued

If the support of $R(T)$ is the integer lattice, then the continuity corrections $\mathbb{P}_{D1}$ and $\mathbb{P}_{D2}$ for $\mathbb{P}\{R(T) \geq m \mid T = n\}$ with m a positive integer are given below as variations on the continuous formula above.

Approximation $\mathbb{P}_{D1}$ uses the saddlepoint pairs $(\hat{r}, \hat{s})$ and $(0, \hat{s}_0)$ from the continuous setting determined by solving (57) with $x = m$. Then

$$\mathbb{P}_{D1}\{R(T) \geq m \mid T = n\} = 1 - \Phi(\hat{w}) - \phi(\hat{w})\left(\frac{1}{\hat{w}} - \frac{1}{\tilde{u}_1}\right) \qquad \hat{r} \neq 0 \tag{58}$$

where $\hat{w}$ is given in (55). The value of $\tilde{u}_1$ which makes the continuity correction is

$$\tilde{u}_1 = (1 - e^{-\hat{r}})\sqrt{|\mathcal{K}''(\hat{r}, \hat{s})| \,/\, |K''_{\mathrm{ss}}(0, \hat{s}_0)|}\,. \tag{59}$$

For the second continuity correction, the correction uses an offset value of m or $m^- = m - 1/2$. We define the offset saddlepoint $(\tilde{r}, \tilde{s})$ as solving

$$\begin{aligned} m^- &= \mathcal{K}'_{\mathrm{r}}(\tilde{r}, \tilde{s}) \qquad (\tilde{r}, \tilde{s}) \in \mathcal{B} \cap \mathcal{C} \\ n &= \mathcal{K}'_{\mathrm{s}}(\tilde{r}, \tilde{s}). \end{aligned}$$

The other saddlepoint $\hat{s}_0$ is the same and solves $n = \mathcal{K}'_{\mathrm{s}}(0, \hat{s}_0)$ for $(0, \hat{s}_0) \in \mathcal{B} \cap \mathcal{C}$. Then

$$\mathbb{P}_{D2}\{R(T) \geq m \mid T = n\} = 1 - \Phi(\tilde{w}_2) - \phi(\tilde{w}_2)\left(\frac{1}{\tilde{w}_2} - \frac{1}{\tilde{u}_2}\right) \qquad \tilde{r} \neq 0, \tag{60}$$

where

$$\tilde{w}_2 = \mathrm{sgn}(\tilde{r})\sqrt{2\left[\{\mathcal{K}(0, \hat{s}_0) - \hat{s}_0 n\} - \{\mathcal{K}(\tilde{r}, \tilde{s}) - \tilde{r}m^- - \tilde{s}n\}\right]}$$

$$\tilde{u}_2 = 2\sinh(\tilde{r}/2)\sqrt{|\mathcal{K}''(\tilde{r}, \tilde{s})| \,/\, |\mathcal{K}''_{\mathrm{ss}}(0, \hat{s}_0)|}\,.$$

Proof of Corollary 1. For fixed r, denote the convergence boundary edge solving $1 - \mathcal{H}(e^{\mathrm{r}}, e^{\mathrm{s}}) = 0$ as $\hat{s} = \hat{s}(\mathrm{r})$. Using implicit differentiation,

$$\frac{d\hat{s}(\mathrm{r})}{d\mathrm{r}} = -\frac{\mathcal{H}'_{\mathrm{y}} e^{\mathrm{r}}}{\mathcal{H}'_{\mathrm{z}} e^{\hat{s}}} < 0,$$

where $\mathcal{H}'_{\mathrm{y}} = \partial\mathcal{H}(\mathrm{y},\mathrm{z})/\partial\mathrm{y}|_{\mathrm{y}=e^{\mathrm{r}},\mathrm{z}=e^{\hat{s}}} > 0$, etc. Further implicit differentiation gives

$$-\mathcal{H}'_{\mathrm{z}}\frac{d^2\hat{s}(\mathrm{r})}{d\mathrm{r}^2} = \mathcal{H}'_{\mathrm{y}}e^{\mathrm{r}} + \mathcal{H}'_{\mathrm{z}}e^{\hat{s}}\left\{\frac{d\hat{s}(\mathrm{r})}{d\mathrm{r}}\right\}^2 + \left(1 \;\; \frac{d\hat{s}(\mathrm{r})}{d\mathrm{r}}\right)\mathcal{H}''\left(1 \;\; \frac{d\hat{s}(\mathrm{r})}{d\mathrm{r}}\right)^T, \tag{61}$$

where $\mathcal{H}''$ is the Hessian matrix of $\mathcal{H}(e^{\mathrm{r}}, e^{\mathrm{s}})$ in (r, s) or

$$\mathcal{H}'' = \begin{pmatrix} \mathcal{H}''_{\mathrm{yy}}e^{\mathrm{r}} & \mathcal{H}''_{\mathrm{yz}}e^{\mathrm{r}+\hat{s}} \\ \mathcal{H}''_{\mathrm{yz}}e^{\mathrm{r}+\hat{s}} & \mathcal{H}''_{\mathrm{zz}}e^{\hat{s}} \end{pmatrix}.$$

Since $\mathcal{H}(e^{\mathrm{r}}, e^{\mathrm{s}})$ is a bivariate MGF, it's Hessian is positive definite by the Cauchy-Schwarz inequality. Thus, (61) shows that $d^2\hat{s}(\mathrm{r})/d\mathrm{r}^2 < 0$ along $\partial\mathcal{B}$ and the proof is complete. □

Proof of remainder of Theorem 2. We prove inequality (13). That non-strict inequality holds follows from the triangle inequality. With $p_{jk} = \mathbb{P}\{R = j, F = k\}$ then

$$|\mathcal{H}(\mathrm{y}, \mathrm{z}e^{i\theta})| = \left|\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{jk}\mathrm{y}^j(\mathrm{z}e^{i\theta})^k\right| \le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{jk}\left|\mathrm{y}^j(\mathrm{z}e^{i\theta})^k\right| = \mathcal{H}(\mathrm{y},\mathrm{z}) \qquad (\mathrm{y},\mathrm{z}) \in \mathcal{N}.$$

Therefore, to show the inequality is strict we assume equality exists for some $0 \ne \theta \in (0, 2\pi)$ and show that this leads to a contradiction. If equality exists, there exists $\alpha \in [0, 2\pi)$ such that

$$\begin{aligned} 0 + 0i &= \mathcal{H}(\mathrm{y}, \mathrm{z}e^{i\theta}) - \mathcal{H}(\mathrm{y},\mathrm{z})e^{i\alpha} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{jk}\mathrm{y}^j(\mathrm{z}e^{i\theta})^k - \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{jk}\mathrm{y}^j\mathrm{z}^k e^{i\alpha} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{jk}\mathrm{y}^j\mathrm{z}^k\left[\{\cos(\theta k) - \cos\alpha\} + \{\sin(\theta k) - \sin\alpha\}\, i\right] \\ &= A + Bi. \end{aligned}$$

Since $A = 0 = B$,

$$\begin{aligned} 0 = A\cos\alpha + B\sin\alpha &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{jk}\mathrm{y}^j\mathrm{z}^k\left[\cos(\theta k)\cos\alpha + \sin(\theta k)\sin\alpha - 1\right] \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_{jk}\mathrm{y}^j\mathrm{z}^k\left[\cos(\theta k - \alpha) - 1\right]. \end{aligned} \tag{62}$$

For (62) to hold, $\cos(\theta k - \alpha)$ must be 1 for almost every (a.e.) k or on $\mathcal{P} = \{k \ge 1 : p_{jk} > 0\ \exists j\}$, the support of F. Thus $\theta\mathcal{P} - \alpha := \{\theta k - \alpha : k \in \mathcal{P}\} \subseteq 2\pi\mathbb{N} = \{0, 2\pi, 4\pi, \ldots\}$ and $\mathcal{P} \subseteq \alpha/\theta + (2\pi/\theta)\mathbb{N}$. However, Theorem 2 assumes that $\mathcal{F}(\mathrm{z})$ is aperiodic so that only the integer lattice can cover $\mathcal{P}$. Thus the spacing $2\pi/\theta = \pm 1$ so that $\theta = \pm 2\pi$ and a contradiction is reached. □

Proof of Corollary 2. We derive expansions for the mean and variance of $R(n)$ as $n \to \infty$ which are based on general residue expansions. To derive (18), we first note that the Laurent expansion of the GF for $\mathbb{E}\{R(n)\}$ about the value $\mathrm{z} = 1$ is

$$\frac{\mathcal{H}_{\mathrm{y}}'(1,\mathrm{z})}{\mathcal{F}_S(\mathrm{z})(1-\mathrm{z})^2} = \frac{\xi_{-2}}{(1-\mathrm{z})^2} + \frac{\xi_{-1}}{1-\mathrm{z}} + \mathcal{A}(\mathrm{z}), \tag{63}$$

where $\mathcal{A}(\mathrm{z})$ is analytic in a neighbourhood of $\mathrm{z} = 1$. Here,

$$\xi_{-2} = \frac{\mathcal{H}_{\mathrm{y}}'(1,1)}{\mathcal{F}_S(1)} = \frac{\rho}{\mu}$$

and

$$\xi_{-1} = -\frac{d}{dz}\frac{\mathcal{H}_{\mathrm{y}}'(1,\mathrm{z})}{\mathcal{F}_S(\mathrm{z})}\bigg|_{\mathrm{z}=1} = \frac{-\mathbb{E}\{FR\}}{\mu} + \frac{\rho}{2\mu^2}[\mathbb{E}\{F^2\} - \mu].$$

Thus, we use the inversion formula,

$$\mathbb{E}\{R(n)\} = [\mathrm{z}^n]\frac{\mathcal{H}_{\mathrm{y}}'(1,\mathrm{z})}{\mathcal{F}_S(\mathrm{z})(1-\mathrm{z})^2} = \frac{1}{2\pi i}\int_{C_\varepsilon}\frac{\mathcal{H}_{\mathrm{y}}'(1,\mathrm{z})}{\mathcal{F}_S(\mathrm{z})(1-\mathrm{z})^2}\frac{1}{\mathrm{z}^{n+1}}d\mathrm{z},$$

where C_ε is a circle of small radius $\varepsilon > 0$ centered at 0. Deform the contour C_ε to the integral over $C_{1+\varepsilon}$ which jumps over the 2-pole at $\mathrm{z} = 1$. Then, by Cauchy's residue theorem as in Theorem 1 of [5],

$$\begin{aligned}\mathbb{E}\{R(n)\} &= -\operatorname{Res}\left\{\frac{\mathcal{H}_{\mathrm{y}}'(1,\mathrm{z})}{\mathcal{F}_S(\mathrm{z})(1-\mathrm{z})^2\mathrm{z}^{n+1}};1\right\} + o\{(1+\varepsilon)^{-n}\}\\ &= (n+1)\xi_{-2} + \xi_{-1} + o\{(1+\varepsilon)^{-n}\} \qquad n \to \infty,\end{aligned}$$

where

$$o\{(1+\varepsilon)^{-n}\} = \frac{1}{2\pi i}\int_{C_{1+\varepsilon}}\frac{\mathcal{H}_{\mathrm{y}}'(1,\mathrm{z})}{\mathcal{F}_S(\mathrm{z})(1-\mathrm{z})^2}\frac{1}{\mathrm{z}^{n+1}}d\mathrm{z}.$$

To derive (19), we use the GF for the second factorial moment. From Theorem 1 we can show that

$$\begin{aligned}\mathbb{E}[R(n)\{R(n)-1\}] &= [\mathrm{z}^n]\ \frac{\partial^2}{\partial \mathrm{y}^2}\frac{\mathcal{F}_S(\mathrm{z})}{1-\mathcal{H}(\mathrm{y},\mathrm{z})}\bigg|_{\mathrm{y}=1}\\ &= [\mathrm{z}^n]\left\{\frac{\mathcal{H}_{\mathrm{yy}}''(1,\mathrm{z})}{\mathcal{F}_S(\mathrm{z})(1-\mathrm{z})^2} + \frac{2\{\mathcal{H}_{\mathrm{y}}'(1,\mathrm{z})\}^2}{\mathcal{F}_S(\mathrm{z})^2(1-\mathrm{z})^3}\right\}\\ &= [\mathrm{z}^n]\left\{\frac{\xi_{-3}}{(1-\mathrm{z})^3} + \frac{\xi_{-2}}{(1-\mathrm{z})^2} + \frac{\xi_{-1}}{1-\mathrm{z}} + O(1)\right\}.\end{aligned} \tag{64}$$

The first two coefficients in the Laurent expansion in (64) are

$$\xi_{-3} = \frac{2\rho^2}{\mu^2} \qquad \text{and} \qquad \xi_{-2} = \frac{\mathbb{E}\{R^2\} - \rho}{\mu} - \frac{4\rho^2}{\mu^2}\left\{\frac{\mathbb{E}\{FR\}}{\rho} - \frac{\mathbb{E}\{F^2\} - \mu}{2\mu}\right\}.$$

By the same argument using Cauchy's deformation theorem to determine the expansion of $\mathbb{E}\{R(n)\}$, we get

$$\mathbb{E}[R(n)\{R(n)-1\}] = \frac{(n+2)(n+1)}{2}\xi_{-3} + (n+1)\xi_{-2} + O(1). \tag{65}$$

Using (65),

$$\mathbb{V}\{R(n)\} \sim \frac{(n+2)(n+1)}{2}\xi_{-3} + (n+1)\xi_{-2} + \mathbb{E}\{R(n)\} - [\mathbb{E}\{R(n)\}]^2 \tag{66}$$

to order $O(1)$. Now replace $\mathbb{E}\{R(n)\}$ in (66) with its full expansion to order $O(1)$ in (18) to derive the variance expansion in (19). □

Proof of Corollary 3. In the derivation of residue approximation $\hat{\mathcal{R}}_n(\mathrm{y})$ for $\mathcal{R}_n(\mathrm{y})$ using Cauchy's residue theorem, the error is

$$\mathcal{R}_n(\mathrm{y}) - \hat{\mathcal{R}}_n(\mathrm{y}) = \frac{1}{2\pi i}\int_{C_{1+\eta(\mathrm{y})}} \frac{\mathcal{F}_S(\mathrm{z})}{1-\mathcal{H}(\mathrm{y},\mathrm{z})}\frac{1}{\mathrm{z}^{n+1}}d\mathrm{z} \tag{67}$$

on circle $C_{1+\eta(\mathrm{y})}$ for some small $\eta(\mathrm{y}) > 0$. We show here that this error is uniformly $O\{(1+\eta_0)^{-n}\}$ as $n \to \infty$ for some $\eta_0 > 0$. As y ranges over a compact set $\mathcal{D}_0$, define $\eta_0 = \inf_{\mathrm{y}\in\mathcal{D}_0}\eta(\mathrm{y}) > 0$. Compute the error integral (67) over $C_{1+\eta_0}$ for each $\mathrm{y} \in \mathcal{D}_0$. Over the contour of the integral

$$\max_{\mathrm{z}\in C_{1+\eta_0}} |\mathcal{F}_S(\mathrm{z})| < B_1 < \infty.$$

Since η_0 was chosen so that $1 - \mathcal{H}(\mathrm{y},\mathrm{z})$ has no zeros on $C_{1+\eta_0}$, then over the contour of the error integral and the compact range of z,

$$0 < \min_{\mathrm{y}\in\mathcal{D}_0,\mathrm{z}\in C_{1+\eta_0}} |1-\mathcal{H}(\mathrm{y},\mathrm{z})| = B_2 < \infty.$$

Hence, using the triangle inequality,

$$|\mathcal{R}_n(\mathrm{y}) - \hat{\mathcal{R}}_n(\mathrm{y})| < \frac{1}{2\pi(1+\eta_0)^{n+1}}\int_{C_{1+\eta_0}} \frac{|\mathcal{F}_S(\mathrm{z})|}{|1-\mathcal{H}(\mathrm{y},\mathrm{z})|}d\mathrm{z} < \frac{1}{(1+\eta_0)^n}\frac{B_1}{B_2},$$

for $\mathrm{y} \in \mathcal{D}_0$. Thus,

$$\max_{\mathrm{y}\in\mathcal{D}_0} |\mathcal{R}_n(\mathrm{y}) - \hat{\mathcal{R}}_n(\mathrm{y})| = O\{(1+\eta_0)^{-n}\} \qquad n \to \infty. \tag{68}$$

To convert this into uniformity for the CGF over $\mathrm{s} \in \mathcal{D}$, a neighbourhood of 0, take $\mathcal{D}_0 = \{e^{\mathrm{s}} : \mathrm{s} \in \mathcal{D}\}$. Then the uniform rate of convergence in (22) follows from (68) through the mapping $\mathcal{R}_n(\mathrm{y}) \to \ln \mathcal{R}_n(e^{\mathrm{s}}) = \mathcal{K}_n(\mathrm{s})$ for $\mathrm{y} \in \mathcal{D}_0$. □

Proof of Corollary 4. The proof makes use of the following weak convergence theorem for MGFs due to Curtiss [23].

Proposition 1 (Continuity theorem for MGFs). *Let* $\{Z_n : n \geq 1\}$ *be a sequence of random variables which has the corresponding sequence of MGFs* $\{\mathcal{M}_n(\mathrm{s})\}$ *all of which are convergent on* $|\mathrm{s}| < \varepsilon$ *for some* $\varepsilon > 0$*. Suppose there exists function* $\mathcal{M}(\mathrm{s})$ *finite on* $|\mathrm{s}| < \varepsilon$ *for which* $\lim_{n\to\infty}\mathcal{M}_n(\mathrm{s}) = \mathcal{M}(\mathrm{s})$ *for all* $|\mathrm{s}| < \varepsilon$*. Then* $\mathcal{M}$ *is the moment generating function for some variable* Z *such that* $Z_n \to Z$ *in distribution as* $n \to \infty$*.*

A comparable continuity theorem for sequences of characteristic functions (CFs) is much better known and is given in Feller [25, §XV.3 theorem 2]. The advantage of Proposition 1 is that it only requires local convergence of $\{\mathcal{M}_n(\mathrm{s})\}$ in an arbitrarily small real neighbourhood of 0 whereas the same theorem for sequences of CFs requires convergence for all real argument values of the CF sequence.

The uniformity of the residue expansion as expressed in Corollary 3 allows us to replace the true CGF of $R(n)$ with its residue approximation which we denote as

$$\hat{\mathcal{K}}_{R(n)}(\mathrm{r}) = -(n+1)\ln\hat{z}(e^{\mathrm{r}}) + \ln\mathcal{D}(\mathrm{r}) \qquad \text{with} \qquad \mathcal{D}(\mathrm{r}) = \frac{\mathcal{F}_S\{\hat{z}(e^{\mathrm{r}})\}}{\mathcal{H}'_{\mathrm{z}}\{e^{\mathrm{r}},\hat{z}(e^{\mathrm{r}})\}}.$$

Adapting this approximation for the standardised value of $R(n)$ or $Z_n = \{R(n) - n\rho/\mu\}/\sigma_n$ with $\sigma_n = \sqrt{n}\sigma_{R.F}$, then

$$\hat{\mathcal{K}}_{Z_n}(\mathrm{r}) = -(n+1)\ln\hat{z}(e^{\mathrm{r}/\sigma_n}) - \mathrm{r}\frac{n\rho}{\mu\sigma_n} + \ln\mathcal{D}(\mathrm{r}/\sigma_n). \tag{69}$$

The last term in (69) vanishes as $n \to \infty$ and $\sigma_n \to \infty$ since

$$\lim_{\mathrm{r}\to 0}\mathcal{D}(\mathrm{r}) = \lim_{\mathrm{y}\to 1}\frac{\mathcal{F}_S\{\hat{z}(\mathrm{y})\}}{\mathcal{H}'_{\mathrm{z}}\{\mathrm{y},\hat{z}(\mathrm{y})\}} = \frac{\mu}{\mu} = 1.$$

Taylor expansion of $\hat{z}(e^{\mathrm{r}})$ about $\mathrm{r} = 0$ in the first term in (69) gives

$$\begin{aligned}\hat{z}(e^{\mathrm{r}}) &= 1 + \mathrm{r}\left.\frac{d\hat{z}(e^{\mathrm{r}})}{d\mathrm{r}}\right|_{\mathrm{r}=0} + \frac{\mathrm{r}^2}{2}\left.\frac{d^2\hat{z}(e^{\mathrm{r}})}{d\mathrm{r}^2}\right|_{\mathrm{s}=0} + O(\mathrm{r}^3)\\ &= 1 - \mathrm{r}\frac{\rho}{\mu} - \frac{\mathrm{r}^2}{2}\left(\sigma^2_{R.F} - \frac{\rho^2}{\mu^2}\right) + O(\mathrm{r}^3)\end{aligned}$$

where the derivatives of $\hat{z}(e^{\mathrm{r}})$ have been determined using implicit differentiation of the expression $0 = 1 - \mathcal{H}\{e^{\mathrm{r}},\hat{z}(e^{\mathrm{r}})\}$. Thus,

$$\hat{z}(e^{\mathrm{r}/\sigma_n}) = 1 - \left\{\frac{\mathrm{r}\rho}{\sqrt{n}\mu\sigma_{R.F}} + \frac{\mathrm{r}^2}{2n\sigma^2_{R.F}}\left(\sigma^2_{R.F} - \frac{\rho^2}{\mu^2}\right)\right\} + O(n^{-3/2}).$$

Substituting this into (69) and Taylor expanding $\ln(1-y) \approx -y - y^2/2 - O(y^3)$, then

$$\begin{aligned}\hat{\mathcal{K}}_{Z_n}(\mathrm{r}) &\sim -n\left\{-\frac{\mathrm{r}\rho}{\sqrt{n}\mu\sigma_{R.F}} - \frac{\mathrm{r}^2}{2n\sigma^2_{R.F}}\left(\sigma^2_{R.F} - \frac{\rho^2}{\mu^2}\right) - \frac{1}{2}\left(\frac{\mathrm{r}\rho}{\sqrt{n}\mu\sigma_{R.F}}\right)^2\right\} - \frac{\mathrm{r}\sqrt{n}\rho}{\mu\sigma_{R.F}}\\ &= \frac{\mathrm{r}^2}{2}.\end{aligned}$$

From Proposition 1, Z_n converges weakly to a standard normal distribution. □

Proof of Corollary 6. Following the same argument as used in Corollary 3, the residue approximation for $\mathcal{J}_n(\mathrm{x},\mathrm{y})$, given as $\hat{\mathcal{J}}_n(\mathrm{x},\mathrm{y})$ in (24), has uniform error which is $o\{(1+\varepsilon)^{-n}\}$ as $n \to \infty$ for (x,y) in compact neighbourhoods of $(1,1)$. Thus $\hat{\mathcal{K}}_n(\mathrm{r},\mathrm{s}) = \ln\hat{\mathcal{J}}_n(e^{\mathrm{r}},e^{\mathrm{s}})$ can be used in place of the joint CGF $\mathcal{K}_n(\mathrm{r},\mathrm{s}) = \ln\mathcal{J}_n(e^{\mathrm{r}},e^{\mathrm{s}})$ for proving the central limit result. The residue term is negligible as $n \to \infty$ so that if $Q_n = \left(Q(n) - n\rho_Q/\mu\right)/\sqrt{n}$ etc.,

$$\mathcal{K}_{Q_n,R_n}(\mathrm{r},\mathrm{s}) \sim -n\ln\left[\hat{z}\left\{e^{\mathrm{r}/(\sqrt{n}\sigma_{Q.F})}, e^{\mathrm{s}/(\sqrt{n}\sigma_{R.F})}\right\}\right] - \sqrt{n}\left(\frac{\mathrm{r}\rho_Q}{\mu\sigma_{Q.F}} + \frac{\mathrm{s}\rho_R}{\mu\sigma_{R.F}}\right). \tag{70}$$

The bivariate quadratic expansion

$$\hat{z}(e^{\mathrm{r}}, e^{\mathrm{s}}) \approx 1 - \mathrm{r}\frac{\rho_Q}{\mu} - \mathrm{s}\frac{\rho_R}{\mu} - \frac{\mathrm{r}^2}{2}\left(\sigma^2_{Q.F} - \frac{\rho_Q^2}{\mu^2}\right) - \frac{\mathrm{s}^2}{2}\left(\sigma^2_{R.F} - \frac{\rho_R^2}{\mu^2}\right) - \mathrm{rs}\left(\sigma^2_{QR.F} - \frac{\rho_Q\rho_R}{\mu^2}\right)$$

captures all the asymptotic terms for an expansion of the first term in (70). Following the proof of Corollary 4,

$$\mathcal{K}_{Q_n,R_n}(\mathrm{r},\mathrm{s}) \to \frac{1}{2}(\mathrm{r}^2 + \mathrm{s}^2) + \mathrm{rs}\rho \qquad \text{with} \qquad \rho = \frac{\sigma^2_{QR.F}}{\sigma_{Q.F}\sigma_{R.F}},$$

as $n \to \infty$ in a small neighbourhood $\mathcal{N}$ of $(\mathrm{r},\mathrm{s}) = (0,0)$.

This alone does not prove the bivariate normal limit based on Proposition 1 which only applies to univariate distributions. To extend Proposition 1 to a bivariate distribution, we use the Cramér-Wold theorem as given in Billingsley [21, theorem 29.4]. This device which says that the bivariate limit holds if and only if $L_n = c_1Q_n + c_2R_n$ converges to a Normal $(0, v^2)$ for every vector (c_1, c_2) , where

$$v^2 = (c_1, c_2)\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1^2 + 2c_1c_2\rho + c_2^2.$$

For any such vector, the CGF of L_n is approximately

$$\mathcal{K}_{L_n}(\mathrm{r}) = \mathcal{K}_{Q_n,R_n}(c_1\mathrm{r}, c_2\mathrm{r}) \to \frac{1}{2}(c_1^2\mathrm{r}^2 + c_2^2\mathrm{r}^2) + \mathrm{r}^2\rho c_1c_2 = \frac{1}{2}v^2\mathrm{r}^2$$

which is the CGF for a Normal $(0, v^2)$. This convergence occurs in a neighbourhood of $\mathrm{r} = 0$ consisting of those r-values such that $(c_1\mathrm{r}, c_2\mathrm{r}) \in \mathcal{N}$. By Proposition 1, $L_n = c_1Q_n + c_2R_n$ converges to a Normal $(0, v^2)$ for every (c_1, c_2) so the weak limit is a bivariate normal using the Cramér-Wold device. □

Proof of Corollary 7. To get an expansion for the mean $\mathbb{E}\{P_m\}$ as $m \to \infty$, take a Laurent expansion of its GF about $\mathrm{y} = 1$ so that

$$\frac{\mu}{(1-\mathrm{y})^2\mathcal{E}_{\mathcal{S}}(\mathrm{y})} = \frac{\xi_{-2}}{(\mathrm{y}-1)^2} + \frac{\xi_{-1}}{\mathrm{y}-1} + \mathcal{A}(\mathrm{y}),$$

where $\mathcal{A}(\mathrm{y})$ is analytic at $\mathrm{y} = 1$ and

$$\xi_{-2} = \frac{\mu}{\rho} \qquad \text{and} \qquad \xi_{-1} = -\frac{\mu}{2\rho^2}\left\{\mathbb{E}\{R^2\} - \rho\right\}.$$

We take the y^{m-1} coefficient to give the $o(1)$ expansion for $\mathbb{E}\{P_m\}$ as given in Corollary 7.

To derive an expansion for the variance, we first need to confirm that the second factorial moment of P_m is

$$\mathbb{E}\{P_m(P_m - 1)\} = 2\sum_{n=1}^{\infty} n\mathbb{P}\{P_m > n\}. \qquad (71)$$

The terms that are summed in (71) have a GF such that

$$2n\mathbb{P}\{P_m > n\} = [\mathrm{z}^n]\left\{2\mathrm{z}\frac{d}{d\mathrm{z}}\frac{1-\mathcal{P}_m(\mathrm{z})}{1-\mathrm{z}}\right\}.$$

Now replace $\mathcal{P}_m(\mathrm{z})$ with $[\mathrm{y}^m]\mathcal{P}(\mathrm{y},\mathrm{z})$ from (28) so that

$$\begin{aligned} 2n\mathbb{P}\{P_m > n\} &= [\mathrm{z}^n]\left[2\mathrm{z}\frac{d}{d\mathrm{z}}[\mathrm{y}^m]\left\{\frac{1}{(1-\mathrm{y})(1-\mathrm{z})} - \frac{\mathcal{P}(\mathrm{y},\mathrm{z})}{(1-\mathrm{z})}\right\}\right] \\ &= [\mathrm{y}^m\mathrm{z}^n]\left[2\mathrm{z}\frac{d}{d\mathrm{z}}\left\{\frac{\mathrm{y}\mathcal{F}_{\mathcal{S}}(\mathrm{z})}{(1-\mathrm{y})\{1-\mathcal{H}(\mathrm{y},\mathrm{z})\}}\right\}\right] \\ &= [\mathrm{y}^{m-1}\mathrm{z}^n]\left[2\mathrm{z}\frac{d}{d\mathrm{z}}\left\{\frac{\mathcal{F}_{\mathcal{S}}(\mathrm{z})}{(1-\mathrm{y})\{1-\mathcal{H}(\mathrm{y},\mathrm{z})\}}\right\}\right]. \end{aligned}$$

The second line follows by substituting the expression for $\mathcal{P}(\mathrm{y},\mathrm{z})$ in (28). Evaluating the double generating function in the square brackets at $\mathrm{z} = 1$ and taking the coefficient of y^{m-1} gives the factorial moment as

$$\sum_{n=1}^{\infty} 2n\mathbb{P}\{P_m > n\} = [\mathrm{y}^{m-1}]\ \frac{d}{d\mathrm{z}}\left\{\frac{2\mathcal{F}_{\mathcal{S}}(\mathrm{z})}{(1-\mathrm{y})\{1-\mathcal{H}(\mathrm{y},\mathrm{z})\}}\right\}\Bigg|_{\mathrm{z}=1}.$$

Hence, taking the derivative in z with $f_2 = \mathcal{F}_{\mathcal{S}}'(1) = [\mathbb{E}\{F^2\} - \mu]/2$ and $r_2 = \mathcal{E}_{\mathcal{S}}'(1) = [\mathbb{E}\{R^2\} - \rho]/2$, then

$$\begin{aligned} \mathbb{E}\{P_m(P_m-1)\} &= [\mathrm{y}^{m-1}]\left\{\frac{2\mu\mathcal{H}_{\mathrm{z}}'(\mathrm{y},1)}{(1-\mathrm{y})^3\mathcal{E}_{\mathcal{S}}(\mathrm{y})^2} + \frac{2f_2}{(1-\mathrm{y})^2\mathcal{E}_{\mathcal{S}}(\mathrm{y})}\right\} \\ &= [\mathrm{y}^{m-1}]\left\{\frac{2\mu^2}{(1-\mathrm{y})^3\rho^2} + \frac{B}{(1-\mathrm{y})^2} + O\left(\frac{1}{1-\mathrm{y}}\right)\right\} \end{aligned}$$

as $\mathrm{y} \to 1$, where

$$B = \frac{2}{\rho}f_2 - 2\frac{\mu}{\rho^2}\mathbb{E}\{RF\} + \frac{4\mu^2}{\rho^3}r_2.$$

Applying a residue expansion as in Theorem 1 of [5] shows that as $m \to \infty$,

$$\mathbb{E}\{P_m(P_m-1)\} = \frac{(m+1)m}{2}\frac{2\mu^2}{\rho^2} + mB + O(1).$$

Now substituting the expansion of $\mathbb{E}\{P_m\}$ in Corollary 7 to order $o(1)$ in the expression

$$\mathbb{V}\{P_m\} = \mathbb{E}\{P_m(P_m-1)\} + \mathbb{E}\{P_m\} - [\mathbb{E}\{P_m\}]^2, \tag{72}$$

we get the expansion for $\mathbb{V}\{P_m\}$ in (32) of Corollary 7. □

Proof of Corollary 8. From the normal limits for rewards in Corollary 4, we let $m \to \infty$ and $n \to \infty$ in such a manner that $(m - n\rho/\mu)/(\sqrt{n}\sigma_{R.F}) \to z$. Then,

$$\mathbb{P}\{R(n) < m\} = \mathbb{P}\left\{\frac{R(n) - n\rho/\mu}{\sqrt{n}\sigma_{R.F}} < \frac{m - n\rho/\mu}{\sqrt{n}\sigma_{R.F}}\right\} \to \Phi(z) \qquad n \to \infty.$$

Write the tail probability for P_m in terms of $R(n)$ and standardise P_m to $Z_m = (P_m - m\mu/\rho)/\sqrt{\mathbb{V}\{P_m\}}$ where $\mathbb{V}\{P_m\}$ is the expression in (32). This gives

$$\Phi(z) \leftarrow \mathbb{P}\{R(n) < m\} = \mathbb{P}\{P_m > n\} = \mathbb{P}\left\{Z_m > \frac{n - m\mu/\rho}{\sqrt{m}(\mu/\rho)^{3/2}\sigma_{R.F}}\right\}. \tag{73}$$

As $n, m \to \infty$, then $\sqrt{m\mu/\rho} \sim \sqrt{n}$ so that

$$\frac{n - m\mu/\rho}{\sqrt{m}(\mu/\rho)^{3/2}\sigma_{R.F}} \sim \frac{n\rho/\mu - m}{\sqrt{n}\sigma_{R.F}} \to -z.$$

Thus as $m \to \infty$, the right tail probability in (73) has the same limit as $\mathbb{P}\{Z_m > -z\}$ which converges to $\Phi(z) = 1 - \Phi(-z)$ and the normal limit holds. □

Example 11 continuation (Reward as an interarrival random walk). We prove that

$$\lim_{z\downarrow 0} \hat{\mathcal{P}}_m(z) = 0 = \mathcal{P}_m(0) \qquad \text{for} \qquad m \geq 1.$$

The limit hinges on the behaviour of $z\mathcal{G}_0'\{\hat{y}(z)\}$ in the denominator. First consider the case in which $\mathfrak{r}_0 < \infty$. As $z \downarrow 0$, $\hat{y}(z) \uparrow \mathfrak{r}_0$ but $z = 1/\mathcal{G}_0\{\hat{y}(z)\}$ so that

$$\lim_{z\downarrow 0} z\mathcal{G}_0'\{\hat{y}(z)\} = \lim_{z\downarrow 0} \frac{\mathcal{G}_0'\{\hat{y}(z)\}}{\mathcal{G}_0\{\hat{y}(z)\}} = \lim_{y\uparrow \mathfrak{r}_0} \frac{\mathcal{G}_0'(y)}{\mathcal{G}_0(y)}. \tag{74}$$

If $\mathfrak{r}_0 < \infty$, the support of the mass function of PGF $\mathcal{G}_0(y)$ extends up to $M_U = \infty$, so that

$$\frac{\mathcal{G}_0'(y)y}{\mathcal{G}_0(y)} \sim M_U = \infty \qquad y \uparrow \mathfrak{r}_0$$

is the upper reach for solution of the saddlepoint equation. Thus the limit in (74) is ∞ so that $\hat{\mathcal{P}}_m(z) \to 0$.

Now consider the case $\mathfrak{r}_0 = \infty$ and let $M_U \leq \infty$ be the supremum of the support for R. For any $M_0 < M_U$,

$$z\mathcal{G}_0'\{\hat{y}(z)\}\hat{y}(z) = \frac{\mathcal{G}_0'\{\hat{y}(z)\}}{\mathcal{G}_0\{\hat{y}(z)\}}\hat{y}(z) > M_0,$$

for $\hat{y}(z)$ sufficiently large, as this reflects the right edge for the solvability of the saddlepoint equation. Thus

$$\frac{1}{z\mathcal{G}_0'\{\hat{y}(z)\}} < \frac{\hat{y}(z)}{M_0}$$

and

$$\hat{\mathcal{P}}_m(z) = \frac{1}{\hat{y}(z)^m} \frac{1 - \mathcal{F}(z)}{\{\hat{y}(z) - 1\}\mu z\mathcal{G}_0'\{\hat{y}(z)\}} < \frac{M_1}{\hat{y}(z)^m} \tag{75}$$

for some M_1 sufficiently large. As $z \downarrow 0$, then $\hat{y}(z) \to \infty$ and the upper bound in (75) goes to 0. □

Proof of Theorem 4. Start with the identity

$$\mathbb{P}\{P_x > n\} = \mathbb{P}\{R(n) < x\}. \tag{76}$$

We take the GF in $n \leftrightsquigarrow \mathrm{z}$ on the left hand side and the LT in $x \leftrightsquigarrow \mathrm{r}$ on the right to get

$$[\mathrm{z}^n]\frac{1-\mathcal{P}_x(\mathrm{z})}{1-\mathrm{z}} = [e^{-x\mathrm{r}}]\frac{\mathcal{R}_n(-\mathrm{r})}{\mathrm{r}},$$

where $\mathcal{R}_n(\mathrm{r})$ is the MGF for the mixed distribution of $R(n)$. Taking the LT of the right-hand side of (76) requires some care when considering that the distribution of $R(n)$ has a point mass at 0. Let $g_n(u)du = \mathbb{P}\{R(n) \in (u, u+du \mid R(n) > 0\}$ be the conditional density portion of $R(n)$ and denote $c_n = \mathbb{P}\{F_1 \le n\}$. Then,

$$\mathbb{P}\{R(n) < x\} = 1 - c_n + c_n \int_0^x g_n(u)du$$

has LT in $\mathrm{r} \leftrightsquigarrow x$ as

$$\frac{1-c_n}{\mathrm{r}} + \frac{c_n}{\mathrm{r}}\int_0^\infty e^{-\mathrm{r}u} g_n(u)du = \frac{1}{\mathrm{r}}\mathbb{E}\{e^{\mathrm{r}R(n)}\} = \frac{\mathcal{R}_n(-\mathrm{r})}{\mathrm{r}}.$$

Now, taking the LT in $x \leftrightsquigarrow \mathrm{r}$ on the left side and the GF in $n \leftrightsquigarrow \mathrm{z}$ on the right side leads to

$$[e^{-x\mathrm{r}}\mathrm{z}^n]\left\{\frac{1}{\mathrm{r}(1-\mathrm{z})} - \frac{\mathcal{P}_c(\mathrm{r},\mathrm{z})}{1-\mathrm{z}}\right\} = [\mathrm{z}^n e^{-x\mathrm{r}}]\left\{\frac{\mathcal{R}_c(-\mathrm{r},\mathrm{z})}{\mathrm{r}}\right\}. \tag{77}$$

Substituting in (77) for

$$\mathcal{R}_c(-\mathrm{r},\mathrm{z}) = \frac{\mathcal{F}_S(\mathrm{z})}{1-\mathcal{H}_c(-\mathrm{r},\mathrm{z})},$$

equating the resulting two expressions in curly braces, and solving for $\mathcal{P}_c(\mathrm{r},\mathrm{z})$ gives (41).

To derive (42),

$$\begin{aligned}\mathbb{P}\{P_x \ge n\} &= [\mathrm{z}^n]\frac{1-\mathrm{z}\mathcal{P}_x(\mathrm{z})}{1-\mathrm{z}} = [\mathrm{z}^n]\frac{1-\mathrm{z}[e^{-x\mathrm{r}}]\mathcal{P}_c(\mathrm{r},\mathrm{z})}{1-\mathrm{z}} \\ &= [\mathrm{z}^n e^{-x\mathrm{r}}]\left\{\frac{1}{\mathrm{r}(1-\mathrm{z})} - \frac{\mathrm{z}\mathcal{P}_c(\mathrm{r},\mathrm{z})}{1-\mathrm{z}}\right\}.\end{aligned}$$

Substituting for $\mathcal{P}_c(\mathrm{r},\mathrm{z})$ and some reduction gives

$$\mathbb{P}\{P_x \ge n\} = [\mathrm{z}^n e^{-x\mathrm{r}}]\left[\frac{1}{\mathrm{r}}\left\{1 + \frac{\mathrm{z}\mathcal{F}_S(\mathrm{z})}{1-\mathcal{H}_c(-\mathrm{r},\mathrm{z})}\right\}\right], \tag{78}$$

the result in (42). Since

$$\mathbb{E}\{P_x\} = \sum_{n=1}^\infty \mathbb{P}\{P_x \ge n\},$$

this is computed from the inverse LT of the double transform in (78) evaluated at $\mathrm{z} = 1$. Thus,

$$\begin{aligned}\mathbb{E}\{P_x\} &= \sum_{n=0}^\infty \mathbb{P}\{P_x \ge n\} - 1 = [e^{-x\mathrm{r}}]\left[\frac{1}{r}\left\{1 + \frac{\mathcal{F}_S(1)}{1-\mathcal{H}_c(-\mathrm{r},1)}\right\}\right] - 1 \\ &= [e^{-x\mathrm{r}}]\frac{\mathcal{F}_S(1)}{1-\mathcal{H}_c(-\mathrm{r},1)} = [e^{-x\mathrm{r}}]\frac{\mu}{\mathrm{r}^2\mathcal{E}_S(\mathrm{r})}. \ \square\end{aligned} \tag{79}$$

Proof of Corollary 9. To derive the moment expansions, we use the residue expansions in [2, theorem 1 and lemma 1] that involve poles of order greater than 1. The inversion of (79) involves a 2-pole and its inversion leads to the expansion for $\mathbb{E}\{P_x\}$ in (44).

Derivation of the expansion for $\mathbb{V}\{P_x\}$ in (45) follows the same approach as used in Corollary 7 wherein the LT for the factorial moment $\mathbb{E}\{P_x(P_x-1)\}$ is first derived. After some derivation, we get

$$\mathbb{E}\{P_x(P_x-1)\} = [e^{-x\mathrm{r}}]\left\{\frac{2\mu\mathcal{H}'_{c\mathrm{z}}(-\mathrm{r},1)}{\mathrm{r}^3\mathcal{E}_S(\mathrm{r})^2} + \frac{2f_2}{\mathrm{r}^2\mathcal{E}_S(\mathrm{r})} + O(1/\mathrm{r})\right\},$$

as $\mathrm{r}\to 0$, where $f_2 = \mathcal{F}'_S(1) = [\mathbb{E}\{F^2\} - \mu]/2$. Residue expansion as $x\to\infty$ of this LT leads to

$$\mathbb{E}\{P_x(P_x-1)\} = \frac{x^2}{2}\frac{2\mu^2}{\rho^2} + x\left(\frac{2}{\rho}f_2 - 2\frac{\mu}{\rho^2}\mathbb{E}\{RF\} + \frac{2\mu^2}{\rho^3}\mathbb{E}\{R^2\}\right) + O(1),$$

as $x\to\infty$. Using (72), we add the expansion for $\mathbb{E}\{P_x\}$ and subtract the square of the expansion for $\mathbb{E}\{P_x\}$ to get the expansion for $\mathbb{V}\{P_x\}$ in (45). □

Proof of Theorem 5. The justification of the residue approximation is based on [6, lemma 4]. As such, the approximation is $e^{\hat{r}(\mathrm{z})x}\xi_{-1}(\mathrm{z})$, where $\xi_{-1}(\mathrm{z})$ is the residue for the simple pole that $\mathcal{P}_c(\mathrm{r},\mathrm{z})$ has at $\hat{r}(\mathrm{z})$ and is computed as

$$\xi_{-1}(\mathrm{z}) = \lim_{\mathrm{r}\to\hat{r}(\mathrm{z})}\left[\{\mathrm{r}-\hat{r}(\mathrm{z})\}\,\mathcal{P}_c(\mathrm{r},\mathrm{z})\right] = \frac{1-\mathcal{F}(\mathrm{z})}{-\hat{r}(\mathrm{z})\mathcal{H}'_{c\mathrm{r}}\{-\hat{r}(\mathrm{z}),\mathrm{z}\}}.$$

We now justify the conditions of Lemma 4 in [6]. For any $\mathrm{z} < \mathfrak{r}$, a value $\hat{r}(\mathrm{z})$ exists for the following reason. Since

$$\mathcal{H}_c(\mathrm{r},\mathrm{z}) = \mathbb{E}\{e^{\mathrm{r}R}\mathrm{z}^F\},$$

$\mathcal{H}_c(0,\mathrm{z}) < \infty$ and values for $\mathcal{H}_c(\mathrm{r},\mathrm{z})$ range from 0 at $\mathrm{r} = -\infty$ to ∞ as r increases monotonically and approaches the boundary of the open convergence region for $\mathcal{H}_c$. Thus, $\hat{r}(\mathrm{z})$ exists and is uniquely defined so that $(\hat{r}(\mathrm{z}),\mathrm{z})$ lies in the convergence region of $\mathcal{H}_c$. Since $\mathcal{H}_c$ is analytic at $(\hat{r}(\mathrm{z}),\mathrm{z})$, $0 < \mathcal{H}'_{c\mathrm{r}}\{-\hat{r}(\mathrm{z}),\mathrm{z}\} < \infty$ so that all factors of $\xi_{-1}(\mathrm{z})$ and hence $\hat{\mathcal{P}}_x(\mathrm{z})$ are well-defined.

Condition $\mathcal{X}$ in Lemma 4 of [6] requires that their exist a $\eta > 0$ such that over the closed interval $V = [\hat{r}(\mathrm{z})-\eta, \hat{r}(\mathrm{z})]$

$$\max_{v\,\in V}|\mathcal{P}_c(v+iw,\mathrm{z})| \to 0 \qquad |w|\to\infty, \tag{80}$$

for each $\mathrm{z} < \mathfrak{r}$. Since $(\hat{r}(\mathrm{z}),\mathrm{z})$ is in the open convergence domain of $\mathcal{H}_c$, the factor of $\mathcal{P}_c$ in curly braces in (41) remains bounded with this limit; the other factor $1/\mathrm{r} = 1/(v+iw)$ ensures that (80) holds.

To satisfy condition $\mathcal{Z}$ in Lemma 4 of [6], we show that the modulus derivative $|\mathcal{P}'_{c\mathrm{r}}\{\hat{r}(\mathrm{z}) - \eta + iw,\mathrm{z}\}|$ is integrable in w. From (41),

$$\mathcal{P}'_{c\mathrm{r}}(\mathrm{r},\mathrm{z}) = -\frac{1}{\mathrm{r}^2}\left\{1 - \frac{1-\mathcal{F}(\mathrm{z})}{1-\mathcal{H}_c(-\mathrm{r},\mathrm{z})}\right\} + \frac{1}{\mathrm{r}}\frac{1-\mathcal{F}(\mathrm{z})}{\{1-\mathcal{H}_c(-\mathrm{r},\mathrm{z})\}^2}\mathcal{H}'_{c\mathrm{r}}(-\mathrm{r},\mathrm{z}). \tag{81}$$

The first term is absolutely integrable since the modulus of the term in curly braces remains bounded over the integration range and the modulus of the leading factor is $O(|w|^{-2})$. For the second term, by assumption $|\mathcal{H}'_{c\mathrm{r}}(-\hat{r}(\mathrm{z})-\eta+iw,\mathrm{z})| = o(|w|^{-\varepsilon})$ as $|w|\to\infty$. Thus, the second term overall is $o(|w|^{-(1+\varepsilon)})$ and is therefore absolutely integrable. Thus, $|\mathcal{P}'_{c\mathrm{r}}(\hat{r}(\mathrm{z})-\eta+iw,\mathrm{z})|$ is integrable in w and condition $\mathcal{Z}$ in Lemma 4 of [6] holds. □

Example 14 continuation (Reward as an interarrival random walk). We first show that condition $\mathcal{Z}$ of Theorem 5 pertaining to $\mathcal{H}_c(\mathrm{r},\mathrm{z})$ in (50) holds for this example. With $\mathcal{H}_c(\mathrm{r},\mathrm{z})$ given in (50), then

$$\mathcal{H}_c(\mathrm{r},\mathrm{z}) = O(e^{2-2\sqrt{1-2\mathrm{r}}}) \qquad \text{and} \qquad \mathcal{H}'_{c\mathrm{r}}(\mathrm{r},\mathrm{z}) = O(e^{2-2\sqrt{1-2\mathrm{r}}}/\sqrt{1-2\mathrm{r}}) \qquad |\mathrm{r}| \to \infty$$

so that

$$\mathcal{H}'_{c\mathrm{r}}(-a+iy,\mathrm{z}) = o(e^{-2\sqrt{1-2(-a+iy)}}) \qquad |y| \to \infty. \tag{82}$$

Holding $a > -1/2$ fixed, then $\sqrt{1-2(-a+iy)} = \sqrt{r}e^{i\theta/2}$ with

$$r = \sqrt{(1+2a)^2+4y^2} > \max(2|y|, 1+2a) \qquad \text{and} \qquad \theta = \tan^{-1}\frac{-2y}{1+2a} \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

Then,

$$\left|e^{-2\sqrt{1-2(-a+iy)}}\right| = e^{-2\sqrt{r}\cos(\theta/2)} < e^{-\sqrt{2}\max(\sqrt{2|y|},\sqrt{1+2a})}, \tag{83}$$

since $\cos(\theta/2) > \cos(\pi/4) = 1/\sqrt{2}$. For fixed a, the expression in (83) is $O(e^{-2\sqrt{|y|}})$ as $|y| \to \infty$. □

Example 14 continuation (Infinite residue expansion is exact). We now prove that the infinite residue expansion using all the poles in (52) is pointwise convergent and leads to the exact PGF $\mathcal{P}_x(\mathrm{z})$. The proof uses [2, §2.7, corollary 2.2] converted so that it inverts a LT rather a MGF. According to Corollary 2.2, it suffices that a sequence $\{a_m\} \subset \mathbb{R}$ exists with $a_m \to \infty$ such that

$$R_m(x) = \frac{1}{2\pi i}\int_{-a_m-i\infty}^{-a_m+i\infty} \mathcal{P}_c(\mathrm{r},\mathrm{z})e^{x\mathrm{r}}d\mathrm{r} \to 0 \qquad m \to \infty.$$

We show this holds for any sequence $\{a_m\}$ such that $a_m \to \infty$ and the values of $-a_m$ avoid the real portions of the poles in (52). We write

$$R_m(x) = \frac{1}{2\pi}e^{-a_m x}\int_{-\infty}^{\infty} \mathcal{P}_c(-a_m+iy,\mathrm{z})e^{ixy}dy.$$

Using integration by parts with $u(y) = \mathcal{P}_c(-a_m+iy,\mathrm{z})$ and $dw(y) = e^{ixy}dy$ leads to $w(y) = e^{ixy}/(ix)$ and

$$2\pi e^{a_m x}R_m(x) = \mathcal{P}_c(-a_m+iy,\mathrm{z})\frac{e^{ixy}}{ix}\Bigg|_{y=-\infty}^{y=\infty} - \frac{1}{x}\int_{-\infty}^{\infty}\mathcal{P}'_{c\mathrm{r}}(-a_m+iy,\mathrm{z})e^{ixy}dy.$$

In the first term $|\mathcal{P}_c(-a_m+iy,\mathrm{z})| \to 0$ as $|y| \to \infty$ so it is 0 and only the second term needs consideration. We show that $|\mathcal{P}'_{c\mathrm{r}}(-a_m+iy,\mathrm{z})|$ is integrable in y uniformly for large m. Using the product rule,

$$\mathcal{P}'_{c\mathrm{r}}(\mathrm{r},\mathrm{z}) = -\frac{1}{\mathrm{r}^2}\left\{1-\frac{1-\mathcal{F}(\mathrm{z})}{1-\mathcal{H}_c(-\mathrm{r},\mathrm{z})}\right\} + \frac{1}{\mathrm{r}}\left\{\frac{1-\mathcal{F}(\mathrm{z})}{[1-\mathcal{H}_c(-\mathrm{r},\mathrm{z})]^2}\right\}\mathcal{H}'_{c\mathrm{r}}(-\mathrm{r},\mathrm{z}).$$

Since $|\mathcal{H}_c(a_m-iy,\mathrm{z})| \to 0$ as $m \to \infty$ or $|y| \to \infty$, the moduli of the factors in curly braces are all uniformly bounded in $y = \mathrm{Im}(-\mathrm{r}) \in \mathbb{R}$ and in sufficiently large m. The first term with factor $1/\mathrm{r}^2$ is $O(|y|^{-2})$ along the line of integration $\mathrm{r} = -a_m + iy$ and integrable uniformly in m. Using arguments that lead to (82) and (83), then

$$\left| e^{-2\sqrt{1-2(a_m - iy)}} \right| < e^{-\sqrt{2}\max(\sqrt{2|y|}, \sqrt{2a_m - 1})}$$

so that

$$|\mathcal{H}'_{c\mathrm{r}}(a_m - iy, \mathrm{z})| < ce^{-\sqrt{2}\max(\sqrt{2|y|}, \sqrt{2a_m - 1})}\},$$

for some $c > 0$. Thus,

$$\begin{aligned} \int_{-\infty}^{\infty} |\mathcal{H}'_{c\mathrm{r}}(a_m - iy, \mathrm{z})| dy &< c \int_{-\infty}^{\infty} e^{-\sqrt{2}\max(\sqrt{2|y|}, \sqrt{2a_m - 1})} dy \\ &= 2c(a_m - 1/2)e^{-\sqrt{4a_m - 2}} + 2c \int_{a_m - 1/2}^{\infty} e^{-2\sqrt{y}} dy \\ &= ce^{-\sqrt{(4a_m - 2)}} \left(2a_m + \sqrt{4a_m - 2} \right). \end{aligned}$$

This converges to 0 as $m \to \infty$. Thus $R_m(x) = o(e^{-a_m x}) \to 0$ as $m \to \infty$. □

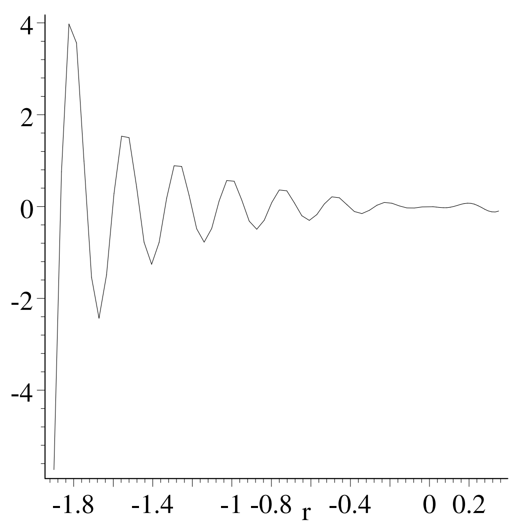

Figure 9. A plot of $10^{32} \times |\hat{\mathcal{E}}_1(e^{\mathrm{r}})|/\hat{\mathcal{P}}_x(e^{\mathrm{r}})$ vs. r showing the relative size and the $O(10^{-32})$ order of the second-order pair of residue terms $|\hat{\mathcal{E}}_1(e^{\mathrm{r}})|$ as compares to the leading term $\hat{\mathcal{P}}_x(e^{\mathrm{r}})$.

Example 15 (Reward as an interarrival random walk). Suppose $R|F = n \sim$ Gamma $(\alpha n, \beta)$ with mean $\alpha n/\beta$ and $F \sim$ Geometric (p) with mean $1/p$. Then $\mathcal{G}_0(\mathrm{r}) = (1 - \mathrm{r}/\beta)^{-\alpha}$ and $\mathcal{F}(\mathrm{z}) = p\mathrm{z}/(1 - q\mathrm{z})$ and

$$\mathcal{H}_c(\mathrm{r}, \mathrm{z}) = \frac{p\mathrm{z}(1 - \mathrm{r}/\beta)^{-\alpha}}{1 - q\mathrm{z}(1 - \mathrm{r}/\beta)^{-\alpha}}.$$

We take $\alpha = 2, \beta = 1$, and $p = 1/3$ so that $\mathbb{E}\{R\} = 6$ and $\mathbb{V}\{R\} = 30$. The residue approximation for the MGF of the reward at time n has $\hat{z}(\mathrm{r}) = (1 - \mathrm{r})^2$ as the root of $1 - \mathcal{H}_c(\mathrm{r}, \mathrm{z}) = 0$ so that

$$\hat{\mathcal{R}}_{cn}(\mathrm{r}) = (1 - \mathrm{r})^{-2n} \frac{1}{3 - 2(1 - \mathrm{r})^2} \qquad -0.224 < \mathrm{r} < 1.$$

With $\alpha = 2$, the expression for $\mathcal{R}_c(\mathrm{r}, \mathrm{z})$ in (16) is rational in z and given by

$$\mathcal{R}_c(\mathrm{r}, \mathrm{z}) = \frac{(1-\mathrm{r})^2 - 2\mathrm{z}/3}{(1 - 2\mathrm{z}/3)\{(1-\mathrm{r})^2 - \mathrm{z}\}} \tag{84}$$

The value $\hat{z}(\mathrm{r}) = (1-\mathrm{r})^2$ accounts for the smallest dominant pole while the other pole occurs at $\mathrm{z} = 3/2$. Using the method of partial fractions, the exact MGF is

$$\mathbb{E}\{e^{\mathrm{r}R(n)}\} = \hat{\mathcal{R}}_{cn}(\mathrm{r}) + \left(\frac{2}{3}\right)^n \frac{(1-\mathrm{r})^2 - 1}{(1-\mathrm{r})^2 - 3/2} \qquad \mathrm{r} < 1. \tag{85}$$

The point mass at 0 has probability $(2/3)^n$ and so the conditional MGF of $R(n)$ given $R(n) > 0$ is

$$\mathcal{R}_{n+}(\mathrm{r}) = \frac{\mathbb{E}\{e^{\mathrm{r}R(n)}\} - (2/3)^n}{1 - (2/3)^n} = \frac{\hat{\mathcal{R}}_{cn}(\mathrm{r}) + (2/3)^n \frac{1/2}{(1-\mathrm{r})^2 - 3/2}}{1 - (2/3)^n}. \tag{86}$$

This is the MGF of a continuous distribution and its survival function when scaled by the factor $c_n = 1 - (2/3)^n$ gives us the exact value for $\mathbb{P}\{R(n) > x\}$.

In determining the survival function of $R(10)$, we compare two saddlepoint approximations in the left panel of Figure 10. The solid line shows a single-saddlepoint approximation based on $\mathcal{R}_{10+}(\mathrm{r})$ in (86) and scaled by factor $1 - (2/3)^{10}$. This represents the best we can do using saddlepoint approximations since it is using the exact MGF with the point mass removed. This transform is seldom computable in other applications. The dashed red line shows a single-saddlepoint approximation based on residue approximation $\hat{\mathcal{R}}_{c,10}(\mathrm{r})$. The dotted line is a Normal $(16, 22)$ approximation based on the moment approximations of Corollary 2. The right panel plots the percentage relative error of the approximation using $\hat{\mathcal{R}}_{c,10}(\mathrm{r})$ taking the approximation based on $\mathcal{R}_{10+}(\mathrm{r})$ as the standard. What this plot shows is that little is lost in using the residue approximation over the approximation based on knowing $\mathcal{R}_{10+}(\mathrm{r})$.

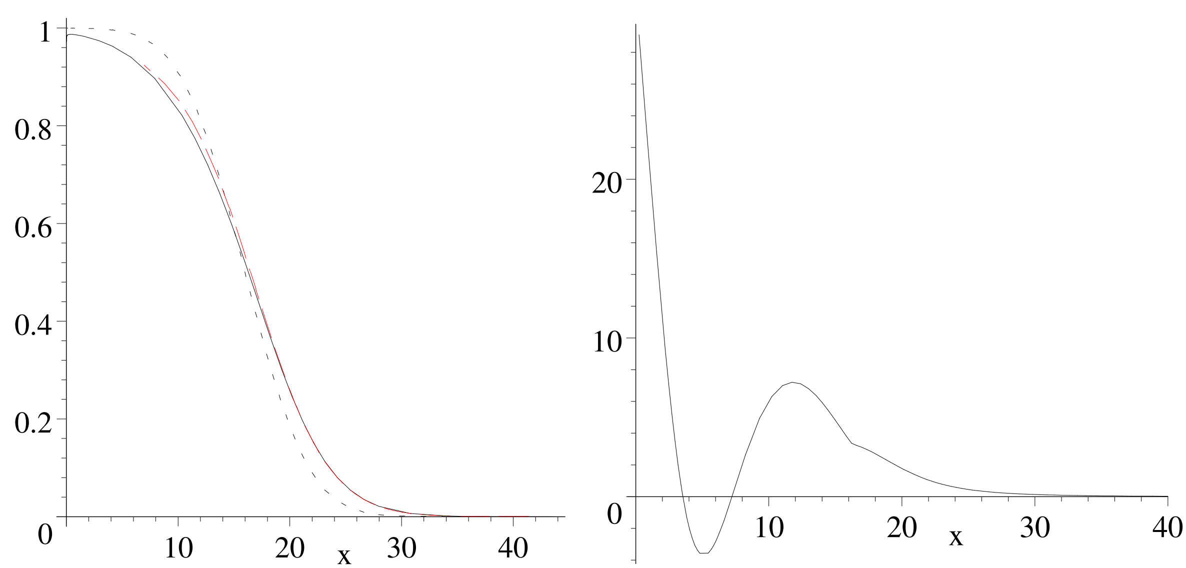

Figure 10. (Left panel) Approximations for the survival function of $R(10)$ using single-saddlepoint inversion of $\mathcal{R}_{10+}(\mathrm{r})$ (solid black line), single-saddlepoint inversion of $\hat{\mathcal{R}}_{c,10}(\mathrm{r})$ (dashed red line), and a Normal $(16, 22)$ approximation (dotted line). (Right panel) Plot of percentage relative errors respecting the tails for inversion using $\hat{\mathcal{R}}_{c,10}(\mathrm{r})$ as compares with inversion using $\mathcal{R}_{10+}(\mathrm{r})$.

Example 16 (Reward as an interarrival random walk). Difficulties arise in Example 15 when using the residue approximation for the PGF of first-passage to reward x. There are two solutions to $\mathcal{G}_0(-\mathrm{r}) = (1+\mathrm{r})^{-2} = 1/\mathrm{z}$ which occur at $\hat{r}(\mathrm{z}) = -1 \pm \sqrt{\mathrm{z}}$ with the plus solution used to compute the residue approximation as

$$\hat{\mathcal{P}}_x(\mathrm{z}) = e^{-(1-\sqrt{\mathrm{z}})x}\frac{\left(1+\sqrt{\mathrm{z}}\right)\sqrt{\mathrm{z}}}{6(1-2\mathrm{z}/3)} \qquad \mathrm{z} < 3/2.$$

The problem with this expression is that it is not analytic at $\mathrm{z} = 0$ and so it cannot serve as a PGF surrogate since there is no Taylor expansion in z about 0 for this approximation. It is analytic in $\sqrt{\mathrm{z}}$ so a Taylor expansion in $\sqrt{\mathrm{z}}$ can be found but the weights are far from what they should be. An exact expression for $\mathcal{P}_x(\mathrm{z})$ may be computed using partial fractions since the expression for $\mathcal{R}_c(\mathrm{r}, \mathrm{z})$ in (84) is rational with the second pole at $-1 - \sqrt{\mathrm{z}}$. This leads to an exact expression for $\mathcal{P}_x(\mathrm{z})$ using partial fractions as

$$\mathcal{P}_x(\mathrm{z}) = e^{-(1-\sqrt{\mathrm{z}})x}\frac{\left(1+\sqrt{\mathrm{z}}\right)\sqrt{\mathrm{z}}}{6(1-2\mathrm{z}/3)} - e^{-(1+\sqrt{\mathrm{z}})x}\frac{\left(1-\sqrt{\mathrm{z}}\right)\sqrt{\mathrm{z}}}{6(1-2\mathrm{z}/3)} \tag{87}$$

$$= e^{-x}\frac{1}{6(1-2\mathrm{z}/3)}\left\{2\sqrt{\mathrm{z}}\sinh(x\sqrt{\mathrm{z}}) + 2\mathrm{z}\cosh(x\sqrt{\mathrm{z}})\right\}. \tag{88}$$

The two individual terms in (87) are not analytic in z but rather expand in powers of $\sqrt{\mathrm{z}}$. When combined, the half-powers in z for the two terms cancel and result in a PGF that is analytic in z as expressed in (88) where both terms in the curly braces are analytic in z at $\mathrm{z} = 0$. Expression (88) shows that $\mathcal{P}_x(\mathrm{z})$ has a simple dominant pole at $\mathrm{z} = 3/2$ and an essential singularity at $\mathrm{z} = \infty$. The use of $\hat{\mathcal{P}}_x(\mathrm{z})$ as a surrogate for $\mathcal{P}_x(\mathrm{z})$ fails because both poles $-1 \pm \sqrt{\mathrm{z}}$ need to be included in a residue expansion for it to be an adequate approximation of the PGF. □

If in Example 16 we assume α is a general integer $\alpha \in \mathbb{N}$, then $\hat{\mathcal{P}}_x(\mathrm{z})$ fails to adequately approximate $\mathcal{P}_x(\mathrm{z})$ except when $\alpha = 1$. For $\alpha \geq 2$ there are α roots equally spaced on a circle in $\mathbb{C}$ and residue expansion terms using all α roots are required to adequately approximate $\mathcal{P}_x(\mathrm{z})$.

References

[21] Billingsley, P. *Probability and Measure.* Wiley: New York, 2012.

[22] Bleistein, N. Uniform asymptotic expansions of integrals with stationary point near algebraic singularity. Comm. Pure Appl. Math. **1966,** XIX, 353–370.

[23] Curtiss, J.H. A note on the theory of moment generating functions. *Ann. Math. Statist.* **1942,** 13, 430–433.

[24] Daniels, H. E. Tail probability approximations. International Statistical Review **1987,** 55, 37–48.

[25] Feller, W. *An Introduction to Probability Theory and its Applications,* Vol. II, 2nd ed. Wiley: New York, 1971.