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SUPPLEMENTARY MATERIAL: FELLER AND ERGODIC PROPERTIES OF JUMP-MOVE PROCESSES WITH APPLICATIONS TO INTERACTING PARTICLE SYSTEMS

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Abstract

This supplementary material contains the proofs of some results of the main article. It also describes some topological properties of the space E endowed with the distance d_1 in the case of interacting particles in $\mathbb{R}^d$, as introduced in Section 2.4 of the main article. All numbering and references in this supplementary material begin with the letter S, the other references referring to the main article.

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1. Proofs of Section 2.3 about the Kolmogorov backward equation

1.1. Proof of Theorem 1

On the one hand, for any $x \in E$, $t > 0$ and $A \in \mathcal{E}$,

$$\begin{aligned}\mathbb{P}_x(X_t \in A, \tau_1 > t) &= \mathbb{E}_x\left[\mathbb{E}_x(\mathbf{1}_{X_t \in A}\,\mathbf{1}_{\tau_1 > t} | (Y_u^{(0)})_{u \geq 0})\right] \\ &= \mathbb{E}_x\left[\mathbb{E}_x(\mathbf{1}_{Y_t^{(0)} \in A}\,\mathbf{1}_{\tau_1 > t} | Y^{(0)})\right] \\ &= \mathbb{E}_x\left[\mathbf{1}_{Y_t^{(0)} \in A}\,\mathbb{E}_x(\mathbf{1}_{\tau_1 > t} | Y^{(0)})\right] \\ &= \mathbb{E}_x\left[\mathbf{1}_{Y_t^{(0)} \in A}\,\mathrm{e}^{-\int_0^t \alpha(Y_u^{(0)})\,\mathrm{d}u}\right] \\ &= \mathbb{E}_x^Y\left[\mathbf{1}_{Y_t \in A}\,\mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u}\right]. \end{aligned} \tag{1.1}$$

On the other hand, by construction of the process

$$\mathbb{E}_x[\mathbf{1}_{X_t \in A} | \mathcal{F}_{\tau_1}]\mathbf{1}_{\tau_1 \leq t} = Q_{t-\tau_1}(X_{\tau_1}, A)\mathbf{1}_{\tau_1 \leq t}$$

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where $\mathcal{F}_{\tau_1} = \{F \in \mathcal{F} : F \cap \{\tau_1 \le t\} \in \mathcal{F}_t,\ \forall t \ge 0\}$. Then

$$\begin{aligned}
\mathbb{P}_x(X_t \in A, \tau_1 \le t) &= \mathbb{E}_x[\mathbb{E}_x[\mathbf{1}_{X_t\in A}|\mathcal{F}_{\tau_1}]\mathbf{1}_{\tau_1\le t}] \\
&= \mathbb{E}_x[Q_{t-\tau_1}(X_{\tau_1}, A)\mathbf{1}_{\tau_1\le t}] \\
&= \mathbb{E}_x[\mathbb{E}_x[Q_{t-\tau_1}(X_{\tau_1}, A)\mathbf{1}_{\tau_1\le t}|\tau_1, Y^{(0)}]] \\
&= \mathbb{E}_x\left[\int_{y\in E} K(Y^{(0)}_{\tau_1}, dy)Q_{t-\tau_1}(y, A)\mathbf{1}_{\tau_1\le t}\right] \\
&= \mathbb{E}_x\left[\mathbb{E}_x[\int_{y\in E} K(Y^{(0)}_{\tau_1}, dy)Q_{t-\tau_1}(y, A)\mathbf{1}_{\tau_1\le t}|Y^{(0)}]\right] \\
&= \mathbb{E}_x\left[\int_0^t\int_{y\in E} K(Y^{(0)}_s, dy)Q_{t-s}(y, A)\alpha(Y^{(0)}_s)\mathrm{e}^{-\int_0^s \alpha(Y^{(0)}_u)\,\mathrm{d}u}\,\mathrm{d}s\right] \\
&= \int_0^t\int_E Q_{t-s}(y, A)\mathbb{E}_x\left[K\left(Y^{(0)}_s, dy\right)\,\alpha(Y^{(0)}_s)\mathrm{e}^{-\int_0^s \alpha(Y^{(0)}_u)\,\mathrm{d}u}\right]\mathrm{d}s \\
&= \int_0^t\int_E Q_{t-s}(y, A)\mathbb{E}^Y_x\left[K\left(Y_s, dy\right)\,\alpha(Y_s)\mathrm{e}^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u}\right]\mathrm{d}s. \qquad (1.2)
\end{aligned}$$

The result then follows gathering (1.1) and (1.2).

1.2. Proof of Proposition 1

The proof is made up from Lemmas 1, 2 and 3, the approach being similar to [FellerFeller1971]. In Lemma 1 we built a solution $Q_{t,\infty}(x, A)$ of (2.5) for any $x \in E$ and $A \in \mathcal{E}$, while Lemmas 2 and 3 will imply the unicity of the solution.

Lemma 1. *For all $x \in E$ and $A \in \mathcal{E}$, the function $t \in \mathbb{R}_+ \mapsto Q_{t,\infty}(x, A)$ is a solution of (2.5).*

Proof. We will proceed as in the proof of Theorem 1. First

$$\mathbb{P}_x(X_t \in A, T_{p+1} > t, \tau_1 > t) = \mathbb{P}_x(X_t \in A, \tau_1 > t) = \mathbb{E}^Y_x\left[\mathbf{1}_{Y_t\in A}\,\mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u}\right].$$

Secondly, if the process jumps once before t (at time τ_1) and is in A at time t with at most $p+1$ jumps, the process has at most p jumps after the time τ_1. By construction of the process, the law of X_t given $\tau_1 < +\infty$ and X_{T_1} is the same as the one of $X_{t-\tau_1}$ given $X_0 = X_{\tau_1}$. We then obtain

$$\mathbb{E}_x[\mathbf{1}_{X_t\in A}\mathbf{1}_{\tau_1\le t<T_{p+1}}|\mathcal{F}_{\tau_1}] = Q_{t-\tau_1,p}(X_{\tau_1}, A)\mathbf{1}_{\tau_1\le t}.$$

This leads to

$$\begin{aligned}
\mathbb{P}_x(X_t \in A, T_{p+1} > t, \tau_1 \le t) &= \mathbb{E}_x[Q_{t-\tau_1,p}(X_{\tau_1}, A)\mathbf{1}_{\tau_1\le t}] \\
&= \mathbb{E}_x[\mathbb{E}_x[Q_{t-\tau_1,p}(X_{\tau_1}, A)\mathbf{1}_{\tau_1\le t}|Y^{(0)}, \tau_1]] \\
&= \mathbb{E}_x[\int_{y\in E} Q_{t-\tau_1,p}(y, A)\mathbf{1}_{\tau_1\le t}K(Y^{(0)}_{\tau_1}, dy)] \\
&= \mathbb{E}_x[\mathbb{E}_x[\int_{y\in E} Q_{t-\tau_1,p}(y, A)\mathbf{1}_{\tau_1\le t}K(Y^{(0)}_{\tau_1}, dy)|Y^{(0)}]] \\
&= \mathbb{E}_x[\int_0^t\int_{y\in E} Q_{t-s,p}(y, A)\mathbf{1}_{\tau_1\le t}K(Y^{(0)}_s, dy)\alpha(Y^{(0)}_s)\mathrm{e}^{-\int_0^s \alpha(Y^{(0)}_u)\,\mathrm{d}u}]] \\
&= \int_0^t\int_E Q_{t-s,p}(y, A)\mathbb{E}^Y_x\left[K\left(Y_s, dy\right)\,\alpha(Y_s)\mathrm{e}^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u}\right]\mathrm{d}s.
\end{aligned}$$

We then obtain the induction formula

$$Q_{t,p+1}(x,A) = \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A}\, \mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} \right] + \int_0^t \int_E Q_{t-s,p}(y,A) \mathbb{E}_x^Y \left[K(Y_s, dy)\ \alpha(Y_s) \mathrm{e}^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u} \right] \mathrm{d}s.$$

This leads by monotone convergence to

$$Q_{t,\infty}(x,A) = \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A}\, \mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} \right] + \int_0^t \int_E Q_{t-s,\infty}(y,A) \mathbb{E}_x^Y \left[K(Y_s, dy)\ \alpha(Y_s) \mathrm{e}^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u} \right] \mathrm{d}s.$$

Lemma 2. *$Q_{t,\infty}$ is called the minimal solution of (2.5) in the sense that for any non-negative solution Q_t of (2.5), we have $Q_t \geq Q_{t,\infty}$.*

Proof. Let Q_t be a non-negative solution of (2.5). Then for any $x \in E$ and $A \in \mathcal{E}$

$$Q_t(x,A) \geq Q_{t,1}(x,A) = \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A}\, \mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} \right].$$

We then proceed by induction. If $Q_t \geq Q_{t,p}$ then

$$\begin{aligned} Q_t(x,A) &= \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A}\, \mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} \right] + \int_0^t \int_E Q_{t-s}(y,A) \mathbb{E}_x^Y \left[K(Y_s, dy)\ \alpha(Y_s) \mathrm{e}^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u} \right] \mathrm{d}s \\ &\geq \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in A}\, \mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} \right] + \int_0^t \int_E Q_{t-s,p}(y,A) \mathbb{E}_x^Y \left[K(Y_s, dy)\ \alpha(Y_s) \mathrm{e}^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u} \right] \mathrm{d}s \\ &= Q_{t,p+1}(x,A). \end{aligned}$$

Finally $Q_t(x,A) \geq Q_{t,p}(x,A)$ for every $p \geq 1$ and the result follows by letting p go to infinity.

Lemma 3. *The minimal solution $Q_{t,\infty}$ is stochastic, i.e. $Q_{t,\infty}(x,E) = 1$.*

Proof. Recall that α is bounded by $\alpha^* > 0$. It is then enough to show by induction that $Q_{t,p}(x,E) \geq 1 - (1 - \mathrm{e}^{-\alpha^* t})^p$ for any $p \geq 1$. First

$$Q_{t,1}(x,E) = \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in E}\, \mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} \right] = \mathbb{E}_x^Y \left[\mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} \right] \geq \mathbb{E}_x \left(\mathrm{e}^{-\alpha^* t} \right) = \mathrm{e}^{-\alpha^* t}.$$

Then notice that

$$\mathbb{P}_x(\tau_1 \leq t) = \mathbb{E}_x^Y \left[1 - \mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} \right] \leq 1 - \mathrm{e}^{-\alpha^* t}.$$

We then obtain by induction

$$\begin{aligned} Q_{t,p+1}(x,E) &= \mathbb{E}_x^Y \left[\mathbf{1}_{Y_t \in E}\, \mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} \right] + \int_0^t \int_E Q_{t-s,p}(y,A) \mathbb{E}_x^Y \left[K(Y_s, dy)\ \alpha(Y_s) \mathrm{e}^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u} \right] \mathrm{d}s \\ &\geq \mathbb{E}_x^Y \left[\mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} \right] + \int_0^t \left(1 - (1 - \mathrm{e}^{-\alpha^*(t-s)})^p \right) \mathbb{E}_x^Y \left[\int_E K(Y_s, dy)\ \alpha(Y_s) \mathrm{e}^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u} \right] \mathrm{d}s \\ &\geq \mathbb{E}_x^Y \left[\mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} \right] + \int_0^t \left(1 - (1 - \mathrm{e}^{-\alpha^* t})^p \right) \mathbb{E}_x^Y \left[\alpha(Y_s) \mathrm{e}^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u} \right] \mathrm{d}s \\ &= \mathbb{P}_x(\tau_1 > t) + \left(1 - (1 - \mathrm{e}^{-\alpha^* t})^p \right) \mathbb{P}_x(\tau_1 \leq t) \\ &= 1 - (1 - \mathrm{e}^{-\alpha^* t})^p\, \mathbb{P}_x(\tau_1 \leq t) \geq 1 - (1 - \mathrm{e}^{-\alpha^* t})^{p+1}. \end{aligned}$$

By bringing together the last three lemmas it does not take long to prove Proposition 1. By Lemma 1, $Q_{t,\infty}$ is a solution of (2.5). We now prove the unicity. Let Q_t be a non-negative sub-stochastic solution of (2.5). Lemma 2 entails $Q_t(x, A) \geq Q_{t,\infty}(x, A)$ and $Q_t(x, E\backslash A) \geq Q_{t,\infty}(x, E\backslash A)$ for every $A \in \mathcal{E}$. We get then

$$1 \geq Q_t(x, E) = Q_t(x, A) + Q_t(x, E\backslash A) \geq Q_{t,\infty}(x, A) + Q_{t,\infty}(x, E\backslash A) = Q_{t,\infty}(x, E) = 1$$

by Lemma 3 so $Q_t(x, A) = Q_{t,\infty}(x, A)$ for every $A \in \mathcal{E}$.

1.3. Proof of Proposition 2

We first show that $Q_{t,(\infty)}$ is a solution of (2.5). Let $n \geq 0$, $x \in E_n$ and $p \geq n$. If there is no jump before t, then

$$\mathbb{P}_x(X_t \in A, \tau_1 > t, \forall s \in [0, t]\ n(X_s) \leq p) = \mathbb{P}_x(X_t \in A, \tau_1 > t).$$

By construction of the process, if the first jump before t is a death,

$$\mathbb{P}_x(X_t \in A, \forall s \in [0, t]\ n(X_s) \leq p \,|\, \mathcal{F}_{\tau_1}, \text{ a death occurs at } \tau_1)\mathbf{1}_{\tau_1 \leq t} = Q_{t-\tau_1,(p)}(X_{\tau_1}, A)\mathbf{1}_{\tau_1 \leq t},$$

and if the first jump before t is a birth,

$$\mathbb{P}_x(X_t \in A, \forall s \in [0, t]\ n(X_s) \leq p \,|\, \mathcal{F}_{\tau_1}, \text{ a birth occurs at } \tau_1)\mathbf{1}_{\tau_1 \leq t} = Q_{t-\tau_1,(p)}(X_{\tau_1}, A)\mathbf{1}_{\tau_1 \leq t}\mathbf{1}_{p>n}.$$

Following the same computations as in the proof of Theorem 1, we obtain

$$\begin{aligned} Q_{t,(p)}(x, A) &= \mathbb{E}_x^Y\left[\mathbf{1}_{Y_t \in A}\, \mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u}\right] \\ &\quad + \int_0^t \int_{E_{n+1}} Q_{t-s,(p)}(y, A)\, \mathbb{E}_x^Y\left[\beta(Y_s)\, K_\beta(Y_s, dy)\ \mathrm{e}^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u}\right] \mathrm{d}s\, \mathbf{1}_{p>n} \\ &\qquad + \int_0^t \int_{E_{n-1}} Q_{t-s,(p)}(y, A)\, \mathbb{E}_x^Y\left[\delta(Y_s)\, K_\delta(Y_s, dy)\ \mathrm{e}^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u}\right] \mathrm{d}s, \end{aligned}$$

and $Q_{t,(\infty)}(x, A)$ satisfies (2.5) by continuity of the probability. The proof is then complete thanks to the unicity of the solution to (2.5).

2. Proofs of Section 3.1 about Feller properties

2.1. Proof of Proposition 3

Both results of the proposition are based on the following calculation, for any $f \in M_b(E)$:

$$\begin{aligned} Q_t f(x) - f(x) &= \mathbb{E}_x^Y\left[f(Y_t)\mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u}\right] - f(x) + \mathbb{E}_x\left[f(X_t)\mathbf{1}_{N_t \geq 1}\right] \\ &= Q_t^Y f(x) - f(x) + \mathbb{E}_x^Y\left[f(Y_t)\left(\mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} - 1\right)\right] + \mathbb{E}_x\left[f(X_t)\mathbf{1}_{N_t \geq 1}\right]. \end{aligned}$$

The last two terms goes uniformly to 0 when $t \to 0$. Indeed,

$$\begin{aligned} \left|\mathbb{E}_x^Y\left[f(Y_t)\left(\mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u} - 1\right)\right] + \mathbb{E}_x\left[f(X_t)\mathbf{1}_{N_t \geq 1}\right]\right| &\leq ||f||_\infty \alpha^* t + ||f||_\infty \mathbb{P}_x(N_t \geq 1) \\ &= ||f||_\infty \alpha^* t + ||f||_\infty \mathbb{E}_x^Y\left[\left(1 - \mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u}\right)\right] \\ &\leq 2\alpha^* t ||f||_\infty. \end{aligned}$$

So we obtain directly the second point of the proposition. For the first point remark that when $f \in C_b(E)$, by continuity of $f \circ Y$ and the dominated convergence theorem, $\lim_{t\to 0} Q^Y_t f(x) = f(x)$.

2.2. Proof of Theorem 2 (part 1)

The proof of the Feller continuous property of $(X_t)_{t\geq 0}$ is based on the following Lemma 4 that exploits the Feller continuous property of Q^Y_t, and on Lemma 5 which in addition makes use of the Feller continuous property of the jump kernel K.

Lemma 4. *Assume that for any $t \geq 0$, $Q^Y_t C_b(E) \subset C_b(E)$. Then for any $p \geq 1$, $f_1, \dots f_p \in C_b(E)$ and $0 \leq t_1 < \cdots < t_p$ the function $x \mapsto \mathbb{E}^Y_x \left[f_1(Y_{t_1}) \dots f_p(Y_{t_p})\right]$ is continuous. Furthermore, for any $f \in C_b(E)$ the function $x \mapsto \mathbb{E}^Y_x [f(Y_t)\mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u}]$ is continuous.*

Proof. To prove the first statement, we proceed first by induction on $p \geq 1$. Since $x \mapsto \mathbb{E}^Y_x \left[f_1(Y_{t_1})\right] = Q^Y_{t_1} f_1(x)$, the property is satisfied for $p = 1$ because $Q^Y_t C_b(E) \subset C_b(E)$ for any $t \geq 0$ by assumption. Suppose now that the property is true for some $p \geq 1$. Let $f_1, \dots, f_{p+1} \in C_b(E)$ and $0 \leq t_1 < \cdots < t_{p+1}$. Then

$$\begin{aligned}\mathbb{E}^Y_x \left[f_1(Y_{t_1}) \dots f_{p+1}(Y_{t_{p+1}})\right] &= \mathbb{E}^Y_x \left[\mathbb{E}^Y_x \left(f_1(Y_{t_1}) \dots f_{p+1}(Y_{t_{p+1}}) \middle| Y_{t_1}, \dots, Y_{t_p}\right)\right] \\ &= \mathbb{E}^Y_x \left[f_1(Y_{t_1}) \dots f_p(Y_{t_p}) \mathbb{E}^Y_x \left(f_{p+1}(Y_{t_{p+1}}) | Y_{t_p}\right)\right] \\ &= \mathbb{E}^Y_x \left[f_1(Y_{t_1}) \dots f_p(Y_{t_p}) Q^Y_{t_{p+1}-t_p} f_{p+1}(Y_{t_p})\right].\end{aligned}$$

The function $f_p \times Q^Y_{t_{p+1}-t_p} f_{p+1}$ is continuous by assumption so we can apply the induction hypothesis.

Regarding the second statement of the lemma, let us take $f \in C_b(E)$ and $t \geq 0$. We have

$$\begin{aligned}\mathbb{E}^Y_x \left[f(Y_t)\mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u}\right] &= \mathbb{E}^Y_x \left[f(Y_t) \sum_{k\geq 0} \frac{(-1)^k}{k!} \left(\int_0^t \alpha(Y_u)\,\mathrm{d}u\right)^k\right] \\ &= \sum_{k\geq 0} \frac{(-1)^k}{k!} \int_{u_1=0}^t \cdots \int_{u_k=0}^t \mathbb{E}^Y_x \left[f(Y_t)\alpha(Y_{u_1}) \dots \alpha(Y_{u_k})\right] \mathrm{d}u_1 \dots \mathrm{d}u_k\end{aligned}$$

which is valid because $f \times \alpha^k$ is bounded. For any $u_1 \geq 0, \dots, u_k \geq 0$, the function $x \in E \mapsto \mathbb{E}^Y_x \left[f(Y_t)\alpha(Y_{u_1}) \dots \alpha(Y_{u_k})\right]$ is continuous by the first part of the proof and this expression is bounded uniformly in x by $\|f\|_\infty \times (\alpha^*)^k \in L^1([0,t]^k)$. Again, by normal convergence, we obtain the expected result.

Lemma 5. *Assume that $Q^Y_t C_b(E) \subset C_b(E)$ for any $t \geq 0$ and that $K C_b(E) \subset C_b(E)$. Let $t > 0$. Then for any $k \geq 1$, for any bounded measurable function φ on $E \times \mathbb{R}_+$ such that $\varphi(.,u)$ is continuous for any $u \leq t$, the function $x \mapsto \mathbb{E}_x[\varphi(X_{T_k}, T_k)\mathbf{1}_{T_k \leq t}]$ is continuous.*

Proof. We shall proceed by induction. For $k = 1$,

$$\begin{aligned}
\mathbb{E}_x[\varphi(X_{T_1}, T_1)\mathbf{1}_{T_1 \le t}] &= \mathbb{E}_x[\mathbf{1}_{T_1 \le t}\, \mathbb{E}_x[\varphi(X_{T_1}, T_1)|Y^{(0)}, T_1]] \\
&= \mathbb{E}_x[\int_E K(Y^{(0)}_{T_1}, dz)\varphi(z, T_1)\mathbf{1}_{T_1 \le t}] \\
&= \mathbb{E}_x[\int_E \int_0^t K(Y^{(0)}_{t_1}, dz)\varphi(z, t_1)\alpha(Y^{(0)}_{t_1})e^{-\int_0^{t_1} \alpha(Y^{(0)}_u)\,\mathrm{d}u}\,\mathrm{d}t_1] \\
&= \int_0^t \mathbb{E}^Y_x[H(Y_{t_1}, t_1)\alpha(Y_{t_1})e^{-\int_0^{t_1} \alpha(Y_u)\,\mathrm{d}u}]\,\mathrm{d}t_1
\end{aligned}$$

where $H(x, u) = \int_E K(x, dz)\varphi(z, u)$. Since $z \mapsto \varphi(z, t_1)$ belongs to $C_b(E)$ for every $t_1 \le t$, the Feller continuous property of K entails the continuity of $x \mapsto H(x, t_1)$ for every $t_1 \le t$. Consequently the function

$$x \mapsto \mathbb{E}^Y_x[H(Y_{t_1}, t_1)\alpha(Y_{t_1})e^{-\int_0^{t_1} \alpha(Y_u)\,\mathrm{d}u}]$$

is continuous for every t_1 by Lemma 4. The functions H and α being bounded, the dominated convergence theorem yields the continuity of $x \mapsto \mathbb{E}_x[\varphi(X_{T_1}, T_1)\mathbf{1}_{T_1 \le t}]$, proving the statement for $k = 1$. Assume now that the property holds for $k \ge 1$. We compute similarly

$$\begin{aligned}
\mathbb{E}_x[\varphi(X_{T_{k+1}}, T_{k+1})\mathbf{1}_{T_{k+1} \le t}] &= \mathbb{E}_x[\mathbb{E}_x[\int_E K(Y^{(k)}_{T_{k+1}-T_k}, dz)\varphi(z, T_{k+1})\mathbf{1}_{T_{k+1} \le t}|\mathcal{F}_{T_k}, Y^{(k)}]] \\
&= \mathbb{E}_x[\mathbb{E}^Y_{X_{T_k}}[\int_0^{t-T_k} \int_E K(Y_\tau, dz)\varphi(z, \tau + T_k)\alpha(Y_\tau)e^{-\int_0^\tau \alpha(Y_u)\,\mathrm{d}u}]\mathbf{1}_{T_k \le t}] \\
&= \mathbb{E}_x[\tilde{\varphi}(X_{T_k}, T_k)\mathbf{1}_{T_k \le t}],
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\varphi}(x, u) &= \mathbb{E}^Y_x[\int_0^{t-u} \int_E K(Y_\tau, dz)\varphi(z, \tau + u)\alpha(Y_\tau)e^{-\int_0^\tau \alpha(Y_u)\,\mathrm{d}u}\,\mathrm{d}\tau] \\
&= \int_0^{t-u} \mathbb{E}^Y_x[H(Y_\tau, \tau + u)\alpha(Y_\tau)e^{-\int_0^\tau \alpha(Y_u)\,\mathrm{d}u}]\,\mathrm{d}\tau.
\end{aligned}$$

By Lemma 4, $x \mapsto \mathbb{E}^Y_x[H(Y_\tau, \tau + u)\alpha(Y_\tau)e^{-\int_0^\tau \alpha(Y_u)\,\mathrm{d}u}]$ is continuous for each u, τ, so $\tilde{\varphi}(., u)$ is continuous for every $u \le t$. We then obtain the result applying the induction hypothesis.

We are now in position to prove the first part of Theorem 2 about the Feller continuous property of $(X_t)_{t \ge 0}$. We compute for $t > 0$, $x \in E$ and $f \in C_b(E)$

$$\begin{aligned}
Q_t f(x) &= \sum_{k=0}^{\infty} \mathbb{E}_x[f(X_t)\mathbf{1}_{N_t = k}] \\
&= \mathbb{E}^Y_x[f(Y_t)e^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u}] + \sum_{k \ge 1} \mathbb{E}_x[f(X_t)\mathbf{1}_{T_k \le t < T_{k+1}}] \\
&= \psi(x, t) + \sum_{k \ge 1} \mathbb{E}_x[f(X_t)\mathbf{1}_{T_{k+1} - T_k > t - T_k}\mathbf{1}_{T_k \le t}]
\end{aligned}$$

where $\psi(x,t) = \mathbb{E}^Y_x[f(Y_t)e^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u}]$. We get from Lemma 4 that $\psi(.,t)$ belongs to $C_b(E)$ for every $t > 0$. Then

$$\begin{aligned}
\mathbb{E}_x[f(X_t)\mathbf{1}_{T_{k+1}-T_k>t-T_k}\mathbf{1}_{T_k\le t}] &= \mathbb{E}_x[\mathbf{1}_{T_k\le t} f(Y^{(k)}_{t-T_k})\,\mathbb{E}_x[\mathbf{1}_{T_{k+1}-T_k>t-T_k}|\mathcal{F}_{T_k}, Y^{(k)}]] \\
&= \mathbb{E}_x[f(Y^{(k)}_{t-T_k})\mathrm{e}^{-\int_0^{t-T_k}\alpha(Y^{(k)}_u)\,\mathrm{d}u}\mathbf{1}_{T_k\le t}] \qquad (2.1)\\
&= \mathbb{E}_x[\mathbb{E}^Y_{X_{T_k}}[f(Y_{t-T_k})e^{-\int_0^{t-T_k}\alpha(Y_u)\,\mathrm{d}u}]\mathbf{1}_{T_k\le t}] \qquad (2.2)\\
&= \mathbb{E}_x[\psi(X_{T_k}, t-T_k)\mathbf{1}_{T_k\le t}],
\end{aligned}$$

so Lemma 5 entails that $x \mapsto \mathbb{E}_x[\psi(X_{T_k}, t-T_k)\mathbf{1}_{T_k\le t}]$ is continuous for every $k \ge 1$. The domination

$$\begin{aligned}
\left|\mathbb{E}_x[\psi(X_{T_k}, t-T_k)\mathbf{1}_{T_k\le t}]\right| &\le \|f\|_\infty \mathbb{P}_x(T_k \le t)\\
&\le \|f\|_\infty \mathbb{P}(N^*_t \ge k)
\end{aligned}$$

where $N^*t \sim \mathcal{P}(\alpha^* t)$ (by (2.3)) allows us to conclude that $x \mapsto Q_t f(x)$ is continuous.

2.3. Proof of Theorem 2 (part 2)

Our aim is to prove the Feller property of $(X_t)_{t\ge 0}$ assuming that for every $t > 0$, $Q^Y_t C_0(E) \subset C_0(E)$ and that $K\,C_0(E) \subset C_0(E)$. We follow the same steps as for the proof of Theorem 2 (part 1), by first inspecting the consequences of $Q^Y_t C_0(E) \subset C_0(E)$ in Lemma 6 and second the additional effect of $K\,C_0(E) \subset C_0(E)$ in Lemma 7.

Lemma 6. *Suppose that for every $t > 0$, $Q^Y_t C_0(E) \subset C_0(E)$. Then*

(i). *for any $f \in C_0(E)$, $\lim_{t\to 0} \|Q^Y_t f - f\|_\infty = 0$,*

(ii). *for any $t > 0$, $\mathrm{supp}_{s\in[0,t]}\, Q^Y_s C_0(E) \subset C_0(E)$,*

(iii). *for any $f \in C_0(E)$ the function $x \mapsto \mathbb{E}^Y_x[f(Y_t)\mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u}]$ is continuous.*

Proof. By continuity of $(Y_t)_{t\ge 0}$, $\lim_{t\to 0} Q^Y_t f(x) = f(x)$ for every $f \in C_0(E)$ and every $x \in E$. As proved in [Revuz and YorRevuz and Yor1991], this is equivalent when $Q^Y_t C_0(E) \subset C_0(E)$ to $\lim_{t\to 0} \|Q^Y_t f - f\|_\infty = 0$, which proves the first statement of the lemma.

Concerning the second property, let $\varepsilon > 0$ and $f \in C_0(E)$. Fix $\eta(f) > 0$ such that for every $s < \eta(f)$, $\|Q^Y_s f - f\|_\infty \le \varepsilon$ and $s(x) \in [0,t]$ satisfying $\mathrm{supp}_{s\in[0,t]}\, Q^Y_s f(x) = Q^Y_{s(x)} f(x)$. Then we have

$$Q^Y_{\frac{\lfloor 2^n s(x)/t\rfloor t}{2^n}} f(x) \le \max_{k=0,\dots,2^n} Q^Y_{\frac{kt}{2^n}} f(x) \le \mathrm{supp}_{s\in[0,t]}\, Q^Y_s f(x).$$

So

$$\left|\operatorname{supp}_{s\in[0,t]} Q^Y_s f(x) - \max_{k=0,\ldots,2^n} Q^Y_{\frac{kt}{2^n}} f(x)\right| \le \left|\operatorname{supp}_{s\in[0,t]} Q^Y_s f(x) - Q^Y_{\frac{\lfloor 2^n s(x)/t\rfloor t}{2^n}} f(x)\right|$$
$$= \left|Q^Y_{s(x)} f(x) - Q^Y_{\frac{\lfloor 2^n s(x)/t\rfloor t}{2^n}} f(x)\right|$$
$$= \left|Q^Y_{\frac{\lfloor 2^n s(x)/t\rfloor t}{2^n}} (Q^Y_{s(x)-\frac{\lfloor 2^n s(x)/t\rfloor t}{2^n}} f(x) - f(x))\right|$$
$$\le \|Q^Y_{s(x)-\frac{\lfloor 2^n s(x)/t\rfloor t}{2^n}} f - f\|_\infty$$
$$\le \varepsilon$$

whenever $t2^{-n} \le \eta(f)$. This leads to $\lim_{n\to\infty} \| \operatorname{supp}_{s\in[0,t]} Q^Y_s f - \max_{k=0,\ldots,2^n} Q^Y_{kt2^{-n}} f\|_\infty = 0$. Since $\max_{k=0,\ldots,2^n} Q^Y_{kt2^{-n}} f \in C_0(E)$ for $f \in C_0(E)$ by assumption and $C_0(E)$ is a closed subset of $M_b(E)$ for $\|.\|_\infty$, we deduce that $\operatorname{supp}_{s\in[0,t]} Q^Y_s f \in C_0(E)$.

We finally prove the third point of the lemma in a similar way as in the proof of Lemma 4. First we show by induction on $p \ge 1$ that for any $t \ge 0$ and $0 \le u_1 \le \cdots \le u_p \le t$ and $f \in C_0(E)$ the function $x \mapsto \mathbb{E}^Y_x \left[f(Y_t)\alpha(Y_{u_1})\ldots\alpha(Y_{u_p})\right]$ is in $C_0(E)$. Indeed for $p = 1$

$$\mathbb{E}_x \left[f(Y_t)\alpha(Y_{u_1})\right] = \mathbb{E}_x \left[\alpha(Y_{u_1})\mathbb{E}_x \left[f(Y_t)|\mathcal{F}_{u_1}\right]\right] = \mathbb{E}_x \left[\alpha(Y_{u_1})Q^Y_{t-u_1} f(Y_{u_1})\right] = Q^Y_{u_1} \left(\alpha \times Q^Y_{t-u_1} f\right)(x)$$

and $Q^Y_{u_1} \left(\alpha \times Q^Y_{t-u_1} f\right) \in C_0(E)$ by assumption. For the induction step we just write

$$\mathbb{E}^Y_x \left[f(Y_t)\alpha(Y_{u_1})\ldots\alpha(Y_{u_p})\alpha(Y_{u_{p+1}})\right] = \mathbb{E}^Y_x \left[\alpha(Y_{u_1})\ldots\alpha(Y_{u_p})(Q^Y_{t-u_{p+1}} f \times \alpha)(Y_{u_{p+1}})\right]$$

that is in $C_0(E)$ by assumption and the induction hypothesis. We then obtain the continuity of the function

$$x \mapsto \mathbb{E}^Y_x [f(Y_t)\mathrm{e}^{-\int_0^t \alpha(Y_u)\,\mathrm{d}u}]$$

similarly as in the proof of Lemma 4.

Lemma 7. *Assume that for every $t > 0$, $Q^Y_t C_0(E) \subset C_0(E)$ and that $K\,C_0(E) \subset C_0(E)$. Let $t > 0$. Then for every $k \ge 1$ and all $g \in C_0(E)$, $x \mapsto \mathbb{E}_x[g(X_{T_k})\mathbf{1}_{T_k\le t}]$ vanishes at infinity.*

Proof. Let us prove the result by induction. For $k = 1$,

$$\left|\mathbb{E}_x[g(X_{T_1})\mathbf{1}_{T_1\le t}]\right| = \left|\int_0^t \mathbb{E}_x[Kg(Y_s)\alpha(Y_s)e^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u}]\,\mathrm{d}s\right|$$
$$\le \alpha^* \int_0^t \mathbb{E}_x[K|g|(Y_s)]\,\mathrm{d}s$$
$$\le \alpha^* t \operatorname{supp}_{s\in[0,t]} Q^Y_s K|g|(x).$$

Since $K\,C_0(E) \subset C_0(E)$, the function $K|g|$ belongs to $C_0(E)$, so $\operatorname{supp}_{s\in[0,t]} Q^Y_s K|g| \in C_0(E)$ by Lemma 6. This entails in particular that $x \mapsto \mathbb{E}_x[g(X_{T_1})\mathbf{1}_{T_1\le t}]$ vanishes at infinity. Let now $k \ge 1$ and assume that $x \mapsto \mathbb{E}_x[g(X_{T_k})\mathbf{1}_{T_k\le t}])$ vanishes at infinity. We compute similarly

$$\mathbb{E}_x[g(X_{T_{k+1}})\mathbf{1}_{T_{k+1}\le t}] = \mathbb{E}_x[\mathbb{E}^Y_{X_{T_k}}[\int_0^{t-T_k} \int_E K(Y_s, dz)g(z)\alpha(Y_s)e^{-\int_0^s \alpha(Y_u)\,\mathrm{d}u}]\mathbf{1}_{T_k\le t}]$$

and

$$
\begin{aligned}
\left|\mathbb{E}_x[g(X_{T_{k+1}})\mathbf{1}_{T_{k+1}\le t}]\right| &\le \alpha^*\mathbb{E}_x[\mathbb{E}_{X_{T_k}}[\int_0^t K|g|(Y_s)\,\mathrm{d}s]\mathbf{1}_{T_k\le t}] \\
&= \alpha^*\mathbb{E}_x[\int_0^t Q_s^Y K|g|(X_{T_k})\,\mathrm{d}s\mathbf{1}_{T_k\le t}] \\
&\le \alpha^* t\mathbb{E}_x[\mathrm{supp}_{s\in[0,t]}\, Q_s^Y K|g|(X_{T_k})\mathbf{1}_{T_k\le t}].
\end{aligned}
$$

Since $\mathrm{supp}_{s\in[0,t]}\, Q_s^Y K|g| \in C_0(E)$, the result follows from the induction hypothesis.

In order to prove Theorem 2 (part 2), first remark that $x \mapsto Q_t f(x)$ is continuous for $f \in C_0(E)$ and any $t \ge 0$. This follows by the same arguments as in the proof of Theorem 2 (part 1) taking $f \in C_0(E)$. Indeed, using the same notation as in the proof of Lemma 5, we obtain that the function $H(.,u)$ belongs to $C_0(E)$ for any $u \le t$ by the assumption $K\,C_0(E) \subset C_0(E)$ and Lemma 6 (item 3.). The conclusion of Lemma 5 then follows by the same proof, using Lemma 6 instead of Lemma 4. Similarly, the proof of Theorem 2 (part 1) with the same substitution entails that $Q_t f \in C_b(E)$.

The strong continuity of Q_t follows by Proposition 3 and the first statement of Lemma 6.

It remains to prove that $x \mapsto Q_t f(x)$ vanishes at infinity. By the same decomposition of $Q_t f$ as in the proof of Theorem 2 (part 1), we obtain using in particular (2.2) that for any $j \ge 1$

$$
|Q_t f(x)| \le Q_t^Y |f|(x) + \sum_{k=1}^{j} \mathbb{E}_x[\mathrm{supp}_{s\in[0,t]}\, Q_s^Y |f|(X_{T_k})\mathbf{1}_{T_k\le t}] + \|f\|_\infty \mathbb{P}(N_t^* \ge j) \tag{2.3}
$$

where $N_t^* \sim \mathcal{P}(\alpha^* t)$. Let $\varepsilon > 0$. First, $Q_t^Y|f| \in C_0(E)$ by assumption, so that $Q_t^Y|f|(x) \le \varepsilon/3$ for x outside a compact set. Second, since $\lim_{j\to\infty}\mathbb{P}(N_t^* \ge j) = 0$, there exists $j_0 \ge 1$ such that $\|f\|_\infty \mathbb{P}(N_t^* \ge j) \le \varepsilon/3$. Third, Lemma 7 entails that for every $k \le j_0$ the function

$$
x \mapsto \mathbb{E}_x[\mathrm{supp}_{s\in[0,t]}\, Q_s^Y |f|(X_{T_k})\mathbf{1}_{T_k\le t}]
$$

vanishes at infinity because $\mathrm{supp}_{s\in[0,t]}\, Q_s^Y|f| \in C_0(E)$ by Lemma 6. It is therefore bounded by ϵ/j_0 for x outside a compact set. Combining these three results in (2.3) concludes the proof.

3. Proof of Theorem 3 about the infinitesimal generator

Let $f \in L_0^Y$, $x \in E$ and $h > 0$. We decompose $\frac{1}{h}(Q_h f(x) - f(x))$ as

$$
\begin{aligned}
\frac{1}{h}\left(Q_h f(x) - f(x)\right) &= \frac{1}{h}\left(\mathbb{E}_x^Y\left[f(Y_h)\mathrm{e}^{-\int_0^h \alpha(Y_u)\,\mathrm{d}u}\right] - f(x) + \mathbb{E}_x\left[f(X_h)\mathbf{1}_{N_h=1}\right] + \mathbb{E}_x\left[f(X_h)\mathbf{1}_{N_h\ge 2}\right]\right) \\
&= \mathbb{E}_x^Y\left[\frac{f(Y_h)-f(x)}{h}\right] + T(x),
\end{aligned}
$$

where

$$
\begin{aligned}
T(x) = -\frac{1}{h}\mathbb{E}_x^Y\left[f(Y_h)\int_0^h \alpha(Y_u)\,\mathrm{d}u\right] &+ \frac{1}{h}\mathbb{E}_x^Y\left[f(Y_h)\left(\mathrm{e}^{-\int_0^h \alpha(Y_u)\,\mathrm{d}u} - 1 + \int_0^h \alpha(Y_u)\,\mathrm{d}u\right)\right] \\
&+ \frac{1}{h}\mathbb{E}_x\left[f(X_h)\mathbf{1}_{N_h=1}\right] + \frac{1}{h}\mathbb{E}_x\left[f(X_h)\mathbf{1}_{N_h\ge 2}\right].
\end{aligned} \tag{3.1}
$$

To prove the theorem, we thus need to show that for any $f \in L_0^Y$

$$\sup_{x\in E} |T(x) + \alpha(x)f(x) - \alpha(x)Kf(x)| \xrightarrow[h\searrow 0]{} 0.$$

Following (3.1), we denote $T(x) = T_1(x) + T_2(x) + T_3(x) + T_4(x)$ and we shall prove that

$$\sup_{x\in E} |T_1(x) + \alpha(x)f(x)| \xrightarrow[h\searrow 0]{} 0, \tag{3.2}$$

$$\sup_{x\in E} |T_2(x)| \xrightarrow[h\searrow 0]{} 0, \tag{3.3}$$

$$\sup_{x\in E} |T_3(x) - \alpha(x)Kf(x)| \xrightarrow[h\searrow 0]{} 0, \tag{3.4}$$

$$\sup_{x\in E} |T_4(x)| \xrightarrow[h\searrow 0]{} 0. \tag{3.5}$$

For (3.2), we compute for $h > 0$ and $x \in E$,

$$\begin{aligned}
T_1(x) + \alpha(x)f(x) &= \alpha(x)f(x) - \frac{1}{h}\mathbb{E}_x^Y\left[f(Y_h)\int_0^h \alpha(Y_u)\,\mathrm{d}u\right]\\
&= \frac{1}{h}\int_0^h \mathbb{E}_x^Y\left[\alpha(x)f(x) - Q_{h-u}^Y f(Y_u)\alpha(Y_u)\right]\,\mathrm{d}u\\
&= \frac{1}{h}\int_0^h \mathbb{E}_x^Y\left[\alpha(x)f(x) - f(Y_u)\alpha(Y_u)\right]\,\mathrm{d}u + \frac{1}{h}\int_0^h \mathbb{E}_x^Y\left[f(Y_u)\alpha(Y_u) - Q_{h-u}^Y f(Y_u)\alpha(Y_u)\right]\,\mathrm{d}u\\
&= \int_0^1 \left(f\times\alpha - Q_{hv}^Y(f\times\alpha)\right)(x)\,\mathrm{d}v + \int_0^1 \mathbb{E}_x^Y\left[\alpha(Y_{hv})\left(f - Q_{h(1-v)}^Y f\right)(Y_{hv})\right]\,\mathrm{d}v.
\end{aligned}$$

So,

$$|T_1(x) + \alpha(x)f(x)| \le \int_0^1 \|Q_{hv}^Y(f\times\alpha) - f\times\alpha\|_\infty\,\mathrm{d}v + \alpha^*\int_0^1 \|Q_{h(1-v)}^Y f - f\|_\infty\,\mathrm{d}v,$$

that does not depend on $x \in E$ and converges to zero when $h \searrow 0$ by the dominated convergence theorem, the fact that $f \in L_0^Y$ and the assumption $\alpha \times f \in L_0^Y$. This proves (3.2).

Now for $f \in L_0^Y$ and $x \in E$

$$|T_2(x)| \le \frac{\|f\|_\infty}{2h}\mathbb{E}_x^Y\left[\left(\int_0^h \alpha(Y_u)\,\mathrm{d}u\right)^2\right] \le \frac{\|f\|_\infty(\alpha^*)^2}{2}\,h,$$

that does not depend on $x \in E$ and converges to zero when $h \searrow 0$, proving (3.3).

For (3.4), we have for any $f \in L_0^Y$,

$$\begin{aligned}
T_3(x) &= \frac{1}{h}\mathbb{E}_x\left[f(X_h)\mathbf{1}_{\tau_1\le h}\mathbf{1}_{\tau_2>h-\tau_1}\right]\\
&= \frac{1}{h}\mathbb{E}_x\left[f(Y_{h-\tau_1}^{(1)})\mathbf{1}_{\tau_1\le h}\mathbb{P}_x\left(\tau_2 > h-\tau_1\,\middle|\,\mathcal{F}_{\tau_1}, Y^{(1)}\right)\right]\\
&= \frac{1}{h}\mathbb{E}_x\left[f(Y_{h-\tau_1}^{(1)})\mathbf{1}_{\tau_1\le h}\mathrm{e}^{-\int_0^{h-\tau_1}\alpha\left(Y_u^{(1)}\right)\mathrm{d}u}\right]\\
&= \frac{1}{h}\mathbb{E}_x\left[\mathbf{1}_{\tau_1\le h}\mathbb{E}_{X_{\tau_1}}^Y\left[f(Y_{h-\tau_1})\mathrm{e}^{-\int_0^{h-\tau_1}\alpha(Y_u)\,\mathrm{d}u}\right]\right]\\
&= \frac{1}{h}\mathbb{E}_x\left[\mathbf{1}_{\tau_1\le h}\mathbb{E}_{X_{\tau_1}}^Y\left[f(Y_{h-\tau_1})\right]\right] + \frac{1}{h}\mathbb{E}_x\left[\mathbf{1}_{\tau_1\le h}\mathbb{E}_{X_{\tau_1}}^Y\left[f(Y_{h-\tau_1})\left(\mathrm{e}^{-\int_0^{h-\tau_1}\alpha(Y_u)\,\mathrm{d}u} - 1\right)\right]\right].
\end{aligned} \tag{3.6}$$

The second term above converges uniformly to 0 when $h \searrow 0$ because

$$\left|\frac{1}{h}\mathbb{E}_x\left[\mathbf{1}_{\tau_1\le h}\mathbb{E}^Y_{X_{\tau_1}}\left[f(Y_{h-\tau_1})\left(\mathrm{e}^{-\int_0^{h-\tau_1}\alpha(Y_u)\,\mathrm{d}u}-1\right)\right]\right]\right| \le \frac{h\alpha^*\|f\|_\infty}{h}\mathbb{P}_x(\tau_1\le h) \le (\alpha^*)^2\|f\|_\infty h.$$

Let us now consider the first term in (3.6) and prove that it converges uniformly to $\alpha(x)Kf(x)$.

$$\begin{aligned}
\frac{1}{h}\mathbb{E}_x\left[\mathbf{1}_{\tau_1\le h}\mathbb{E}^Y_{X_{\tau_1}}\left[f(Y_{h-\tau_1})\right]\right] &= \frac{1}{h}\mathbb{E}_x\left[\mathbf{1}_{\tau_1\le h}\mathbb{E}_x\left[\mathbb{E}^Y_{X_{\tau_1}}\left[f(Y_{h-\tau_1})\right]\Big|Y^{(0)},\tau_1\right]\right]\\
&= \frac{1}{h}\mathbb{E}_x\left[\mathbf{1}_{\tau_1\le h}\int_E \mathbb{E}^Y_z\left[f(Y_{h-\tau_1})\right]K\left(Y^{(0)}_{\tau_1},\mathrm{d}z\right)\right]\\
&= \frac{1}{h}\mathbb{E}_x\left[\mathbf{1}_{\tau_1\le h}\int_E Q^Y_{h-\tau_1}f(z)K\left(Y^{(0)}_{\tau_1},\mathrm{d}z\right)\right]\\
&= \frac{1}{h}\mathbb{E}_x\left[\int_0^h\int_E Q^Y_{h-s}f(z)K\left(Y^{(0)}_s,\mathrm{d}z\right)\alpha(Y^{(0)}_s)\mathrm{e}^{-\int_0^s\alpha(Y^{(0)}_u)\,\mathrm{d}u}\,\mathrm{d}s\right]\\
&= \mathbb{E}^Y_x\left[\int_0^1\int_E Q^Y_{h(1-v)}f(z)K\left(Y_{hv},\mathrm{d}z\right)\alpha(Y_{hv})\mathrm{e}^{-\int_0^{hv}\alpha(Y_u)\,\mathrm{d}u}\,\mathrm{d}v\right]\\
&= \mathbb{E}^Y_x\left[\int_0^1\int_E Q^Y_{h(1-v)}f(z)K\left(Y_{hv},\mathrm{d}z\right)\alpha(Y_{hv})\,\mathrm{d}v\right]\\
&\quad + \mathbb{E}^Y_x\left[\int_0^1\int_E Q^Y_{h(1-v)}f(z)K\left(Y_{hv},\mathrm{d}z\right)\alpha(Y_{hv})\left(\mathrm{e}^{-\int_0^{hv}\alpha(Y_u)\,\mathrm{d}u}-1\right)\,\mathrm{d}v\right].
\end{aligned}$$

On one hand,

$$\begin{aligned}
\left|\mathbb{E}^Y_x\left[\int_0^1\int_E Q^Y_{h(1-v)}f(z)K\left(Y_{hv},\mathrm{d}z\right)\alpha(Y_{hv})\left(\mathrm{e}^{-\int_0^{hv}\alpha(Y_u)\,\mathrm{d}u}-1\right)\,\mathrm{d}s\right]\right| &\le \alpha^*\|f\|_\infty\mathbb{E}^Y_x\left[\int_0^1\int_0^{hv}\alpha(Y_u)\,\mathrm{d}u\,\mathrm{d}v\right]\\
&\le (\alpha^*)^2\|f\|_\infty\, h,
\end{aligned}$$

which tends uniformly to 0 when $h \searrow 0$. And on the other hand,

$$\begin{aligned}
&\left|\mathbb{E}^Y_x\left[\int_0^1\int_E Q^Y_{h(1-v)}f(z)K\left(Y_{hv},\mathrm{d}z\right)\alpha(Y_{hv})\,\mathrm{d}v\right]-\alpha(x)Kf(x)\right|\\
&\le \int_0^1\left|\mathbb{E}^Y_x\left[\alpha(Y_{hv})\,KQ^Y_{h(1-v)}f(Y_{hv})-\alpha(Y_{hv})\,Kf(Y_{hv})\right]\right|\,\mathrm{d}v + \int_0^1\left|\mathbb{E}^Y_x\left[\alpha(Y_{hv})\,Kf(Y_{hv})-\alpha(x)Kf(x)\right]\right|\,\mathrm{d}v\\
&\le \alpha^*\int_0^1\|KQ^Y_{h(1-v)}f-Kf\|_\infty\,\mathrm{d}v + \int_0^1\left|Q^Y_{hv}(\alpha\times Kf)(x)-(\alpha\times Kf)(x)\right|\,\mathrm{d}v\\
&\le \alpha^*\int_0^1\|Q^Y_{h(1-v)}f-f\|_\infty\,\mathrm{d}v + \int_0^1\|Q^Y_{hv}(\alpha\times Kf)-(\alpha\times Kf)\|_\infty\,\mathrm{d}v,
\end{aligned}$$

converges to 0 when $h \searrow 0$ by the dominated convergence theorem and the fact that $f \in L^Y_0$ and $\alpha\times Kf \in L^Y_0$. The latter is implied by the fact that by assumption $g := Kf \in L^Y_0$, implying $\alpha\times g \in L^Y_0$. This proves (3.4).

To complete the proof, it remains to remark that (3.5) follows from the following, using (2.3),

$$|T_4(x)| \le \frac{\|f\|_\infty}{h}\mathbb{P}_x(N_h\ge 2) \le \frac{\|f\|_\infty\,(\alpha^*)^2}{2}\,h + \underset{h\searrow 0}{o}(h).$$

4. Topological results for systems of interacting particles in $\mathbb{R}^d$

We detail the topological properties of the state space E for systems of interacting particles in $W \subset \mathbb{R}^d$, introduced in Section 2.4. Remember that in this setting $E = \cup_{n=0}^{\infty} E_n$ where $E_n = \pi_n(W^n)$ with $\pi_n((x_1, \dots, x_n)) = \{x_1, \dots, x_n\}$, and we have equipped the space E with the distance d_1 defined for $x = \{x_1, \dots, x_{n(x)}\}$ and $y = \{y_1, \dots, y_{n(y)}\}$ in E such that $n(x) \le n(y)$ by

$$d_1(x, y) = \frac{1}{n(y)} \left(\min_{\sigma \in \mathcal{S}_{n(y)}} \sum_{i=1}^{n(x)} (\|x_i - y_{\sigma(i)}\| \wedge 1) + (n(y) - n(x)) \right),$$

with $d_1(x, \emptyset) = 1$ and where $\mathcal{S}_n$ denotes the set of permutations of $\{1, \dots, n\}$.

We verify in this section that if W is a closed subset of $\mathbb{R}^d$ (possibly $W = \mathbb{R}^d$), then (E, d_1) is a locally compact and complete set, strengthening results already obtained in [Schuhmacher and XiaSchuhmacher and Xia2008]. We also show that $n(.)$ and $\pi_n(.)$ are continuous under this topology, as claimed in Section 2.4. We continue with the proof of Proposition 4, which clarifies the meaning of converging sequences in (E, d_1), and of Proposition 5 that describes the compact sets of E_n and E, along with some useful corollaries. We finally show that the Hausdorff distance is not appropriate in our setting, not the least because it does not make $n(.)$ continuous.

In the following, we will often use in a equal way the spaces $\left(\mathbb{R}^{nd}, \|.\|\right)$ and $\left((\mathbb{R}^d)^n, \|.\|_n\right)$ where

$$\|x\|_n = \frac{1}{n} \sum_{i=1}^{n} \|x_i\|.$$

Indeed, introducing the natural bijection $\psi_n : z \in \mathbb{R}^{nd} \mapsto (z_1, \dots, z_n) \in (\mathbb{R}^d)^n$ we observe that for any $z \in \mathbb{R}^{nd}$, $\|z\|/n \le \|\psi_n(z)\|_n \le \|z\|/\sqrt{n}$ by the Cauchy-Schwarz inequality. The norms being equivalent, we henceforth abusively confuse z and $\psi_n(z)$. Similarly, any function from $\mathbb{R}^{nd}$ to $\mathbb{R}^d$ can be seen as a function from $(\mathbb{R}^d)^n$ to $\mathbb{R}^d$ and we will confuse the two points of view.

We start in the following lemmas with the continuity of $n(.)$ and $\pi_n(.)$. We will use the following straightforward property, for all $x, y \in E$,

$$d_1(x, y) \ge \frac{|n(y) - n(x)|}{n(x) \vee n(y)}. \tag{4.1}$$

Lemma 8. *The function $n(.) : (E, d_1) \to (\mathbb{N}, |.|)$ is continuous.*

Proof. Take $x \in E$ and a sequence $(x^{(p)})_{p \ge 0}$ such that $d_1(x^{(p)}, x) \to 0$ as $p \to \infty$. Assume that the sequence $(n(x^{(p)}))_{p \ge 0}$ is not bounded. We then may define a subsequence $(n(x^{(p')}))_{p' \ge 0}$ such that $n(x^{(p')}) \to \infty$, and by (4.1) we obtain

$$d_1(x, x^{(p')}) \ge \frac{|n(x) - n(x^{(p')})|}{n(x) \vee n(x^{(p')})} \underset{p' \to \infty}{\longrightarrow} 1,$$

which is a contradiction. The sequence $(n(x^{(p)}))_{p \ge 0}$ is therefore bounded by some $M > 0$, which gives again by (4.1)

$$|n(x^{(p)}) - n(x)| \le (M \vee n(x))\, d_1(x^{(p)}, x) \underset{p \to \infty}{\longrightarrow} 0,$$

that is

$$n(x^{(p)}) \underset{p \to \infty}{\longrightarrow} n(x).$$

Lemma 9. *The projection $\pi_n : (W^n, \|.\|_n) \to (E_n, d_1)$ is continuous.*

Proof. Let $x, y \in W^n$. Then

$$d_1(\pi_n(x), \pi_n(y)) = \frac{1}{n}\left(\min_{\sigma\in\mathcal{S}_n}\sum_{i=1}^n(\|x_i - y_{\sigma(i)}\| \wedge 1)\right) \le \frac{1}{n}\sum_{i=1}^n(\|x_i - y_i\| \wedge 1) \le \|x - y\|_n.$$

From Lemma 9 we deduce that (E, d_1) is a locally compact space.

Corollary 1. *Let W a closed subset of $\mathbb{R}^d$. Then (E, d_1) is a locally compact space.*

Proof. First recall that $d_1(x, \emptyset) = 1$ so $\{\emptyset\}$ is a compact neighborhood of $\emptyset$. Now take $x = \{x_1, \ldots, x_n\} \in E_n$ with $n \ge 1$. The space W^n is locally compact so there exists $K \subset W^n$ a compact neighborhood of $(x_1, \ldots, x_n)$. Now set $\tilde{K} = \pi_n(K)$. Then, $x \in \tilde{K}$ and $\tilde{K}$ is a compact set by Lemma 9. We show that there is an open set containing x which is included in $\tilde{K}$. By definition there exists $\varepsilon \in (0, \frac{1}{2})$ such that $B_{\|.\|_n}((x_1, \ldots, x_n), \varepsilon) \cap W^n \subset K$, where $B_{\|.\|_n}((x_1, \ldots, x_n), \varepsilon)$ is the open ball centred at $(x_1, \ldots, x_n)$ with radius ε for the norm $\|.\|_n$. If $z \in B_{d_1}(x, \varepsilon) \cap E_n$ there exists $\sigma \in \mathcal{S}_n$ such that

$$\frac{1}{n}\sum_{i=1}^n \|x_i - z_{\sigma(i)}\| < \varepsilon,$$

so $z = \pi_n((z_{\sigma(1)}, \ldots, z_{\sigma(n)}))$ and $(z_{\sigma(1)}, \ldots, z_{\sigma(n)}) \in B_{\|.\|_n}((x_1, \ldots, x_n), \varepsilon) \cap W^n$. To sum up,

$$B_{d_1}(x, \varepsilon) \cap E_n \subset \pi_n\left(B_{\|.\|_n}((x_1, \ldots, x_n), \varepsilon) \cap W^n\right) \subset \tilde{K},$$

so $\tilde{K}$ is a compact neighborhood of x in E_n and so in E.

A further consequence of Lemma 9 is the following result, that will turn to be useful when considering the Feller continuous property of a process on E.

Corollary 2. *If $f \in C_b(E)$ then for any $n \ge 1$, $f \circ \pi_n \in C_b(W^n)$.*

Proof. For any $n \ge 1$ and $f \in C_b(E)$, the function $f \circ \pi_n$ is well-defined on W^n, continuous as the composition of two continuous functions and bounded by $\|f\|_\infty$.

Let us now prove that (E, d_1) is a complete space.

Proposition 1. *Suppose that W is closed. Then (E, d_1) is a complete space and for any $n \ge 1$, (E_n, d_1) is also complete.*

Proof. Let $(x^{(p)})_{p\ge0}$ be a Cauchy sequence in (E, d_1). First, we show that the sequence $(n(x^{(p)}))_{p\ge0}$ is constant for p large enough. Fix $\varepsilon \in (0, 1)$. There exists $q \ge 0$ such that for any $p \ge q$, $d_1(x^{(p)}, x^{(q)}) < \varepsilon$, so by (4.1)

$$\left|n(x^{(p)}) - n(x^{(q)})\right| \le (n(x^{(p)}) \vee n(x^{(q)}))\,\varepsilon \le (n(x^{(p)}) + n(x^{(q)}))\,\varepsilon,$$

implying that $(1 - \varepsilon)\,n(x^{(p)}) \le (1 + \varepsilon)\,n(x^{(q)})$ and $n(x^{(p)}) \le n(x^{(q)})(1 + \varepsilon)/(1 - \varepsilon)$. This entails that the sequence $(n(x^{(p)}))_{p\ge0}$ is bounded by some $N_0 > 0$. Now take $\varepsilon \in (0, 1)$ and $p_1 \ge 0$ such that for any $p \ge p_1$, $d_1(x^{(p)}, x^{(p_1)}) < \varepsilon/N_0$. Write $n = n(x^{(p_1)})$ for short. Then by (4.1) one has for any $p \ge p_1$

$$\left|n(x^{(p)}) - n\right| \le (n(x^{(p)}) \vee n)\,d_1(x^{(p)}, x^{(p_1)}) \le N_0\,d_1(x^{(p)}, x^{(p_1)}) \le \varepsilon < 1,$$

which implies that $n(x^{(p)}) = n$ for all $p \geq p_1$.

Second, we may fix $p_2 \geq 0$ such that $d_1(x^{(p)}, x^{(q)}) \leq \varepsilon$ for any $p, q \geq p_2$. Finally let $p_0 = \max(p_1, p_2)$, so that for all $p, q \geq p_0$,

$$d_1(x^{(p)}, x^{(q)}) = \frac{1}{n} \min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^{n} \|x_i^{(p)} - x_{\sigma(i)}^{(q)}\| \leq \varepsilon.$$

In particular for $q = p_0$, this leads to $\min_{\sigma \in \mathcal{S}_n} \sum_{i=1}^n \|x_i^{(p_0)} - x_{\sigma(i)}^{(p)}\|/n \leq \varepsilon$ for any $p \geq p_0$. The minimum over σ is reached for some $\sigma_{p_0,p} \in \mathcal{S}_n$, so that we may define the sequence $(\hat{x}^{(p)})_{p \geq p_0}$ in W^n by $\hat{x}^{(p)} = (x^{(p)}_{\sigma_{p_0,p}(1)}, \dots, x^{(p)}_{\sigma_{p_0,p}(n)})$ satisfying $\|\hat{x}^{(p)} - x^{(p_0)}\|_n \leq \varepsilon$ for all $p \geq p_0$. Then for $p, q \geq p_0$, $\|\hat{x}^{(p)} - \hat{x}^{(q)}\|_n \leq 2\varepsilon$. This proves that the sequence $(\hat{x}^{(p)})_{p \geq p_0}$ is a Cauchy sequence in the finite dimensional vector space $((\mathbb{R}^d)^n, \|.\|_n)$, implying its convergence to some $\hat{x} \in W^n$ because W is a closed set. Finally for $p \geq p_0$

$$d_1(x^{(p)}, \pi_n(\hat{x})) = d_1(\pi_n(\hat{x}^{(p)}), \pi_n(\hat{x})) \leq \|\hat{x}^{(p)} - \hat{x}\|_n \leq 2\varepsilon,$$

which proves that $(x^{(p)})_{p \geq 0}$ converges to $\pi_n(\hat{x})$ in E, and so (E, d_1) is complete.

Finally for any $n \geq 1$, (E_n, d_1) is also complete as a closed subset of (E, d_1) by continuity of $n(.)$.

4.1. Proof of Proposition 4

Let $x \in E$ and set $n = n(x)$. By Lemma 8, if $x^{(p)}$ converges to x as $p \to \infty$, i.e. $d_1(x^{(p)}, x) \to 0$, then $n(x^{(p)})$ tends to n, which means that there exists $p_0 \geq 1$ such that $n(x^{(p)}) = n$ for all $p \geq p_0$. From the definition of d_1, for any $p \geq p_0$ there exists a permutation $\sigma_p \in \mathcal{S}_n$ satisfying

$$d_1(x^{(p)}, x) = \frac{1}{n} \sum_{i=1}^{n} (\|x_i - x_{\sigma_p(i)}^{(p)}\| \wedge 1).$$

Assume that there exists $i \in \{1, \dots, n\}$ such that $\limsup_{p \to \infty} \|x_i - x_{\sigma_p(i)}^{(p)}\| > 0$. We then may fix $\eta > 0$ and a subsequence $(\varphi(p))_{p \geq p_0}$, both depending on i, such that for every $p \geq p_0$, $\|x_i - x_{\sigma_{\varphi(p)}(i)}^{(\varphi(p))}\| \geq \eta$. This implies $d_1(x^{(\varphi(p))}, x) \geq (\eta \wedge 1)/n$ and $\limsup_{p \to \infty} d_1(x^{(p)}, x) > 0$ which is a contradiction. Finally, for every $i = 1, \dots, n$, $\limsup_{p \to \infty} \|x_i - x_{\sigma_p(i)}^{(p)}\| = 0$, proving the result.

4.2. Proof of Proposition 5 and corollaries

In order to prove this proposition, we first recall the following definitions and results (see e.g. [BourbakiBourbaki1966]):

- A finite subset L of a metric space (X, d) is called an ε−net, for $\varepsilon > 0$, if the following property is satisfied :

$$\forall x \in X, \ \exists l \in L, \ s.t. \ d(x, l) \leq \varepsilon.$$

- A metric space (X, d) is said to be totally bounded if it contains an ε−net for any $\varepsilon > 0$.

- Let (X, d) a metric space. Then (X, d) is compact if and only if (X, d) is totally bounded and complete.

To prove the first statement of the proposition, let A be a closed subset of (E_n, d_1). We start by assuming that we may fix $\varepsilon \in (0, 1/n)$ and $w \in W$ such that

$$\forall R > 0,\ \exists x = \{x_1, ..., x_n\} \in A,\ \max_{1\le k\le n} \{\|x_k - w\|\} > R + n\varepsilon, \tag{4.2}$$

and we show that A is not a compact set because it does not contain any ε−net. Take $L = \{l^{(1)}, \dots, l^{(N)}\}$ a finite subset of A and let us define

$$R_0 = \max_{1\le i\le N} \max_{1\le k\le n} \{\|l^{(i)}_k - w\|\}.$$

By (4.2) we may define $x \in A$ and $1 \le j \le n$ such that

$$\|x_j - w\| = \max_{1\le k\le n} \{\|x_k - w\|\} > R_0 + n\varepsilon.$$

This leads for all $\sigma \in \mathcal{S}_n$ and $1 \le i \le N$ to

$$\|x_j - l^{(i)}_{\sigma(j)}\| \ge \left| \|x_j - w\| - \|l^{(i)}_{\sigma(j)} - w\| \right| = \|x_j - w\| - \|l^{(i)}_{\sigma(j)} - w\| > n\varepsilon$$

and for any $1 \le i \le N$

$$\begin{aligned} d_1(x, l^{(i)}) &= \frac{1}{n} \min_{\sigma\in\mathcal{S}_n} \sum_{k=1}^{n} \left(\|x_k - l^{(i)}_{\sigma(k)}\| \wedge 1 \right) \\ &\ge \frac{1}{n} \min_{\sigma\in\mathcal{S}_n} \left(\|x_j - l^{(i)}_{\sigma(j)}\| \wedge 1 \right) \\ &> \frac{n\varepsilon}{n} = \varepsilon. \end{aligned}$$

Therefore L cannot be an ε−net and A cannot be a compact set.

Let us now prove the converse. Fix $w \in W$ and assume that there exists a positive R such that for all $x \in A$,

$$\max_{1\le k\le n} \{\|x_k - w\|\} \le R.$$

Under this assumption, A is a subset of

$$C := \{x \in E_n, \frac{1}{n} \sum_{k=1}^{n} \|x_k - w\| \le R\}.$$

Let us show that C is a compact set. To this end we define $\mathbf{w} = (w, \dots, w) \in W^n$ and write $\bar{B}_{\|.\|_n}(\mathbf{w}, R)$ for the closed ball of radius R and center $\mathbf{w}$ for the norm $\|.\|_n$ on the finite dimensional vector space $(\mathbb{R}^d)^n$. The closed set $\bar{B}_{\|.\|_n}(\mathbf{w}, R) \cap W^n$ is then a compact set of W^n and by continuity of the projection π_n, we get that $\pi_n \left(\bar{B}_{\|.\|_n}(\mathbf{w}, R) \cap W^n\right)$ is a compact set of E_n. Let us prove that $\pi_n \left(\bar{B}_{\|.\|_n}(\mathbf{w}, R) \cap W^n\right) = C$ to conclude the proof. First, if $x = \{x_1, \dots, x_n\} \in C$ then $\hat{x} = (x_1, \dots, x_n) \in \bar{B}_{\|.\|_n}(\mathbf{w}, R) \cap W^n$ and $\pi_n(\hat{x}) = x$. Second, if $x = \pi_n(\hat{x})$ with $\hat{x} = (x_1, \dots, x_n) \in \bar{B}_{\|.\|_n}(\mathbf{w}, R) \cap W^n$, then

$$\frac{1}{n} \sum_{k=1}^{n} \|x_k - w\| = \|\hat{x} - \mathbf{w}\|_n \le R$$

which proves the claim. The set C is then compact and so is A because it is a closed set.

Let us finally prove the second statement of Proposition 5 by contradiction. Let A be a compact subset of E and suppose that $\mathcal{P} = \{p \geq 0,\ A \cap E_p \neq \emptyset\}$ is infinite. Then we can construct a sequence $(y_p)_{p\in\mathcal{P}}$ with $y_p \in A \cap E_p$. But A is a compact set so there exists a subsequence $(y_{p'})_{p'\in\mathcal{P}}$ which converges to some $y \in E$ when $p' \to \infty$. But by Lemma 8, $n(y_{p'}) = p' \to n(y)$ as $p' \to \infty$ which is absurd, concluding the proof.

We end this section with two corollaries of Proposition 5.

Corollary 3. *If W is a compact set, then (E_n, d_1) is a compact set for any $n \geq 1$.*

Proof. W is a compact set of $\mathbb{R}^d$ so it is bounded, i.e. we may fix a non-negative R such that $\|w\| \leq R$ for any $w \in W$. Let $w \in W$ and $x \in E_n$. Then

$$\max_{1\leq k\leq n}\{\|x_k - w\|\} \leq \max_{1\leq k\leq n}\|x_k\| + \|w\| \leq 2R.$$

E_n is therefore a compact set by the first statement of Proposition 5.

Corollary 4. *If $f \in C_0(E)$ then for any $n \geq 1$, $f \circ \pi_n \in C_0(W^n)$.*

Proof. Take $f \in C_0(E)$ and $\varepsilon > 0$. There exists a compact set $B \subset E$ such that if $x \notin B$ then $|f(x)| < \varepsilon$. In this case $B_n := B \cap E_n$ is a compact set because E_n is closed so by Proposition 5 there exists $w \in W$ and $R \geq 0$ such that for any $x = \{x_1, \ldots, x_n\} \in B_n$, $\max_{1\leq k\leq n}\|x_k - w\| \leq R$. Then for any $z \notin \bar{B}_{\|.\|_n}(w, R/n)$ we get $|f \circ \pi_n(z)| < \varepsilon$.

4.3. The Hausdorff distance is not appropriate

For systems of particles in $\mathbb{R}^d$, we have equipped E with the distance d_1 defined in (2.7). A common alternative distance between random sets is the Hausdorff distance defined for $x = \{x_1, \ldots, x_{n(x)}\}$ and $y = \{y_1, \ldots, y_{n(y)}\}$ in E by

$$d_H(x, y) = \max\left\{\max_{1\leq i\leq n(x)}\min_{1\leq j\leq n(y)}\|x_i - y_j\|,\ \max_{1\leq j\leq n(y)}\min_{1\leq i\leq n(x)}\|x_i - y_j\|\right\}.$$

Yet we show in this section that this distance does not make the function $n(.)$ continuous, which has serious consequences on the structure of $C_b(E)$ with this topology. In particular, we show that a simple uniform death kernel is not even Feller continuous in this setting.

As a preliminary, for the Hausdorff distance to be a proper distance, we must focus on simple point configurations only. We therefore consider for any $n \geq 1$

$$\tilde{W}_n = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n, \quad i \neq j \implies x_i \neq x_j\right\},$$

and the state space is

$$\tilde{E} = \bigcup_{n\geq 0}\tilde{E}_n,$$

where $\tilde{E}_n = \tilde{\pi}_n(\tilde{W}_n)$ and $\tilde{\pi}_n$ is the same projection function as in Section 2.4 but defined on $\tilde{W}_n$. Then we have

Lemma 10. *The Hausdorff distance d_H is a proper distance function on $\tilde{E}$.*

Proof. Symmetry is obvious and triangle inequality is well known for d_H. We only prove the identity of indiscernibles. Let $x = \{x_1, \ldots, x_{n(x)}\}$ and $y = \{y_1, \ldots, y_{n(y)}\}$ in $\tilde{E}$ satisfying $d_H(x, y) = 0$. This implies

$$\min_{1\leq j\leq n(y)}\|x_i - y_j\| = 0$$

for any $i \in \{1, ..., n(x)\}$, leading for any $i \in \{1, ..., n(x)\}$ to the existence of $j \in \{1, ..., n(y)\}$ such that $x_i = y_j$. Since x and y are simple, we deduce that $n(y) \geq n(x)$. We obtain similarly $n(x) \geq n(y)$ and then $n(x) = n(y)$. We then may define a permutation $\sigma \in \mathcal{S}_n$ such that for all $i \in \{1, ..., n(x)\}$, $x_i = y_{\sigma(i)}$ which means that $x = y$ in $\tilde{E}$.

We now verify that $n(.)$ is not continuous for this topology.

Lemma 11. *Assume that* $\mathring{W} \neq \emptyset$. *Then the function* $n(.)$ *is not continuous on* $(\tilde{E}, d_H)$.

Proof. Assume without loss of generality that $0 \in \mathring{W}$. Let $k \geq 1$ and $y \in \mathbb{R}^d$ such that $\|y\| = 1/k$. Take k large enough so that $y \in W$. Then $|n(\{0, y\}) - n(\{0\})| = 1$ and $d_H(\{0, y\}, \{0\}) = 1/k \to 0$ as $k \to \infty$, proving the result.

This result reveals a singularity caused by the distance d_H. As a consequence, a simple uniform death kernel is not even Feller continuous, as proved in the following lemma.

Lemma 12. *Assume that* $\mathring{W} \neq \emptyset$ *and consider for* $f \in M_b(\tilde{E})$ *the kernel*

$$Kf(x) = \frac{1}{n(x)} \sum_{i=1}^{n(x)} f(x \setminus x_i).$$

Then $KC_b(\tilde{E})$ *is not included in* $C_b(\tilde{E})$, *i.e.* K *is not Feller continuous.*

Proof. Consider the function $f(x) = \max_{1 \leq i \leq n(x)} x_{i,1} \wedge 1$ where $x_{i,1}$ is the first coordinate of $x_i \in W$. This function is bounded and satisfies for any $x, y \in \tilde{E}$,

$$|f(x) - f(y)| \leq \left| \max_{1 \leq i \leq n(x)} x_{i,1} - \max_{1 \leq j \leq n(y)} y_{j,1} \right|, \tag{4.3}$$

for any $x, y \in \tilde{E}$. Let us show that the latter bound is lower than $d_H(x, y)$. Let $\mathcal{I}_0 = \text{argmax}_{1 \leq i \leq n(x)} x_{i,1}$ and $\mathcal{J}_0 = \text{argmax}_{1 \leq j \leq n(y)} y_{j,1}$. This follows from the fact that for any $i_0 \in \mathcal{I}_0$ and $j_0 \in \mathcal{J}_0$,

$$d_H(x, y) \geq \max_{1 \leq i \leq n(x)} \min_{1 \leq j \leq n(y)} \|x_i - y_j\| \geq \min_{1 \leq j \leq n(y)} \|x_{i_0} - y_j\| \geq \min_{1 \leq j \leq n(y)} |x_{i_0,1} - y_{j,1}| = |x_{i_0} - y_{j_0}|.$$

So by (4.3) $|f(x) - f(y)| \leq d_H(x, y)$, proving that $f \in C_b(\tilde{E})$.

Assume without loss of generality that $0 \in \mathring{W}$. Let $a \in W$, $a \neq 0$, and $a_k = (1/k, 0, \ldots, 0) \in \mathbb{R}^d$ with k large enough to ensure $a_k \in W$. Consider the sequence $x^{(k)} = \{0, a, a_k\}$ and let $x = \{0, a\}$ so that $d_H(x^{(k)}, x) = 1/k$ tends to 0 as $k \to \infty$. On the one hand,

$$Kf(x^{(k)}) = \frac{1}{3} \left[f(\{0, a_k\}) + f(\{a, a_k\}) + f(\{0, a\}) \right] = \frac{(1/k) + (1/k) \vee a_1 + a_1}{3} \underset{k \to \infty}{\longrightarrow} \frac{2a_1}{3},$$

and on the other hand,

$$Kf(x) = \frac{1}{2} \left(f(\{0\}) + f(\{a\}) \right) = \frac{a_1}{2}$$

whereby $Kf \notin C_b(\tilde{E})$.

5. Proof of Proposition 6

First we show that if $(Z_t^{|n})_{t\geq 0}$ is a Feller continuous process on W^n for every $n \geq 1$ then $(Y_t)_{t\geq 0}$ is a Feller continuous process on E. Indeed, let $x \in E$ and a sequence $(x^{(p)})_{p\geq 0}$ converging to x. By Proposition 4 we may fix $p_0 \geq 1$ such that $n(x^{(p)}) = n(x) := n$ for any $p \geq p_0$ and a sequence of permutations σ_p of $\{1, \ldots, n\}$ such that for any $1 \leq i \leq n$, $x^{(p)}_{\sigma_p(i)} \to x_i$ as $p \to \infty$. We then obtain for any $f \in C_b(E)$ and $p \geq p_0$, using the permutation equivariance property of $(Z_t^{|n})_{t\geq 0}$ (that allows us to arbitrarily choose the ordering of its initial value), the continuity of its transition kernel, and Corollary 2, that

$$\begin{aligned}
\mathbb{E}\left(f(Y_t)\,|\,Y_0 = x^{(p)}\right) &= \mathbb{E}\left(f(Y_t^{|n})\,|\,Y_0 = x^{(p)}\right)\\
&= \mathbb{E}\left(f\circ\pi_n(Z_t^{|n})\,|Z_0^{|n} = (x^{(p)}_{\sigma_p(1)}, \ldots, x^{(p)}_{\sigma_p(n)})\right)\\
&\underset{p\to\infty}{\longrightarrow} \mathbb{E}\left(f\circ\pi_n(Z_t^{|n})\,|Z_0^{|n} = (x_1, \ldots, x_n)\right)\\
&= \mathbb{E}\left(f(Y_t)\,|\,Y_0 = x\right).
\end{aligned}$$

Second, let us prove that if $(Z_t^{|n})_{t\geq 0}$ is a Feller process on W^n for every $n \geq 1$ then $(Y_t)_{t\geq 0}$ is a Feller process on E. Let $f \in C_0(E)$. We start by the strong continuity. Take $\varepsilon > 0$. By the second statement of Proposition 5 there exists $n_0 \geq 0$ such that $n(x) > n_0 \Rightarrow |f(x)| < \frac{\varepsilon}{4}$. So for any $x \in E$,

$$\begin{aligned}
\left|Q_t^Y f(x) - f(x)\right| &\leq \left|Q_t^Y f(x) - f(x)\right| \mathbf{1}_{n(x)\leq n_0} + \mathbb{E}_x[|f(Y_t)|]\mathbf{1}_{n(x)>n_0} + f(x)\mathbf{1}_{n(x)>n_0}\\
&\leq \sum_{n=0}^{n_0}\left|Q_t^{Y^{|n}} f(x) - f(x)\right| \mathbf{1}_{x\in E_n} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}\\
&\leq \sum_{n=1}^{n_0}\left|\mathbb{E}\left(f(\pi_n(Z_t^{|n}))\,|\,Z_0^{|n} = (x_1, \ldots, x_n)\right) - f(\pi_n((x_1, \ldots, x_n)))\right| \mathbf{1}_{x\in E_n} + \frac{\varepsilon}{2}\\
&\leq \sum_{n=1}^{n_0}\|Q_t^{Z^{|n}}(f\circ\pi_n) - f\circ\pi_n\|_\infty + \frac{\varepsilon}{2}.
\end{aligned}$$

By Corollary 4, for any $n = 1, \ldots, n_0$, there exists $t_n > 0$ such that

$$t \in (0, t_n) \implies \|Q_t^{Z^{|n}}(f\circ\pi_n) - f\circ\pi_n\|_\infty < \frac{\varepsilon}{2n_0}.$$

So for any $t \in (0, t(\varepsilon))$ where $t(\varepsilon) = \min_{1\leq n\leq n_0} t_n$, we get $\|Q_t^Y f - f\|_\infty < \varepsilon$, which proves the strong continuity of Q_t^Y at 0.

It remains to show that $Q_t^Y C_0(E) \subset C_0(E)$. Continuity follows from above. Take now $f \in C_0(E)$ and fix $\varepsilon > 0$ and $B \subset E$ a compact set such that $x \notin B \Rightarrow |f(x)| < \frac{\varepsilon}{2}$. By Proposition 5 there exists $n_0 \geq 0$ such that $x \in B \Rightarrow n(x) \leq n_0$. Also by Corollary 4 we can fix for any $n = 1, \ldots, n_0$ a compact set A_n of W^n such that $z \notin A_n \Rightarrow \left|Q_t^{Z^{|n}}(f\circ\pi_n)(z)\right| < \varepsilon/(2n_0)$. Then, $A = \bigcup_{n=1}^{n_0} \pi_n(A_n)$ is a compact set of E and for any $x \notin \{\emptyset\} \cup A \cup B$

$$\|Q_t^Y f(x)\| \leq \sum_{n=1}^{n_0}\|Q_t^{Z^{|n}}(f\circ\pi_n)((x_1, \ldots, x_n))\| + \frac{\varepsilon}{2} \leq \varepsilon.$$

6. Proof of Theorem 5

We recall and complete notations introduced in Section 4.1 of the main article regarding the coupling between X and η. The coupled process is $\check{C} = (X', \eta')$, where from Theorem 4, X' and η' have the same distributions as X and η. We denote by T_j and t_j the jump times of X and η. Similarly we denote by T'_j and t'_j the jump times of X' and η'. To prove Theorem 5, we start with the following lemma where $s_0 := \inf\{t \geq t_1,\ \eta_t = 0\}$ is the time of the first return of η in the state 0 and $S_\emptyset := \inf\{t \geq T_1,\ X_t = \emptyset\}$ is the time of the first return of $(X_t)_{t\geq 0}$ in the state $\emptyset$.

Lemma 13. *Suppose that* 0 *is an ergodic state for the simple process* η*, that is* $\mathbb{E}_0(s_0) < \infty$. *Then* $\lim_{t\to\infty} Q_t(\emptyset, A)$ *exists for all* $A \in \mathcal{E}$. *Suppose moreover that for all* $n \geq 0$, $\mathbb{E}_n(s_0) < \infty$. *Then,* $\lim_{t\to\infty} Q_t(x, A)$ *exists for all* $x \in E$, $A \in \mathcal{E}$, *and is independent of* x.

Proof. Let $\check{s}_0 := \inf\{t \geq t'_1,\ \check{C}_t \in E \times \{0\}\}$. Using the first statement of Theorem 4, we can prove that $\mathbb{P}_{(\emptyset,0)}(\check{s}_0 > t) = \mathbb{P}_0(s_0 > t)$. Similarly, by the second statement of this theorem, $\mathbb{P}_{(\emptyset,0)}(\check{S}_\emptyset > t) = \mathbb{P}_\emptyset(S_\emptyset > t)$ where $\check{S}_\emptyset := \inf\{t \geq T'_1,\ \check{C}_t \in \{\emptyset\} \times \mathbb{N}\}$. We thus have

$$\mathbb{P}_\emptyset(S_\emptyset > t) = \mathbb{P}_{(\emptyset,0)}(\check{S}_\emptyset > t) \leq \mathbb{P}_{(\emptyset,0)}(\check{s}_0 > t) = \mathbb{P}_0(s_0 > t),$$

where the inequality comes from Proposition 7.

By the assumptions of Lemma 13, this implies that $S_\emptyset < \infty$ $\mathbb{P}_\emptyset - a.s.$ and that

$$\mathbb{E}_\emptyset(S_\emptyset) = \int_0^\infty \mathbb{P}_\emptyset(S_\emptyset > t)\,\mathrm{d}t \leq \int_0^\infty \mathbb{P}_0(s_0 > t)\,\mathrm{d}t < \infty,$$

proving that $\emptyset$ is an ergodic state for the process $(X_t)_{t\geq 0}$. Note moreover that $S_\emptyset$ has a density with respect to the Lebesgue measure, that we denote by $\mu_\emptyset$. This comes from the fact that τ_j has a density for any j, so does T_j, whereby given a Lebesgue null set $I \in \mathcal{B}(\mathbb{R})$, $\mathbb{P}_\emptyset(S_\emptyset \in I) \leq \sum_{j=1}^\infty \mathbb{P}_\emptyset(T_j \in I) = 0$.

We have the following equation

$$\begin{aligned}
Q_t(\emptyset, A) &= \mathbb{P}_\emptyset(X_t \in A, S_\emptyset > t) + \int_0^t \mathbb{P}_\emptyset(X_t \in A, S_\emptyset \in \mathrm{d}s) \\
&= \mathbb{P}_\emptyset(X_t \in A, S_\emptyset > t) + \int_0^t \mathbb{P}_\emptyset(X_t \in A | S_\emptyset = s)\mu_\emptyset(s)\,\mathrm{d}s \\
&= \mathbb{P}_\emptyset(X_t \in A, S_\emptyset > t) + \int_0^t Q_{t-s}(\emptyset, A)\mu_\emptyset(s)\,\mathrm{d}s.
\end{aligned}$$

This is a renewal equation and we may apply the renewal theorem given in [FellerFeller1971, Chapter XI]. To this end, denote by $\mathcal{Z}(t) = Q_t(\emptyset, A)$, $\xi(t) = \mathbb{P}_\emptyset(X_t \in A, S_\emptyset > t)$ and $F\{I\} = \mathbb{P}_\emptyset(S_\emptyset \in I)$. Remark that $\mathcal{Z}$ is bounded, ξ is non-negative, bounded by 1 and directly Riemann integrable on $\mathbb{R}_+$ because it is dominated by the monotone integrable function $t \mapsto \mathbb{P}_\emptyset(S_\emptyset > t)$. Moreover, $0 < \mathbb{E}_\emptyset(S_\emptyset) < \infty$ and since $S_\emptyset$ has a density, F is not arithmetic. Then, by the renewal theorem, we obtain:

$$Q_t(\emptyset, A) = \mathcal{Z}(t) \underset{t\to\infty}{\longrightarrow} \frac{1}{\mathbb{E}_\emptyset(S_\emptyset)} \int_0^\infty \xi(u)\,\mathrm{d}u = \frac{1}{\mathbb{E}_\emptyset(S_\emptyset)} \int_0^\infty \mathbb{P}_\emptyset(X_u \in A, S_\emptyset > u)\,\mathrm{d}u \tag{6.1}$$

which proves the first statement of Lemma 13.

Let now turn to the second part of Lemma 13. Let $x \in E_n$. By the arguments as in the beginning of the proof, we get that $S_\emptyset < \infty,\ \ \mathbb{P}_x - a.s.$ and that $\mathbb{E}_x(S_\emptyset) \leq \mathbb{E}_n(s_0) < \infty$. We have

$$\begin{aligned}
Q_t(x,A) &= \mathbb{P}_x(X_t \in A) \\
&= \mathbb{P}_x(X_t \in A, S_\emptyset > t) + \int_0^t \mathbb{P}_x(X_t \in A | S_\emptyset = s)\mu_\emptyset(s)\ \mathrm{d}s \\
&= \mathbb{P}_x(X_t \in A, S_\emptyset > t) + \int_0^t \mathbb{P}_\emptyset(X_{t-s} \in A)\mu_\emptyset(s)\ \mathrm{d}s \\
&= \mathbb{P}_x(X_t \in A, S_\emptyset > t) + \int_0^t Q_{t-s}(\emptyset, A)\mu_\emptyset(s)\ \mathrm{d}s.
\end{aligned}$$

The first term tends to 0 as $t \to \infty$ because it is dominated by $\mathbb{P}_x(S_\emptyset > t)$ and we know that $\mathbb{P}_x(S_\emptyset < \infty) = 1$. For the second term, for all $s \geq 0$, we have by (6.1)

$$Q_{t-s}(\emptyset, A)\mathbf{1}_{[0,t]}(s) \underset{t\to\infty}{\longrightarrow} \frac{1}{\mathbb{E}_\emptyset(S_\emptyset)} \int_0^\infty \mathbb{P}_\emptyset(X_u \in A, S_\emptyset > u)\ \mathrm{d}u.$$

Moreover $|Q_{t-s}(\emptyset, A)\mathbf{1}_{[0,t]}(s)\mu_\emptyset(s)| \leq \mu_\emptyset(s)$ which is integrable. So by the dominated convergence theorem,

$$Q_t(x,A) \underset{t\to\infty}{\longrightarrow} \frac{1}{\mathbb{E}_\emptyset(S_\emptyset)} \int_0^\infty \mathbb{P}_\emptyset(X_u \in A, S_\emptyset > u)\ \mathrm{d}u$$

which is independent of x.

We are now in position to prove Theorem 5. The conditions (4.6) or (4.7) of [Karlin and McGregorKarlin and McGregor1957] imply the assumptions made in Lemma 13. We then deduce that $\mu(A) := \lim_{t\to\infty} Q_t(x,A)$ exists for all $x \in E$ and $A \in \mathcal{E}$, and is independent of x. It is a probability measure because for any $t \geq 0$ and $x \in E$, $Q_t(x,.)$ is a probability measure.

Let us prove that μ is an invariant measure. The previous convergence reads

$$\int_E f(y)Q_s(x,dy) \underset{s\to\infty}{\longrightarrow} \int_E f(y)\mu(dy). \tag{6.2}$$

where $f = \mathbf{1}_A$ with $A \in \mathcal{E}$. It is not difficult to extend it to any step function and by limiting arguments to any $f \in M_b^+(E)$. By the Markov property, for all $t, s \geq, x \in E$ and $A \in \mathcal{E}$,

$$Q_{t+s}(x,A) = \int_E Q_t(y,A)\,Q_s(x,dy).$$

Letting s tend to ∞, we obtain that the left hand side converges to $\mu(A)$, while for the right hand side, we may apply (6.2) to $f = Q_t(.,A) \in M_b^+(E)$ to finally obtain

$$\mu(A) = \int_E Q_t(y,A)\mu(dy).$$

Finally, if ν is a probability measure on E, such that for any $A \in \mathcal{E}$

$$\nu(A) = \int_E Q_t(y,A)\nu(dy),$$

then as $Q_t(x, A) \leq 1$, taking $t \to \infty$, we get by the dominated convergence theorem

$$\nu(A) = \int_E \mu(A)\nu(dy) = \mu(A).$$

Hence μ is the unique invariant probability measure.

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