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SUPPLEMENTARY MATERIAL FOR "AN EXTREME WORST-CASE RISK MEASURE BY EXPECTILE"

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The supplementary material contains an auxiliary lemma and the proofs of all theoretical results in the main paper. We also provide some simulations to support our asymptotic results.

Appendix A. An auxiliary lemma

Lemma A.1. *Let f be an increasing integrable function on $(0,1)$, $\beta>1$ and*

$$s\left(t\right)=\int_{0}^{t}f\left(u\right)du+\beta\int_{t}^{1}f\left(u\right)du+\left[\left(\beta-1\right)t-\beta\right]f\left(t\right),$$

then $s\left(t\right)$ is decreasing on $(0,1)$.

Proof. We directly prove it by definition. Let $0<t_1\leq t_2<1$, then

$$\begin{aligned}
&s\left(t_2\right)-s\left(t_1\right)\\
=&\left(1-\beta\right)\int_{t_1}^{t_2}f\left(u\right)du+\left[\left(\beta-1\right)t_2-\beta\right]f\left(t_2\right)-\left[\left(\beta-1\right)t_1-\beta\right]f\left(t_2\right)\\
=&-\left(\beta-1\right)\int_{t_1}^{t_2}f\left(u\right)du+\left(\beta-1\right)\left[t_2f\left(t_2\right)-t_1f\left(t_1\right)\right]-\beta\left[f\left(t_2\right)-f\left(t_1\right)\right]\\
=&\left(\beta-1\right)t_2\left[f\left(t_2\right)-f\left(t_1\right)\right]+\left(\beta-1\right)\left(t_2-t_1\right)f\left(t_1\right)-\left(\beta-1\right)\int_{t_1}^{t_2}f\left(u\right)du\\
&-\beta\left[f\left(t_2\right)-f\left(t_1\right)\right]\\
=&\left[\left(\beta-1\right)t_2-\beta\right]\left[f\left(t_2\right)-f\left(t_1\right)\right]+\left(\beta-1\right)\left[\left(t_2-t_1\right)f\left(t_1\right)-\int_{t_1}^{t_2}f\left(u\right)du\right]\leq 0.
\end{aligned}$$

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□

Appendix B. Proof of main results

B.1. Proof of results in Section 3.1

B.1.1. **Proof of Theorem 3.1** For $\gamma \in (1/\beta, 1)$, let

$$l_p(\gamma) := \|h_\gamma\|_q, \quad k_G(\gamma) := \int_0^1 G^{-1}(u) h_\gamma(u) du,$$

then

$$z_{p,G,\varepsilon}(\gamma) = \varepsilon l_p(\gamma) + k_G(\gamma),$$

we only need to analyze l_p and k_G separately.

By the definition of h_γ, we make the following direct calculation to obtain the explicit expression:

$$h_\gamma(u) = H'_{\gamma-}(u) = \begin{cases} \gamma, & \text{if } 0 < u \le \tau \\ \frac{1-\gamma}{1-\tau} + \gamma, & \text{if } \tau < u < 1 \end{cases} = \begin{cases} \gamma, & \text{if } 0 < u \le \tau \\ \beta\gamma, & \text{if } \tau < u < 1 \end{cases}. \tag{B.1}$$

For the first term $l_p(\gamma)$, when $p > 1$, i.e., $1 < q < \infty$, we have

$$l_p(\gamma) = \|h_\gamma\|_q = \left[\tau\gamma^q + \beta^q\gamma^q(1-\tau)\right]^{1/q} = \gamma\left[\tau + \beta^q(1-\tau)\right]^{1/q}.$$

Obviously $l_p(\gamma)$ is a smooth function. Differentiate to $l_p(\gamma)$, we obtain its first derivative:

$$l'_p(\gamma) = \left[\tau + \beta^q(1-\tau)\right]^{1/q} + q^{-1}\left[\tau + \beta^q(1-\tau)\right]^{1/q-1} \frac{1-\beta^q}{\gamma(\beta-1)}. \tag{B.2}$$

Thus

$$l''_p(\gamma) = \frac{(1-\beta^q)^2}{q(\beta-1)^2\gamma^3}\left[\tau + \beta^q(1-\tau)\right]^{1/q-2}\left(q^{-1}-1\right) < 0.$$

This implies that $l_p(\gamma)$ is a strictly concave function. In addition, we have

$$l'_p(1) = 1 + \frac{1-\beta^q}{q(\beta-1)} = \frac{q(\beta-1) - (\beta^q-1)}{q(\beta-1)} < 0, \tag{B.3}$$

and

$$l'_p(1/\beta) = \frac{\beta^{2-q}}{q(\beta-1)}\left[(q-1)\beta^q - q\beta^{q-1} + 1\right] > 0. \tag{B.4}$$

Now we turn to the function $k_G(\gamma)$. Continuity and Concavity of $k_G(\gamma)$ will be checked subsequently. Indeed, according to (B.1), we have

$$k_G(\gamma) = \int_0^1 G^{-1}(u)h_\gamma(u)du = \gamma \int_0^\tau G^{-1}(u)\,du + \beta\gamma \int_\tau^1 G^{-1}(u)\,du. \tag{B.5}$$

By the integrability of G^{-1} and the absolute continuity of integration on G^{-1}, we obtain $k_G(\gamma)$ is continuous. We emphasize here that no extra assumptions on the continuity of G^{-1} are put on this theorem, hence including distributions whose cdf contains flat parts, with connection to the jump discontinuity, where $k_G(\gamma)$ is not differentiable. However, it is easy to check that $k_G(\gamma)$ always has left and right derivatives on $(1/\beta, 1)$. Specifically,

$$\begin{aligned} k'_{G+}(\gamma) &= \int_0^\tau G^{-1}(u)\,du + \gamma G^{-1}(\tau^+)\cdot\left[\gamma^2(\beta-1)\right]^{-1} \\ &\quad + \beta\int_\tau^1 G^{-1}(u)\,du - \beta\gamma G^{-1}(\tau^+)\cdot\left[\gamma^2(\beta-1)\right]^{-1} \\ &= \int_0^\tau G^{-1}(u)\,du + \beta\int_\tau^1 G^{-1}(u)\,du - \gamma^{-1}G^{-1}(\tau^+). \end{aligned} \tag{B.6}$$

Similarly,

$$k'_{G-}(\gamma) = \int_0^\tau G^{-1}(u)\,du + \beta\int_\tau^1 G^{-1}(u)\,du - \gamma^{-1}G^{-1}(\tau). \tag{B.7}$$

Here $G^{-1}(\tau^-)$ is replaced by $G^{-1}(\tau)$ due to left continuity of G^{-1}. Now it is sufficient to show that both $k'_{G+}(\gamma)$ and $k'_{G-}(\gamma)$ are decreasing. Since the proof simply rests on the monotonicity of $G^{-1}(\tau^+)$ or $G^{-1}(\tau^-)$, we assume that G^{-1} is continuous for simplicity. Hence, $k'_{G+}(\gamma) = k'_{G-}(\gamma) = k'_G(\gamma)$. Noting that τ is a strictly increasing function of γ, by Lemma A.1 we obtain that

$$k'_G(\gamma(\tau)) = \int_0^\tau G^{-1}(u)\,du + \beta\int_\tau^1 G^{-1}(u)\,du + \left[(\beta-1)\tau - \beta\right]G^{-1}(\tau)$$

is a decreasing function of τ.

Therefore continuity and strict concavity of $z_{p,G,\varepsilon}(\gamma)$ is attained by the above arguments. In addition, $z_{p,G,\varepsilon}(\gamma)$ has left and right derivatives and together with Eqs. (B.3), (B.4), (B.6), (B.7), denoted by $z'_{p,G,\varepsilon+}(\gamma)$ and $z'_{p,G,\varepsilon-}(\gamma)$ respectively. Then we have

$$z'_{p,G,\varepsilon+}(1/\beta) = l'_p(1/\beta) + \beta\left[\int_0^1 G^{-1}(u)\,du - G^{-1}(0^+)\right] > 0,$$

and

$$z'_{p,G,\varepsilon-}\left(1\right)=l'_{p}\left(1\right)+\int_{0}^{1}G^{-1}\left(u\right)du-G^{-1}\left(1\right)<0.$$

Thus the maximum point must lie in the open interval $(1/\beta,1)$, and

$$\gamma^{*}=\underset{\gamma\in\left[\frac{1}{\beta},1\right]}{\operatorname{argmax}}\ z_{p,G,\varepsilon}\left(\gamma\right)=\sup_{\gamma\in\left(\frac{1}{\beta},1\right)}\left\{z'_{p,G,\varepsilon-}\left(\gamma\right)\geq0\right\}.$$

Regarding the second conclusion, on the one hand, it is easy to verify that $F_{p,G,\varepsilon,\gamma^*}\in\mathcal{W}_{p,G,\varepsilon}$. Therefore, by definition, we have $e_\alpha(F_{p,G,\varepsilon,\gamma^*})\leq\mathrm{WCE}_\alpha^{\mathcal{W}_{p,G,\varepsilon}}$. On the other hand, by Proposition 1, we have the opposite

$$e_\alpha(F_{p,G,\varepsilon,\gamma^*})=\max_{\gamma\in[\frac{1}{\beta},1]}\int_0^1F^{-1}_{p,G,\varepsilon,\gamma^*}(u)h_\gamma(u)du\geq\int_0^1F^{-1}_{p,G,\varepsilon,\gamma^*}(u)h_{\gamma^*}(u)du.$$

While the right side of the above inequality is indeed $z_{p,G,\varepsilon}(\gamma^*)$, form which we complete the proof. □

B.1.2. **Proof of Theorem 3.2** When $p=1$, $q=\infty$,

$$l_1(\gamma)=\|h_\gamma\|_\infty=\begin{cases}\beta\gamma, & \text{if } \frac{1}{\beta}\leq\gamma<1\\ 1, & \text{if } \gamma=1\end{cases}.$$

Having obtained the continuity and concavity of $k_G(\gamma)$ on $(1/\beta,1)$, the same properties hold for $z_{1,G,\varepsilon}(\gamma)$ quite obviously, because $l_1(\gamma)$ is linear on $(1/\beta,1)$. As pointed out above Theorem 3.2, we simply need to consider $\sup_{\gamma\in\left(\frac{1}{\beta},1\right)}z_{1,G,\varepsilon}\left(\gamma\right)$, thus the discontinuity of $z_{1,G,\varepsilon}(\gamma)$ on $\gamma=1$ is negligible. Moreover, we have

$$z'_{1,G,\varepsilon+}\left(1/\beta\right)=\varepsilon\beta+\beta\left[\int_0^1G^{-1}\left(u\right)du-G^{-1}\left(0^+\right)\right]>0,$$

and

$$z'_{1,G,\varepsilon-}\left(1^-\right)=\varepsilon\beta+\int_0^1G^{-1}\left(u\right)du-G^{-1}\left(1\right)=\varepsilon\beta+\mu_G-\operatorname{ess\,sup}_G.$$

Consequently, if $\operatorname{ess\,sup}_G\leq\mu_G+\varepsilon\beta$, then $z'_{1,G,\varepsilon-}\left(\gamma\right)$ is always positive on $(1/\beta,1)$, and

$$\mathrm{WCE}_\alpha^{\mathcal{W}_{1,G,\varepsilon}}=\sup_{\gamma\in\left(\frac{1}{\beta},1\right)}z_{1,G,\varepsilon}\left(\gamma\right)=\lim_{\gamma\to1^-}z_{1,G,\varepsilon}\left(\gamma\right)=\varepsilon\beta+\mu_G.$$

It follows that for any sequence $\tau_n \in (0,1)$ such that $\tau_n \to 1$, or equivalently, $\gamma(\tau_n) \to 1$,

$$\lim_{n\to\infty} z_{1,G,\varepsilon}(\gamma(\tau_n)) = \mathrm{WCE}_{\alpha}^{\mathcal{W}_{1,G,\varepsilon}},$$

which implies that for any given $\delta > 0$, when n becomes large enough, we have

$$z_{1,G,\varepsilon}(\gamma(\tau_n)) > \mathrm{WCE}_{\alpha}^{\mathcal{W}_{1,G,\varepsilon}} - \delta.$$

Now, Proposition 2 entails that

$$e_\alpha(F_{n,G,\varepsilon}) = e_\alpha(F_{G,\varepsilon,\gamma(\tau_n)}) \geq \int_0^1 F^{-1}_{G,\varepsilon,\gamma(\tau_n))}(u) h_{\gamma(\tau_n)}(u) du = z_{1,G,\varepsilon}(\gamma(\tau_n)).$$

Then the end of the proof of assertion (i) is an opposite inequality $e_\alpha(F_{n,G,\varepsilon}) \leq \mathrm{WCE}_{\alpha}^{\mathcal{W}_{1,G,\varepsilon}}$ by definition. If ess $\sup_G > \mu_G + \varepsilon\beta$, then $z'_{1,G,\varepsilon-}(1^-) < 0$, which indicates that $\gamma^* \in (1/\beta, 1)$. The subsequent proof is essentially the same as that of Theorem 3.1 and is omitted here.

□

B.1.3. **Proof of Proposition 4** The conclusion for the situation when G is degenerate is obvious due to assertion (i) in Theorem 3.1. Now if G is non-degenerate and $\alpha > 1/2$, μ_G is strictly less than $e_\alpha(G)$ according to equation (12) in [1], which yields our conclusion under ess $\sup_G \leq \mu_G + \beta\varepsilon$. While under ess $\sup_G > \mu_G + \beta\varepsilon$, we know form assertion (ii) in Theorem 3.2 that there exists $\gamma^* < 1$ such that

$$\mathrm{WCE}_{\alpha}^{\mathcal{W}_{1,G,\varepsilon}} = z_{1,G,\varepsilon}(\gamma^*) = \int_0^1 G^{-1}(u) h_{\gamma^*}(u) du + \beta\varepsilon\gamma^* < e_\alpha(G) + \beta\varepsilon,$$

where the last inequality uses Proposition 1. Hence we complete the proof. □

B.1.4. **Proof of Theorem 3.3** By Theorem 3.2, we know there exists a unique maximum value point $\gamma^* \in (1/\beta, 1)$. Denote by γ^* the corresponding maximizer, i.e. $\tau^* = (\beta - 1/\gamma^*)/(\beta - 1)$ or see Eq.(3.1) Under the assumption that G^{-1} is continuous, $z_{1,G,\varepsilon}$ is differentiable. Viewing τ as pivot, we rewrite $z'_{1,G,\varepsilon}(\gamma(\tau))$ as

$$z'_{1,G,\varepsilon}(\gamma(\tau)) = \varepsilon\beta + \mu_G + (\beta - 1)\left[\int_\tau^1 G^{-1}(u)\,du - G^{-1}(\tau)(1-\tau)\right] - G^{-1}(\tau). \tag{B.8}$$

By substituting $\tau_0 = G(\varepsilon\beta/2)$ into Eq.(B.8), we obtain

$$z'_{1,G,\varepsilon}(\gamma(\tau_0)) = \varepsilon\beta/2 + \mu_G + (\beta-1)\left[\int_{\tau_0}^1 G^{-1}(u)\,du - G^{-1}(\tau_0)(1-\tau_0)\right] > 0,$$

Taking β large enough, then we have $\tau^* \in (G(\varepsilon\beta/2), 1)$. Thus

$$\begin{aligned}
\mathrm{WCE}_\alpha^{\mathcal{W}_{1,G,\varepsilon}} &= z_{1,G,\varepsilon}(\gamma^*) = G^{-1}(\tau^*) \\
&= \mu_G + \varepsilon\beta + (\beta-1)\Big[\int_{\tau^*}^1 G^{-1}(u)\,du - G^{-1}(\tau^*)(1-\tau^*)\Big] \\
&= \mu_G + \varepsilon\beta + \beta\cdot O\Big(\int_{\tau^*}^1 G^{-1}(u)\,du\Big) \\
&= \mu_G + \varepsilon\beta + \beta\cdot O\Big(\int_{G(\varepsilon\beta/2)}^1 G^{-1}(u)\,du\Big) \\
&= \mu_G + \varepsilon\beta + O\Big(\beta\mathbb{E}^G\Big[Y\mathbb{1}_{\{Y>\varepsilon\beta/2\}}\Big]\Big),
\end{aligned}$$

where the second and third equalities are simple transformations of $z'_{1,G,\varepsilon}(\gamma(\tau^*)) = 0$. Note that $\beta\mathbb{E}^G\Big[Y\mathbb{1}_{\{Y>\varepsilon\beta/2\}}\Big] = o(1)$ due to $\mathbb{E}^G[(Y^+)^2] < \infty$, which completes the proof. □

B.1.5. **Proof of Example 3.1** It is easy to verify that $G^{-1}(\tau) = 1/\sqrt{1-\tau}$ and $\int_\tau^1 G^{-1}(u)du = 2\sqrt{1-\tau}$, so

$$z'_{1,G,\varepsilon}(\gamma(\tau)) = \varepsilon\beta + 2 + (\beta-1)(1-\tau)^{1/2} - (1-\tau)^{-1/2}.$$

Setting it to zero, we obtain

$$\mathrm{WCE}_\alpha^{\mathcal{W}_{1,G,\varepsilon}} = G^{-1}(\tau^*) = \frac{2(\beta-1)}{-(2+\varepsilon\beta) + \sqrt{(\varepsilon\beta+2)^2 + 4(\beta-1)}}.$$

Let $a = 2(\beta-1)$, $b = \varepsilon\beta + 2$, we have

$$\begin{aligned}
\mathrm{WCE}_\alpha^{\mathcal{W}_{1,G,\varepsilon}} &= \frac{a}{-b + \sqrt{b^2+2a}} \\
&= \frac{1}{2}\left(\sqrt{b^2+2a} + b\right) \\
&= \frac{b}{2}\Big(1 + \sqrt{1 + \frac{2a}{b^2}}\Big) \\
&= \Big(\frac{\varepsilon\beta+2}{2}\Big)\Big(2 + \frac{2(\beta-1)}{(\varepsilon\beta+2)^2} + o(1/\beta)\Big) \\
&= \varepsilon\beta + 2 + 1/\varepsilon + \mathrm{o}(1).
\end{aligned}$$

□

Before giving the concrete proof of Theorems 3.4-3.7, we provide a sketch of our strategy. As indicated in the proof of Example 1 (Eq.(B.24) below), we know that the maximizer $\gamma^* \approx 1/p$ when G is a degenerate distribution function. From this point of intuition, we wish to maintain this asymptotic behavior of the maximizer by imposing some additional conditions on the reference distribution G. Technically, we describe in as much detail as possible the rate at which γ^* tends to $1/p$ through tedious but elementary asymptotic expansion. Ultimately, this asymptotic maximizer is carried into the optimized function to acquire the result after simplification. In this process, we constantly and flexibly use the Taylor expansion and test for the range of maximizer by Eq. (3.1). The detailed proofs are as follows.

B.1.6. **Proof of Theorem 3.4** For convenience, denote $\Lambda(\tau) := z'_{1,G,\varepsilon}(\gamma(\tau))$. The maximizer τ^* is the unique zero point of Λ on $(0,1)$. Next, we rewrite $\Lambda(\tau)$ by the following steps.

$$
\begin{aligned}
\int_\tau^1 G^{-1}(u)du &= \int_{G^{-1}(\tau)}^\infty x dG(x) = \mathbb{E}^G\left[Y\mathbb{1}_{\{Y>G^{-1}(\tau)\}}\right] \\
&= \int_0^\infty \mathbb{P}\left(Y\mathbb{1}_{\{Y>G^{-1}(\tau)\}} > t\right) dt \\
&= \int_0^\infty \mathbb{P}\left(Y > t, Y > G^{-1}(\tau)\right) dt \\
&= G^{-1}(\tau)\overline{G}(G^{-1}(\tau)) + \int_{G^{-1}(\tau)}^\infty \overline{G}(t)dt \\
&= G^{-1}(\tau)(1-\tau) + \int_{G^{-1}(\tau)}^\infty \overline{G}(t)dt,
\end{aligned}
$$

which implies that

$$
\Lambda(\tau) = \varepsilon\beta + \mu_G + (\beta - 1)\int_{G^{-1}(\tau)}^\infty \overline{G}(t)dt - G^{-1}(\tau). \tag{B.9}
$$

Before evaluating τ^*, we need the following two limits:

$$\begin{aligned}
\lim_{x\to\infty}\frac{\int_x^\infty \overline{G}(x)dx}{\overline{G}(x)x} &= \lim_{x\to\infty}\frac{-\overline{G}(x)}{\overline{G}'(x)x+\overline{G}(x)}\\
&= \lim_{x\to\infty}\frac{-x^{-\theta}L(x)}{-\theta x^{-\theta}L(x)+x^{1-\theta}L'(x)+x^{-\theta}L(x)}\\
&= \frac{1}{\theta-1},
\end{aligned} \tag{B.10}$$

and

$$\begin{aligned}
&\lim_{x\to\infty} x\left(\int_x^\infty \overline{G}(t)dt - \frac{x\overline{G}(x)}{\theta-1}\right)\\
=&\lim_{x\to\infty}\frac{-\overline{G}(x)-\frac{\overline{G}(x)}{\theta-1}-\frac{\overline{G}'(x)x}{\theta-1}}{-\frac{1}{x^2}}\\
=&\lim_{x\to\infty} x^2\left(\frac{\theta}{\theta-1}x^{-\theta}L(x)+\frac{x}{\theta-1}\left(-\theta x^{-\theta-1}L(x)+x^{-\theta}L'(x)\right)\right)\\
=&\frac{1}{\theta-1}\lim_{x\to\infty}x^{3-\theta}L'(x)=0.
\end{aligned} \tag{B.11}$$

Actually, Eq. (B.11) is stronger than Eq. (B.10), and both of them are special cases of Karamata's Theorem (Proposition 1.5.10 in [2]). We use the different two equations flexibly for different purposes.

In the following, we obtain that

$$\frac{G^{-1}(\tau^*)}{\beta} = \varepsilon + c\beta^{1-\theta} + \mathrm{o}(\beta^{1-\theta}),\ \text{where } c = \begin{cases}\mu_G+\zeta/\varepsilon, & \theta=2\\ \varepsilon^{1-\theta}\zeta/(\theta-1), & \theta<2\end{cases}, \tag{B.12}$$

and only give the proof when $\theta < 2$. In fact, let c_1 be a real number such that $c_1 < c$, and $\tau_1 = G(\varepsilon\beta + c_1\beta^{2-\theta})$. Substituting Eq. τ_1 into Eq. (B.9), together with Eq. (B.10), we obtain

$$
\begin{aligned}
\Lambda(\tau_1) &= \varepsilon\beta + \mu_G + \beta \int_{G^{-1}(\tau_1)}^{\infty} \overline{G}(t)dt - G^{-1}(\tau_1) + o(1) \\
&= \frac{\beta}{\theta-1} G^{-1}(\tau_1)(1-\tau_1) + o\left(\beta G^{-1}(\tau_1)(1-\tau_1)\right) - c_1\beta^{2-\theta} \\
&= \frac{\beta}{\theta-1} (\varepsilon\beta + c_1\beta^{2-\theta})(\varepsilon\beta + c_1\beta^{2-\theta})^{-\theta} L(\varepsilon\beta + c_1\beta^{2-\theta}) - c_1\beta^{2-\theta} + o\left(\beta G^{-1}(\tau_1)(1-\tau_1)\right) \\
&= \frac{\varepsilon^{1-\theta}\beta^{2-\theta}}{\theta-1} \left(1 + c_1\varepsilon^{-1}\beta^{1-\theta}\right) \left(1 + c_1\varepsilon^{-1}\beta^{1-\theta}\right)^{-\theta} (\zeta + o(1)) - c_1\beta^{2-\theta} + o(\beta^{2-\theta}) \\
&= \beta^{2-\theta} \left(\frac{\zeta\varepsilon^{1-\theta}}{\theta-1} - c_1 + o(1) \right) \to \infty,
\end{aligned}
$$

which means $\tau_1 < \tau^*$, or equivalently, $\left(G^{-1}(\tau^*)/\beta - \varepsilon\right)\beta^{\theta-1} > c_1$ when β is large enough. Taking $c_2 > c$, we can similarly obtain the opposite inequality. Then let $\beta \to \infty$:

$$
c_1 \le \liminf_{\beta\to\infty} \left(\frac{G^{-1}(\tau^*)}{\beta} - \varepsilon \right) \beta^{\theta-1} \le \limsup_{\beta\to\infty} \left(\frac{G^{-1}(\tau^*)}{\beta} - \varepsilon \right) \beta^{\theta-1} \le c_2.
$$

Finally, letting $c_1 \uparrow c$, $c_2 \downarrow c$, we complete the proof of Eq. (B.12). Note that Eq. (B.12) is actually the conclusion when $\theta = 2$. When $\theta < 2$, together with Eq. (B.11) and the assumption $\theta > 3/2$, we obtain

$$
\begin{aligned}
\mathrm{WCE}_{\alpha}^{\mathcal{W}_{1,G,\varepsilon}} =& G^{-1}(\tau^*) = \varepsilon\beta + \mu_G + \beta \int_{G^{-1}(\tau^*)}^{\infty} \overline{G}(t)dt + o(1) \\
=& \varepsilon\beta + \mu_G + \frac{\beta}{\theta-1} G^{-1}(\tau^*)(1-\tau^*) \\
&+ \frac{\beta}{G^{-1}(\tau^*)} G^{-1}(\tau^*) \Big(\int_{G^{-1}(\tau^*)}^{\infty} \overline{G}(t)dt - \frac{G^{-1}(\tau^*)\overline{G}(G^{-1}(\tau^*))}{\theta-1} \Big) + o(1) \\
=& \varepsilon\beta + \mu_G + \frac{\varepsilon^{1-\theta}\beta^{2-\theta}}{\theta-1} \left(1 + c\varepsilon^{-1}\beta^{1-\theta} + o(\beta^{1-\theta})\right) \left(1 + c\varepsilon^{-1}\beta^{1-\theta} + o(\beta^{1-\theta})\right)^{-\theta} \\
&(\zeta + o(\beta^{\theta-2})) + (\varepsilon + o(1))o(1) + o(1) \\
=& \varepsilon\beta + \mu_G + \frac{\zeta\varepsilon^{1-\theta}}{\theta-1}\beta^{2-\theta} + o(1),
\end{aligned}
$$

where the fourth equality holds due to the fact that $\zeta - L(x) = o(x^{\theta-2})$, which is a conclusion from the assumption $\lim_{x\to\infty} x^{3-\theta} L'(x) = 0$. We thus complete the proof. $\square$

B.1.7. **Proof of Theorem 3.5** For convenience, denote $\omega(\tau) := z'_{p,G,\varepsilon}(\gamma(\tau))$. Therefore, the maximizer τ^* satisfies

$$\tau^* = \sup_{\tau\in(0,1)} \{\omega(\tau) \geq 0\}.$$

Simplifying the result in Eq. (B.7) and combining it with Eq. (B.2), we find that

$$\begin{aligned}\omega(\tau) =& \varepsilon\Big\{\big[\tau + \beta^q(1-\tau)\big]^{1/q} + q^{-1}(1-\beta^q)\big[\tau + \beta^q(1-\tau)\big]^{1/q-1}\big(1-\tau+(\beta-1)^{-1}\big)\Big\} \\ &+ \mu_G + G^{-1}(\tau) + (\beta-1)\Big[\int_\tau^1 G^{-1}(u)\,du - G^{-1}(\tau)(1-\tau)\Big].\end{aligned}$$

First we prove that

$$1-\tau^* = s\beta^{-1} + d\beta^{-1/p-1} + \mathrm{o}\left(\beta^{-1/p-1}\right), \tag{B.13}$$

or equivalently,

$$[(1-\tau^*)\beta - s]\beta^{1/p} \to d, \tag{B.14}$$

where $s = p-1$, $d = \varepsilon^{-1}qs^{2-1/q}\Delta G$, and $\Delta G = \operatorname{ess\,sup}_G - \mu_G$. To do so, we take d_1 such that $d_1 > d$, and let $\tau_2 = 1 - s\beta^{-1} - d_1\beta^{-1/p-1}$. Now substituting τ_2 into $\omega(\tau)$, we obtain

$$\begin{aligned}\omega(\tau_2) =& \varepsilon\big[\beta^{q-1}\Big(s + d_1\beta^{-1/p}\Big) - s\beta^{-1} - d_1\beta^{-1/p-1} + 1\big]^{1/q} \\ &+ \varepsilon q^{-1}\big[\tau_2 + \beta^q(1-\tau_2)\big]^{1/q-1}\big[s\beta^{-1} + d_1\beta^{-1/p-1} + (\beta-1)^{-1}\big] \\ &- \varepsilon q^{-1}\beta^q\big[\beta^{q-1}(s + d_1\beta^{-1/p}) - s\beta^{-1} - d_1\beta^{-1/p-1} + 1\big]^{1/q-1}\big[1-\tau_2+(\beta-1)^{-1}\big] \\ &+ \mu_G + G^{-1}(\tau_2) + (\beta-1)\Big[\int_{\tau_2}^1 G^{-1}(u)\,du - G^{-1}(\tau_2)(1-\tau_2)\Big] \\ =&: A_1 + A_2 - A_3 + A_4 + A_5.\end{aligned} \tag{B.15}$$

Before analyzing the five terms above, an elementary observation is $q-1-p^{-1} = q + q^{-1} - 2 > 0$, and further $\beta^{1-q} = o\left(\beta^{-1/p}\right)$, which is significant for our calculation when ignoring higher order infinitesimal. For the first term A_1,

$$
\begin{aligned}
A_1 &= \varepsilon\Big[\beta^{q-1}\big(s + d_1\beta^{-1/p}\big) - s\beta^{-1} - d_1\beta^{-1/p-1} + 1\Big]^{1/q} \\
&= \varepsilon s^{1/q}\beta^{1/p}\Big[1 + d_1 s^{-1}\beta^{-1/p} + s^{-1}\beta^{1-q} - \beta^{-q} - d_1 s^{-1}\beta^{-1/p-q}\Big]^{1/q} \\
&= \varepsilon s^{1/q}\beta^{1/p}\Big[1 + d_1 s^{-1}\beta^{-1/p} + o\big(\beta^{-1/p}\big)\Big]^{1/q} \\
&= \varepsilon s^{1/q}\beta^{1/p}\Big[1 + d_1 s^{-1}q^{-1}\beta^{-1/p} + o\big(\beta^{-1/p}\big)\Big] \\
&= \varepsilon s^{1/q}\beta^{1/p} + \varepsilon d_1 q^{-1}s^{1/q-1} + o\left(1\right). \qquad \text{(B.16)}
\end{aligned}
$$

For A_2, we notice that $\tau + \beta^q\left(1-\tau\right) \geq 1$, so $A_2 \leq \varepsilon q^{-1}\big[s\beta^{-1} + d_1\beta^{-1/p-1} + (\beta-1)^{-1}\big]$, which implies

$$A_2 = o\left(1\right). \qquad \text{(B.17)}$$

For A_3, similar to the analysis of A_1, we have

$$
\begin{aligned}
A_3 =& \varepsilon q^{-1}\big[\tau_2 + \beta^q\left(1-\tau_2\right)\big]^{1/q-1}\Big[s\beta^{-1} + d_1\beta^{-1/p-1} + (\beta-1)^{-1}\Big] \\
=& \varepsilon q^{-1}\beta^q s^{1/q-1}\beta^{(q-1)(1/q-1)}\Big[1 + d_1 s\beta^{-1/p} + o\big(\beta^{-1/p}\big)\Big]^{1/q-1} \\
& \Big[s\beta^{-1} + d_1\beta^{-1/p-1} + (\beta-1)^{-1}\Big] \\
=& \varepsilon q^{-1}\beta^q s^{1/q-1}\beta^{(q-1)(1/q-1)}\Big[1 + d_1 s\beta^{-1/p}\left(1/q-1\right) + o\big(\beta^{-1/p}\big)\Big] \\
& \Big[p\beta^{-1} + d_1\beta^{-1/p-1} + o\big(\beta^{-1/p-1}\big)\Big] \\
=& \varepsilon q^{-1}\beta^{q-1}s^{1/q-1}\beta^{(q-1)(1/q-1)}\Big[p + \frac{pc_2}{s}\left(1/q-1\right)\beta^{-1/p} + d_1\beta^{-1/p} + o\big(\beta^{-1/p}\big)\Big] \\
=& \varepsilon q^{-1}\beta^{1/p}s^{1/q-1}\Big[p - d_1 s\beta^{-1/p} + d_1\beta^{-1/p} + o\big(\beta^{-1/p}\big)\Big] \\
=& \frac{\varepsilon\beta^{1/p}s^{1/q}p}{q\left(p-1\right)} - \varepsilon d_1 q^{-1}s^{1/q-2} + \varepsilon d_1 q^{-1}s^{1/q-1} + o\left(1\right) \\
=& \varepsilon\beta^{1/p}s^{1/q} - \varepsilon d_1 q^{-1}s^{1/q-2} + \varepsilon d_1 q^{-1}s^{1/q-1} + o\left(1\right). \qquad \text{(B.18)}
\end{aligned}
$$

For A_4 and A_5, due to the left continuity of G^{-1}, we obtain

$$A_4 = -\Delta G + G^{-1}\left(1\right) - G^{-1}\left(\tau_2\right) = -\Delta G + o\left(1\right). \qquad \text{(B.19)}$$

and

$$\begin{aligned}A_5 &= (\beta-1)\left(\int_{\tau_2}^{1} G^{-1}(u)\,du - G^{-1}(\tau_2)(1-\tau_2)\right)\\ &\le (1-\tau_1)(\beta-1)\left(G^{-1}(1)-G^{-1}(\tau_2)\right)\\ &= \left(s+o(1)\right)o(1) = o(1). \qquad \text{(B.20)}\end{aligned}$$

Adding up equation Eqs. (B.16), (B.17), (B.18), (B.19), (B.20), we obtain

$$\omega(\tau_2) = \varepsilon d_1 q^{-1} s^{1/q-2} - \Delta G + \mathrm{o}(1) = \varepsilon(d_1-d)q^{-1}s^{1/q-2} + o(1). \qquad \text{(B.21)}$$

Thus, when β is large enough, we have $\omega(\tau_2) > 0$, in other words, $1-\tau^* \le s\beta^{-1} + d_1\beta^{-1/p-1}$, or equivalently,

$$\left[(1-\tau^*)\beta - s\right]\beta^{1/p} \le d_1.$$

Similarly, for any $d_2 < d$, we have

$$\left[(1-\tau^*)\beta - s\right]\beta^{1/p} \ge d_2.$$

Together with the two equations above and let $\beta \to \infty$:

$$d_2 \le \liminf_{\beta\to\infty}\left[(1-\tau^*)\beta - s\right]\beta^{1/p} \le \limsup_{\beta\to\infty}\left[(1-\tau^*)\beta - s\right]\beta^{1/p} \le d_1.$$

Finally, letting $d_2 \uparrow d$ and $d_1 \downarrow d$, we complete the proof for Eq. (B.14). Next substituting Eq. (B.15) into $z_{p,G,\varepsilon}(\gamma)$, we have

$$\begin{aligned}\gamma^* &= \frac{1}{(1-\tau^*)\beta + \tau^*}\\ &= \frac{1}{p + d\beta^{-1/p} + o\left(\beta^{-1/p}\right)}\\ &= \frac{p^{-1}}{1 + \frac{d}{p}\beta^{-1/p} + o\left(\beta^{-1/p}\right)}\\ &= p^{-1} - dp^{-2}\beta^{-1/p} + o\left(\beta^{-1/p}\right). \qquad \text{(B.22)}\end{aligned}$$

Similar to A_1,

$$\left[\tau^* + \beta^q(1-\tau^*)\right]^{1/q} = s^{1/q}\beta^{1/p} + dq^{-1}s^{1/q-1} + o(1).$$

Due to Eq. (B.22),

$$\mu_G \tau^* = p^{-1} \mu_G + o\left(1\right).$$

Recall Eq. (B.20), we have

$$\begin{aligned}
\left(\beta-1\right)\int_{\tau^*}^{1} G^{-1}\left(u\right)du =& \left(\beta-1\right)G^{-1}\left(\tau^*\right)\left(1-\tau^*\right)+o\left(1\right)\\
=& \Big(G^{-1}\left(\tau^*\right)-G^{-1}\left(1\right)\Big)\left(\beta-1\right)\left(1-\tau^*\right)\\
&+G^{-1}\left(1\right)\left(\beta-1\right)\left(1-\tau^*\right)+o\left(1\right)\\
=& o\left(1\right)\big(s+o\left(1\right)\big)+G^{-1}\left(1\right)\big(s+o\left(1\right)\big)+o\left(1\right)\\
=& sG^{-1}\left(1\right)+o\left(1\right).
\end{aligned}$$

Together with the above equations and Eq. (B.22), we obtain

$$\left(\beta-1\right)\gamma^*\int_{\tau^*}^{1} G^{-1}\left(u\right)du = q^{-1}G^{-1}\left(1\right)+o\left(1\right). \tag{B.23}$$

Combining Eqs. (B.22) and (B.23), we have

$$\begin{aligned}
z_{p,G,\varepsilon}\left(\gamma^*\right) =& \varepsilon\gamma^*\big[\tau^*+\beta^q\left(1-\tau^*\right)\big]^{1/q}+\mu_G\gamma^*+\left(\beta-1\right)\gamma^*\int_{\tau^*}^{1} G^{-1}\left(u\right)du\\
=& \varepsilon\Big[1/p-dp^{-2}\beta^{-1/p}+o(\beta^{-1/p})\Big]\Big[s^{1/q}\beta^{1/p}+dq^{-1}s^{1/q-1}+o\left(1\right)\Big]\\
&+p^{-1}\mu_G+q^{-1}G^{-1}\left(1\right)+o\left(1\right)\\
=& \varepsilon p^{-1}s^{1/q}\beta^{1/p}+\varepsilon dp^{-1}s^{1/q}\left(q^{-1}s^{-1}-p^{-1}\right)+p^{-1}\mu_G+q^{-1}G^{-1}\left(1\right)+o\left(1\right)\\
=& \varepsilon p^{-1}s^{1/q}\beta^{1/p}+p^{-1}\mu_G+q^{-1}G^{-1}\left(1\right)+o\left(1\right),
\end{aligned}$$

which completes the proof of this theorem. □

B.1.8. **Proof of Example 1** The result $\mathrm{WCE}_{\alpha}^{\mathcal{W}_{1,G,\varepsilon}} = \varepsilon\beta + x_0$ is obvious. Hence we consider $p > 1$. When $G(x) = \mathbb{1}_{\{x \geq x_0\}}$, according to Eq. (B.5),

$$k_G(\gamma) = \gamma\tau x_0 + \beta\gamma(1-\tau)x_0 = x_0[\gamma + (\beta-1)\gamma(1-\tau)] = x_0.$$

Regarding l_p, we obtain

$$\begin{aligned} l_p(\gamma) &= \gamma\big[\tau + \beta^q(1-\tau)\big]^{1/q} \\ &= \gamma\Big[\beta^q + \frac{1-\beta^q}{\beta-1}(\beta - 1/\gamma)\Big]^{1/q} \\ &= \Big[\frac{\beta^q-1}{\beta-1}\gamma^{q-1} + \frac{\beta-\beta^q}{\beta-1}\gamma^q\Big]^{1/q} \\ &:= \left(\frac{\upsilon(\gamma)}{\beta-1}\right)^{1/q}. \end{aligned} \tag{B.24}$$

Notice that $k_G(\gamma)$ is a constant, so we only need to maximize $l_p(\gamma)$, or equivalently, $\upsilon(\gamma)$. By the first order condition, we obtain

$$\gamma^* = \frac{1}{p}\left(\frac{\beta^q-1}{\beta^q-\beta}\right),$$

and we can verify that $\gamma^*\,(1/\beta, 1)$. Then substituting it into Eq. (B.24), we obtain

$$\begin{aligned} l_p(\gamma^*) &= \gamma^*\Big[\beta^q + \frac{1-\beta^q}{\beta-1}(\beta - 1/\gamma^*)\Big]^{1/q} \\ &= \frac{1}{p}\left(\frac{\beta^q-1}{\beta^q-\beta}\right)\left[\beta^q + \frac{1-\beta^q}{\beta-1}\Big(\beta - \frac{p(\beta^q-\beta)}{\beta^q-1}\Big)\right]^{1/q} \\ &= \frac{1}{p}\left(1+\frac{\beta-1}{\beta^q-\beta}\right)\left[(p-1)\beta^{q-1}\frac{\beta-\beta^{2-q}}{\beta-1}\right]^{1/q} \\ &= p^{-1}(p-1)^{1/q}\beta^{1/p}\left(1+\frac{\beta-1}{\beta^q-\beta}\right)\left(1+\frac{1-\beta^{2-q}}{\beta-1}\right)^{1/q}. \end{aligned}$$

□

B.1.9. **Proof of Theorem 3.6** As explained after Theorem 3.6, only item (ii) would be checked. Since both $z_{p,G,\varepsilon}$ and $z^{icx}_{p,G,\varepsilon}$ are strictly concave on $[1/\beta]$, we only need to show that $z^{icx}_{p,G,\varepsilon}(\gamma) > z_{p,G,\varepsilon}(\gamma)$ for all $\gamma \in (1/\beta, 1)$. Equivalently, we need to prove that for all $\beta > 1$, $q > 1$ and $\tau \in (0,1)$, it holds that

$$(\tau + \beta^q(1-\tau))^{1/q} < 1 + (\beta-1)(1-\tau)^{1/q}.$$

Let $x = (1-\tau)^{1/q} \in (0,1)$, the above inequality is transformed to

$$1 + (\beta^q - 1)x^q < (1 + (\beta-1)x)^q\,.$$

noticing that the above two functions are equal at $\beta = 1$, we differentiate to β to obtain the desired result. □

B.2. Proof of Section 3.2

B.2.1. **Proof of Theorem 3.7** First, we need to calculate $[h_\gamma]_q$. Let

$$\hat{x} = \underset{x\in R}{\operatorname{argmin}} \int_0^1 |h_\gamma(u) - x|^q du = \underset{x\in R}{\operatorname{argmin}} \Big\{ \tau |x-\gamma|^q + (1-\tau)|\beta\gamma - x|^q \Big\}.$$

Obviously $\hat{x} \in [\gamma, \beta\gamma]$, hence we let $f(x) = \tau(x-\gamma)^q + (1-\tau)(\beta\gamma - x)^q$, take the derivative and make it equal to 0,

$$f'(\hat{x}) = q\tau(\hat{x}-\gamma)^{q-1} - q(1-\tau)(\beta\gamma - \hat{x})^{q-1} = 0. \tag{B.25}$$

We obtain $\tau(\hat{x}-\gamma)^{q-1}(\beta\gamma - \hat{x}) = (1-\tau)(\beta\gamma-\hat{x})^q$, which gives

$$f(\hat{x}) = \tau\gamma(\beta-1)(\hat{x}-\gamma)^{q-1} = (\beta\gamma - 1)(\hat{x}-\gamma)^{q-1}.$$

Rewrite Eq. (B.25) as

$$\frac{\hat{x}-\gamma}{\beta\gamma-\hat{x}} = \left(\frac{1-\tau}{\tau}\right)^{\frac{1}{q-1}} = \left(\frac{1-\gamma}{\beta\gamma-1}\right)^{\frac{1}{q-1}} =: a,$$

which implies $\hat{x} = \left(\frac{a\beta-1}{a+1}\right)\gamma$, and further

$$\hat{x} - \gamma = \frac{a}{a+1}(\beta-1)\gamma = \frac{(1-\gamma)^{\frac{1}{q-1}}}{(1-\gamma)^{\frac{1}{q-1}} + (\beta\gamma-1)^{\frac{1}{q-1}}}(\beta-1)\gamma. \tag{B.26}$$

Substituting Eq. (B.26) into Eq. (B.25), we obtain

$$f(\hat{x}) = \frac{(1-\gamma)(\beta\gamma-1)(\beta-1)^{q-1}\gamma^{q-1}}{\left[(\beta\gamma-1)^{p-1} + (1-\gamma)^{p-1}\right]^{q-1}}.$$

Finally, we obtain

$$[h_\gamma]_q = f(\hat{x})^{1/q} = (\beta-1)^{1/p}\left[\frac{\gamma(\beta\gamma-1)^s(1-\gamma)^s}{(\beta\gamma-1)^s + (1-\gamma)^s}\right]^{1/p}, \tag{B.27}$$

where $s = p-1 > 0$. Next, we only need to solve $\max_{\gamma\in\left[\frac{1}{\beta},1\right]} [h_\gamma]_q$. Let $g(\gamma) = \frac{\gamma(\beta\gamma-1)^s(1-\gamma)^s}{(\beta\gamma-1)^s+(1-\gamma)^s}$. Note that g is a nonnegative function on $[1/\beta, 1]$ with $g(1) =$

$g\left(1/\beta\right)=0$, so it must attain its maximum on $(1/\beta,1)$. Differentiate to it:

$$\begin{aligned}g'\left(\gamma\right)=&\frac{\left(\beta\gamma-1\right)^{s-1}\left(1-\gamma\right)^{s-1}}{\left(\beta\gamma-1\right)^{s}+\left(1-\gamma\right)^{s}}\Big[\left(\beta\gamma-1\right)^{p}\left(1-p\gamma\right)+\left(1-\gamma\right)^{p}\left(p\beta\gamma-1\right)\Big]\\=&\frac{\left(\beta\gamma-1\right)^{s-1}\left(1-\gamma\right)^{s-1}}{\left(\beta\gamma-1\right)^{s}+\left(1-\gamma\right)^{s}}\omega\left(\gamma\right),\end{aligned}$$

where $\omega\left(\gamma\right):=\left(\beta\gamma-1\right)^{p}\left(1-p\gamma\right)+\left(1-\gamma\right)^{p}\left(p\beta\gamma-1\right)$. We claim that $\omega\left(\gamma\right)$ has only one zero point on $(1/\beta,1)$, denoted by γ^*, which also means

$$\gamma^*=\underset{\gamma\in\left[\frac{1}{\beta},1\right]}{\operatorname{argmin}}\left[h_\gamma\right]_q.$$

The proof of $\mathrm{WCE}_\alpha^{\mathcal{M}_{p,\mu,\sigma}}=e_\alpha(F_{p,\mu,\sigma,\gamma^*})$ is totally similar to that of Theorem 3.1 according to Proposition 3. Consequently, we need to analyze the behavior of γ^* and divide this problem into three situations.

We first take $p=2$ into account as a special and important case. At this time,

$$\begin{aligned}\omega\left(\gamma\right)&=\left(\beta\gamma-1\right)^2\left(1-2\gamma\right)+\left(1-\gamma\right)^2\left(2\beta\gamma-1\right)\\&=\left(\beta^2\gamma^2-2\beta\gamma+1\right)\left(1-2\gamma\right)+\left(\gamma^2-2\gamma+1\right)\left(2\beta\gamma-1\right)\\&=\beta^2\gamma^2-2\beta\gamma+1-2\beta^2\gamma^3+4\beta\gamma^2-2\gamma-\gamma^2+2\gamma-1+2\beta\gamma^3-4\beta\gamma^2+2\beta\gamma\\&=\left(\beta-1\right)\gamma^2\left(\beta+1-2\beta\gamma\right).\end{aligned}$$

Immediately, we obtain $\gamma^*=(\beta+1)/2\beta$. And substituting this into Eq. (B.27) we see

$$\left[h_{\gamma^*}\right]_q=\sqrt{\beta-1}\left[\frac{\gamma^*\left(\frac{\beta-1}{2}\right)\left(\frac{\beta-1}{2\beta}\right)}{\left(\beta-1\right)\gamma^*}\right]^{1/2}=\frac{\beta-1}{2\sqrt{\beta}}.$$

Thus the conclusion when $p=2$ is obtained.

Before analyzing the situation when $p\neq2$, an elementary and necessary observation is as follows:

$$\omega\left(1/p\right)=\left(1-1/p\right)^p\left(\beta-1\right)>0. \tag{B.28}$$

Next we consider $p>2$. Let κ be a number between $p+2/p-2$ and $p-1$, taking

β large enough, we have

$$\begin{aligned}
\omega\left(1/p+\beta^{-\kappa}\right) =& -p\beta^{-\kappa}\left(\beta/p+\beta^{1-\kappa}-1\right)^{p}+\left(1-1/p-\ \beta^{-\kappa}\right)^{p}\left(\beta+p\beta^{1-\kappa}-1\right) \\
=& -\frac{\beta^{p-\kappa}}{p^{p-1}}\left(1+p\beta^{-\kappa}-p\beta^{-1}\right)^{p}+\left(1-1/p\right)^{p}\beta\left[1-p\beta^{-\kappa}/(p-1)\right]^{p} \\
&\left(1+p\beta^{-\kappa}-\beta^{-1}\right) \\
=& -\frac{\beta^{p-\kappa}}{p^{p-1}}\left[1+O\left(\beta^{-1}\right)\right]+\left(1-1/p\right)^{p}\beta\left[1+O\left(\beta^{-\kappa}\right)\right]\left[1+O\left(\beta^{-1}\right)\right] \\
=& -\frac{\beta^{p-\kappa}}{p^{p-1}}+\left(1-1/p\right)^{p}\beta+O\left(1\right)+O\left(\beta^{p-\kappa-1}\right).
\end{aligned}$$

Note that when $p>2$, $p-\kappa>1$ and $p-\kappa-1<1-2/p<1$, so as $\beta\to\infty$,

$$\omega\left(1/p+\beta^{-\kappa}\right)<0. \tag{B.29}$$

Together with Eqs. (B.28) and (B.29) , we have

$$\gamma^{*}=1/p+O\left(\beta^{-\kappa}\right).$$

Furthermore,

$$\left(\frac{1-\gamma^{*}}{\beta\gamma^{*}-1}\right)^{p-1}\beta^{p-1}=\left[\frac{\beta/q+O\left(\beta^{1-\kappa}\right)}{\beta/p-1+O\left(\beta^{1-\kappa}\right)}\right]^{p-1}\to(p-1)^{p-1},$$

which implies

$$\left(\frac{1-\gamma^{*}}{\beta\gamma^{*}-1}\right)^{p-1}=O\left(\beta^{1-p}\right)=O\left(\beta^{-\kappa}\right).$$

Due to Eq. (B.27), we finally obtain

$$\begin{aligned}
\left[h_{\gamma^{*}}\right]_{q} &= (\beta-1)^{1/p}\left[\left(1/p+O\left(\beta^{-\kappa}\right)\right)^{1/p}\left(1/q+O\left(\beta^{-\kappa}\right)\right)^{1/q}\right]\left[1+\left(\left(\frac{1-\gamma^{*}}{\beta\gamma^{*}-1}\right)^{p-1}\right)\right]^{-1/p} \\
&= \frac{(\beta-1)^{1/p}}{p^{1/p}q^{1/q}}\left[1+O\left(\beta^{-\kappa}\right)\right]\left[1+O\left(\beta^{-\kappa}\right)\right]\left[1+o\left(\beta^{-\kappa}\right)\right] \\
&= \frac{(\beta-1)^{1/p}}{p^{1/p}q^{1/q}}+O\left(\beta^{1/p-\kappa}\right) \\
&= \frac{(\beta-1)^{1/p}}{p^{1/p}q^{1/q}}+o\left(\beta^{2-p-1/p}\right),
\end{aligned}$$

which completes the proof of $p>2$.

When $p < 2$, the same arguments can be applied but are more tedious. First, we should determine the speed of convergence $\gamma^* \to 1/p$, i.e.,

$$\gamma^* = p^{-1} + m\beta^{1-p} + n\beta^{2(1-p)} + o\big(\beta^{2(1-p)}\big), \tag{B.30}$$

where $m = (p-1)^p/p$, $n = -(p-1)^{2p}\left(1 + 1/p(p-1)\right)$. The proof of the above equation is totally similar to Eq. (B.14). Now we have to substitute Eq. (B.30) into Eq. (B.27), before doing so, we need the following second-order Taylor expansion. For real number λ,

$$(1+x)^\lambda = 1 + \lambda x + \lambda(\lambda-1)x^2/2 + o(x^2). \tag{B.31}$$

Then

$$\begin{aligned}
(\gamma^*)^{1/p} &= \Big(p^{-1} + m\beta^{1-p} + n\beta^{2(1-p)} + o\big(\beta^{2(1-p)}\big)\Big)^{1/p} \\
&= p^{-1/p}\Big(1 + mp\beta^{1-p} + np\beta^{2(1-p)} + o\big(\beta^{2(1-p)}\big)\Big)^{1/p} \\
&= p^{-1/p}\Big(1 + m\beta^{1-p} + n\beta^{2(1-p)} - (2pq)^{-1}\cdot p^2m^2\beta^{2(1-p)} + o\big(\beta^{2(1-p)}\big)\Big) \\
&= p^{-1/p}\Big(1 + m\beta^{1-p} + \beta^{2(1-p)}\big(n - (2q)^{-1}pm^2\big) + o\big(\beta^{2(1-p)}\big)\Big),
\end{aligned} \tag{B.32}$$

where the third equation uses Eq. (B.31). Similarly,

$$(1-\gamma^*)^{1/q} = q^{-1/q}\Big(1 - m\beta^{1-p} - \beta^{2(1-p)}\Big(n + (2p)^{-1}qm^2\Big) + o\big(\beta^{2(1-p)}\big)\Big). \tag{B.33}$$

And again,

$$\begin{aligned}
\Big(\frac{1-\gamma^*}{\beta\gamma^*-1}\Big)^{p-1} =& \beta^{1-p}\Big(\frac{1-\gamma^*}{\gamma^*-1/\beta}\Big)^{p-1} \\
=& \beta^{1-p}\left(\frac{p-1-mp\beta^{1-p} - np\beta^{2(1-p)} + o\big(\beta^{2(1-p)}\big)}{1 + mp\beta^{1-p} + np\beta^{2(1-p)} + o\big(\beta^{2(1-p)}\big) - p\beta^{-1}}\right)^{p-1} \\
=& (p-1)^{p-1}\beta^{1-p}\Big(1 - mp(p-1)^{-1}\beta^{1-p} + o\big(\beta^{1-p}\big)\Big)^{p-1} \\
& \Big(1 + mp\beta^{1-p} + o\left(\beta^{1-p}\right)\Big)^{1-p} \\
=& (p-1)^{p-1}\beta^{1-p}\Big(1 - mp\beta^{1-p} + o\big(\beta^{1-p}\big)\Big) \\
& \Big(1 - mp(p-1)\beta^{1-p} + o\big(\beta^{1-p}\big)\Big) \\
=& (p-1)^{p-1}\beta^{1-p}\Big(1 - mp^2\beta^{1-p} + o\big(\beta^{1-p}\big)\Big).
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
\left[1+\left(\frac{1-\gamma^*}{\beta\gamma^*-1}\right)^{p-1}\right]^{-1/p} =&\Big[1+(p-1)^{p-1}\beta^{1-p}-mp^2(p-1)^{p-1}\beta^{2(1-p)}+o\left(\beta^{2(1-p)}\right)\Big]^{-1/p}\\
=&1-(p-1)^{p-1}p^{-1}\beta^{1-p}+mp(p-1)^{p-1}\beta^{2(1-p)}\\
&+(p+1)(p-1)^{p-1}(2p)^{-2}\beta^{2(1-p)}+o\left(\beta^{2(1-p)}\right),
\end{aligned}
\tag{B.34}
$$

where we use Eq. (B.31) again. Then together with Eqs. (B.27), (B.32), (B.33), (B.34), we obtain

$$
\begin{aligned}
[h_{\gamma^*}]_q =&(\beta-1)^{1/p}(\gamma^*)^{1/p}(1-\gamma^*)^{1/q}\left[1+\left(\frac{1-\gamma^*}{\beta\gamma^*-1}\right)^{p-1}\right]^{-1/p}\\
=&\frac{(\beta-1)^{1/p}}{p^{1/p}q^{1/q}}\left[1-\frac{(p-1)^{p-1}}{p\beta^{p-1}}+\left(-m^2-\frac{pm^2}{q}-\frac{qm^2}{p}\right.\right.\\
&\left.\left.+\frac{(p+1)(p-1)^{p-1}}{2p^2}\right)\beta^{2(1-p)}+o\left(\beta^{2(1-p)}\right)\right]\\
=&\frac{(\beta-1)^{1/p}}{p^{1/p}q^{1/q}}\left[1-\frac{(p-1)^{p-1}}{p\beta^{p-1}}+\left(\frac{(p-1)^{2p-1}}{p^2}+\frac{(p-1)^{2p}}{p}\right.\right.\\
&\left.\left.+\frac{(p-1)^{p-1}(p+1)}{2p^2}\right)\beta^{(2(1-p))}+\mathrm{o}\left(\beta^{2(1-p)}\right)\right],
\end{aligned}
$$

which completes the proof.

□

Appendix C. Additional simulations

In this section, some examples of the worst-case value of expectiles are calculated and the efficiency of our approximation at an extreme level is explored. Specifically, we let the level α vary from 0.9 to 0.99, and the corresponding β lies in $[9, 99]$. Then two aspects are taken into account: one is the behavior of $\mathrm{WCE}_\alpha^{\mathcal{M}}$ with α tending to 1 for various ambiguity sets $\mathcal{M}$, the other is the difference between precise and approximate values of $\mathrm{WCE}_\alpha^{\mathcal{M}}$. The following simulations all demonstrate our theoretical results.

C.1. A simulation study on Theorem 3.3

We illustrate the worst-case value of expectile $\mathrm{WCE}_\alpha^{\mathcal{W}_{1,G,\varepsilon}}$ in Theorem 3.3 for the reference distribution G satisfying the condition $\mathbb{E}^G[(Y^+)^2] < \infty$, which includes the standard normal distribution $\mathcal{N}(0,1)$ and the Pareto distribution $Pa(1,3)$ with density function $3x^{-4}\mathbb{1}_{\{x>1\}}$. In addition to the precise and approximate values of $\mathrm{WCE}_\alpha^{\mathcal{W}_{1,G,\varepsilon}}$, the expectile of reference distribution G is also drawn on the chart. In addition, to balance the computability and aesthetics of the results, we choose ε to be 0.1 and 0.2, respectively. From Figure C.1, we see that the worst-case value of expectile has a dramatic increase near $\alpha = 1$, although the expectile of reference distribution grows much slower, which indicates potential extreme risk due to the ambiguity of distribution. It is worth mentioning that both simulations and theoretical results reveal that the trend of $\mathrm{WCE}_\alpha^{\mathcal{W}_{1,G,\varepsilon}}$ as $\alpha \to 1$ is independent of a particular reference distribution whenever the condition in Theorem 3.3 is satisfied. Finally, our approximation is obviously of good efficiency, implying the correctness of our results, especially in the case of a normal distribution.

(a) Approximate and Precise Values of $\mathrm{WCE}_\alpha^{\mathcal{W}_{1,\mathcal{N}(0,1),0.1}}$ and Expectile of $\mathcal{N}(0,1)$.

(b) Approximate and Precise Values of of $\mathrm{WCE}_\alpha^{\mathcal{W}_{1,Pa(1,3),0.2}}$ and Expectile of $Pa(1,3)$.

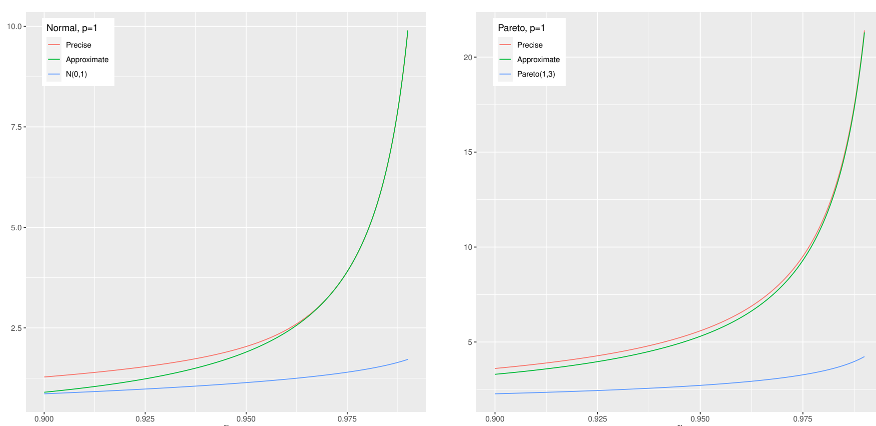


FIGURE C.1: $\mathrm{WCE}_\alpha^{\mathcal{W}_{1,G,\varepsilon}}$ through level $\alpha \in [0.9, 0.99]$: (a). G is the standard normal distribution $\mathcal{N}(0,1)$ and $\varepsilon = 0.1$. (b). G is the Pareto distribution $Pa(1,3)$ and $\varepsilon = 0.2$. The precise value (red line), approximate value (green line) in Theorem 3.3 and expectile of reference distribution G (blue line) are presented.

C.2. A simulation study on Theorem 3.4

We first introduce two important examples of heavy-tailed distributions that will be used later. One is the generalized Pareto distribution with two parameters σ and ξ, whose survival function is

$$\overline{G}_1(x) = \left(1 + \frac{\xi x}{\sigma}\right)^{-1/\xi} = x^{-1/\xi} L_1(x),\ x > 0,\ \text{with } L_1(x) = \left(x^{-1} + \frac{\xi}{\sigma}\right)^{-1/\xi}.$$

Here we assume that $1/2 \leq \xi < 2/3$. It is not difficult to verify that all conditions in Theorem 3.4 are satisfied with $\zeta = (\sigma/\xi)^{1/\xi}$ and $\theta = 1/\xi$. We simply denote this distribution as $\mathrm{GPD}(\sigma, \xi)$. Moreover, its expectation μ_{G_1} is $\sigma/(1-\xi)$.

The other is Fréchet distribution with one parameter θ, whose survival function is

$$\overline{G}_2(x) = 1 - e^{-x^{-\theta}} = x^{-\theta} L_2(x),\ x > 0,\ \text{with } L_2(x) = x^{\theta}\left(1 - e^{-x^{-\theta}}\right).$$

We impose the parameter θ satisfies $3/2 < \theta \leq 2$ so that the conditions in Theorem 3.4 are again satisfied with $\zeta = 1$. Similarly, we denote this distribution as $\mathrm{Frechet}(\theta)$. We can check its expectation μ_{G_2} is $\Gamma(1 - 1/\theta)$.

In Figure C.2, we illustrate the worst-case value of expectile $\mathrm{WCE}_{\alpha}^{\mathcal{W}_{1,G,\varepsilon}}$ in a similar way as Section C.1, but the distributions $\mathrm{GPD}(1, 5/9)$ and $\mathrm{Frechet}(1.6)$ are considered as the reference distributions. Another difference here is that we choose ε to be 1 in both cases to support our theoretical results. As we mentioned before, the general trends of $\mathrm{WCE}_{\alpha}^{\mathcal{W}_{1,G,\varepsilon}}$ with $\alpha \to 1$ are again independent of specific reference distributions, just the same as Theorem 3.4 showing that the main term is $\varepsilon\frac{\alpha}{1-\alpha}$, which has nothing to do with G. The difference is that when $\theta < 2$, there is a diverging quantity $\frac{\zeta\varepsilon^{1-\theta}}{\theta-1}\beta^{2-\theta}$ that relies heavily on the reference distribution G, rather than simply the constant μ_G in Theorem 3.3. Another phenomenon worth noting is that $\mathrm{WCE}_{\alpha}^{\mathcal{W}_{1,G,\varepsilon}}$ still has much more rapid growth than the expectile of the reference distribution, even though G has much heavier tails than before, reflecting the extensive extreme risk caused by ambiguity. Overall, the simulations provide strong evidence for the efficiency of our approximation with extra expansion.

C.3. Simulation of Theorem 3.5

Due to the limitations of our technology, Theorem 3.5 simply concentrates on the situations when $p > 1$ and $G^{-1}(1) < \infty$, where the latter is a strong assumption. Therefore, we choose G to be the most common bounded distribution: Beta distribution with two parameters ψ_1 and ψ_2, denoted by $\mathrm{Beta}(\psi_1, \psi_2)$. Figure C.3 shows higher accuracy for different p and G than previous results when $G^{-1}(1)$ is unbounded. Especially when $\alpha > 0.975$, the two lines representing the exact and approximate values for $\mathrm{WCE}_{\alpha}^{\mathcal{W}_{p,G,\varepsilon}}$ almost coincide. Of course, the distribution G and index ε we chose are to make the results more aesthetic.

(a) Approximate and Precise Values of $\mathrm{WCE}_{\alpha}^{\mathcal{W}_{1,GPD(1,5/9),1}}$ and Expectile of $GPD(1,5/9)$.

(b) Approximate and Precise Values of $\mathrm{WCE}_{\alpha}^{\mathcal{W}_{1,Frechet(1.6),1}}$ and Expectile of $Frechet(1.6)$.

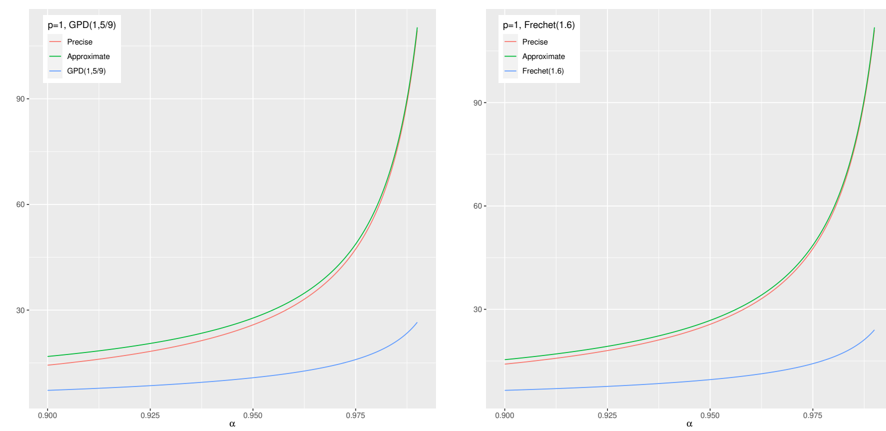


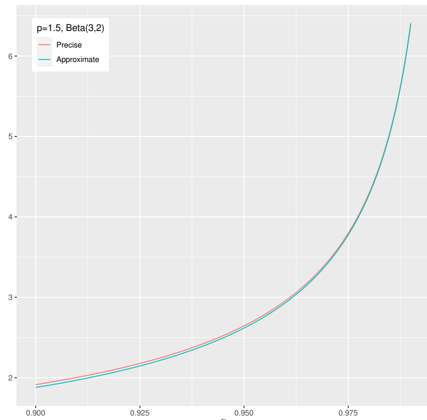
FIGURE C.2: $\mathrm{WCE}_{\alpha}^{\mathcal{W}_{1,G,\varepsilon}}$ through level $\alpha \in [0.9, 0.99]$: (a). G is the generalized Pareto distribution $GPD(1, 5/9)$ and $\varepsilon = 1$. (b) G is the Fréchet distribution with parameter $\theta = 1.6$ and ε is set to 1. Precise value (red line), approximate value (green line), and expectile of reference distribution G (blue line) are presented.

C.4. A simulation study on Theorem 3.6

We perform simulations on Theorem 3.6. Here we assume that $\mu = 0$ and $\sigma = 1$ for the sake of simplicity. We choose different values of p: 4, 3, 1.5, 1.2, situations both when $p > 2$ and $p < 2$ are included. From the above two panels in Figure C.4, we see the efficiency of the one-term expansion in Theorem 3.6. (ii) behaves better as p increases, which is consistent with our portrayal for the residual term, i.e. $o\left(\beta^{2-p-\frac{1}{p}}\right)$. In Figure 4(c), we illustrate the precise and approximate value for $\mathrm{WCE}_{\alpha}^{\mathcal{M}_{1.5,0,1}}$. Noting that when $p = 1.5$, the first correction item $\eta_1 p^{-\frac{1}{p}} q^{-\frac{1}{q}} (\beta - 1)^{\frac{1}{p}} \beta^{1-p}$ is not $o(1)$, it indicates the necessity for the extra expansions in Theorem 3.6. (iii). At last, $\beta^{\frac{1}{p}-2p+2}$ tends to infinity, when $p = 1.2$. Even though Theorem 3.6. (iii) shows that the residual term is $o(\beta^{\frac{1}{p}-2p+2})$, Figure 4(d) reveals that the error does not converge.

In Figure C.5, we demonstrate the difference between approximate and precise

(a) Approximate and Precise Values of $\mathrm{WCE}_{\alpha}^{\mathcal{W}_{1.5,Beta(3,2),0.5}}$.



(b) Approximate and Precise Values of $\mathrm{WCE}_{\alpha}^{\mathcal{W}_{3,Beta(2,3),1}}$.

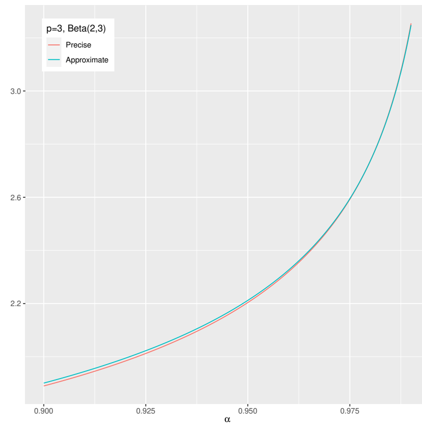


FIGURE C.3: $\mathrm{WCE}_{\alpha}^{\mathcal{W}_{p,G,\varepsilon}}$ through level $\alpha \in [0.9, 0.99]$: (a). G is Beta distribution $Beta(3,2)$ and $\varepsilon = 0.5$, p=1.5. (b) G is the Beta distribution $Beta(2,3)$ and ε is set to 1. Precise (red line), and approximate (green line) values in Theorem 3.5 are presented.

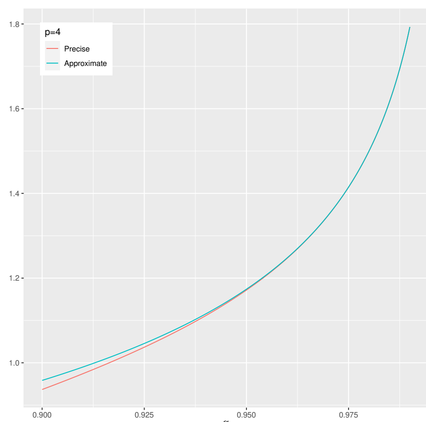
values of $\mathrm{WCE}_{\alpha}^{\mathcal{M}_{p,0,1}}$ for $1.3, 1.25, 1.2, 1.15, 1.1$. It appears that the error term becomes increasingly uncontrollable as p approaches 1, hence it seems that there does not appear to be a unified approximation when $p \to 1$.

References

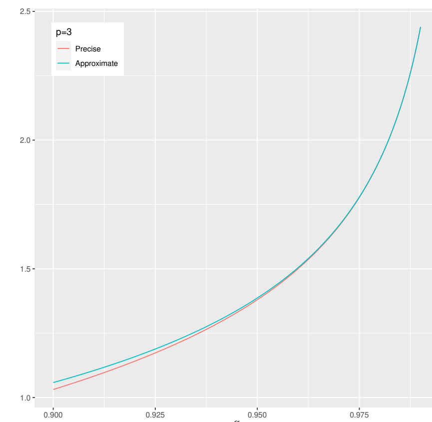
[1] Bellini, F., Klar, B., Müller, A. and Rosazza-Gianin, E. (2014). Generalized quantiles as risk measures. *Insurance Math. Econom.* **54**, 41–48.

[2] Bingham, N. H., Goldie, C. M. and Teugels, J. L. (1989). *Regular variation.* Cambridge university press.

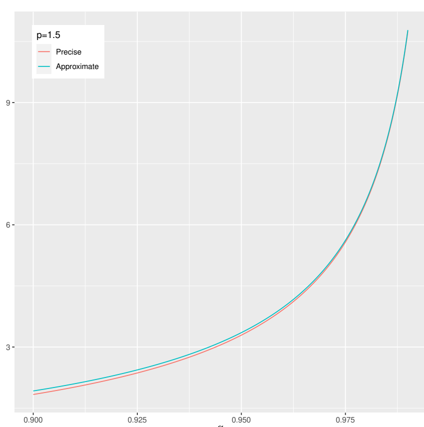
(a) Approximate and Precise Values of $\mathrm{WCE}_\alpha^{\mathcal{M}_{4,0,1}}$



(b) Approximate and Precise Values of $\mathrm{WCE}_\alpha^{\mathcal{M}_{3,0,1}}$



(c) Approximate and Precise Values of $\mathrm{WCE}_\alpha^{\mathcal{M}_{1.5,0,1}}$



(d) Approximate and Precise Values of $\mathrm{WCE}_\alpha^{\mathcal{M}_{1.2,0,1}}$

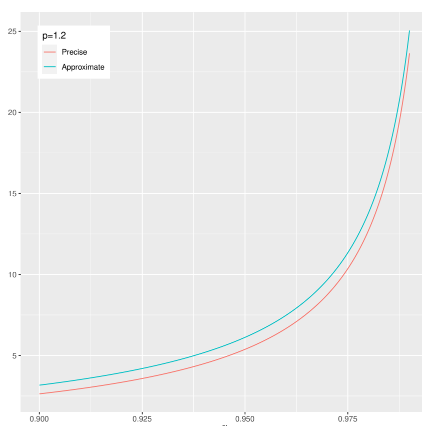


FIGURE C.4: $\mathrm{WCE}_\alpha^{\mathcal{M}_{p,0,1}}$ for $p = 4, 3, 1.5, 1.2$ with varying level $\alpha \in [0.9, 0.99]$. Both precise (red line) and approximate (green line) values in Theorem 3.6 are presented.

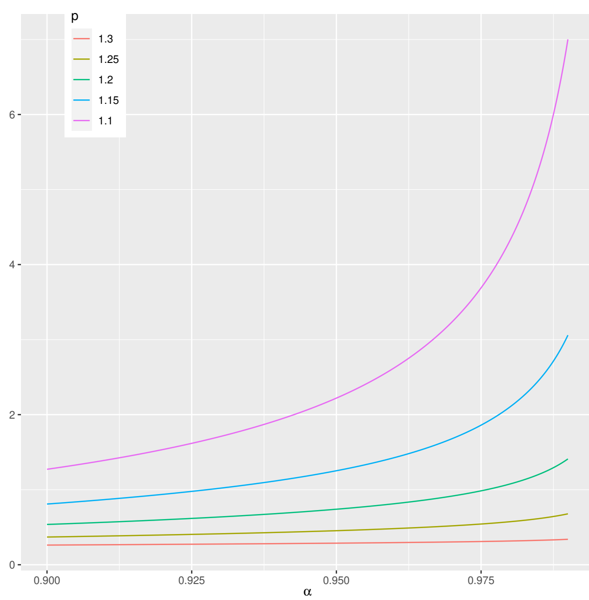


FIGURE C.5: Difference between approximate and precise values of $\mathrm{WCE}_{\alpha}^{\mathcal{M}_{p,0,1}}$ for $p = 1.3, 1.25, 1.2, 1.15, 1.1$ and varying levels $\alpha \in [0.9, 0.99]$