Economic Scenario Generator and Parameter Uncertainty: A Bayesian Approach

Jean-François Bégin[1]

[1]Department of Statistics and Actuarial Science, Simon Fraser University

Supplementary Materials

SM.A Proofs

SM.A.1 Proof of Proposition 1

If we only focus on the inflation dynamics, we have the following posterior density if we use the non-informative prior:

$$\begin{aligned} p\left(\Theta_i \,\middle|\, \{i(t)\}_{t=1}^T\right) &= \mathcal{L}\left(\{i(t)\}_{t=1}^T \,\middle|\, \Theta_i\right)\pi(\Theta_i) \\ &\propto \left(\frac{1}{\sigma_i}\right)^{T+2} \exp\left(-\frac{1}{2}\frac{\sum_{t=1}^T \left(i(t) - \mu_i - a_i\left(i(t-1) - \mu_i\right)\right)^2}{\sigma_i^2}\right)\mathbf{1}_{\{a_i \in (-1,1)\}} \end{aligned}$$

because all the other likelihood expressions do not depend on μ_i, σ_i^2, a_i, and i_0. Using Bayes' theorem, it is possible to find the full conditional density of μ_i, σ_i^2 or a_i. For instance, ignoring the terms that are not in μ_i gives

$$\begin{aligned} &\pi\left(\mu_i \,\middle|\, \sigma_i^2, a_i, \{i(t)\}_{t=0}^T\right) \\ &\propto \ \exp\left(-\frac{1}{2}\frac{\sum_{t=1}^T \left(i(t) - \mu_i - a_i\left(i(t-1) - \mu_i\right)\right)^2}{\sigma_i^2}\right) \\ &\propto \exp\left(-\frac{1}{2}\frac{\sum_{t=1}^T \left(\mu_i(1-a_i) - (i(t) - a_i i(t-1))\right)^2}{\sigma_i^2}\right) \\ &\propto \exp\left(-\frac{1}{2}\frac{T\mu_i^2(1-a_i)^2 - 2\sum_{t=1}^T \mu_i(1-a_i)\left(i(t) - a_i i(t-1)\right) + \sum_{t=1}^T \left(i(t) - a_i i(t-1)\right)^2}{\sigma_i^2}\right) \\ &\propto \frac{1}{\sqrt{\frac{\sigma_i^2}{T(1-a_i)^2}}} \exp\left(-\frac{1}{2}\frac{\left(\mu_i - \frac{\sum_{t=1}^T (i(t) - a_i i(t-1))}{T(1-a_i)}\right)^2}{\frac{\sigma_i^2}{T(1-a_i)^2}}\right) \end{aligned} \tag{A.1}$$

given that $a_i \in (-1, 1)$. Equation (A.1) is indeed a Gaussian probability density function (pdf) such that

$$\mu_i \,\middle|\, \sigma_i^2, a_i, \{i(t)\}_{t=0}^T \sim \mathcal{N}\left(\frac{\sum_{t=1}^T i(t) - a_i i(t-1)}{T(1-a_i)}, \frac{\sigma_i^2}{T(1-a_i)^2}\right).$$

It is slightly different for parameter σ_i^2:

$$\begin{aligned}
&\pi\left(\sigma_i^2 \,\middle|\, \mu_i, a_i, \{i(t)\}_{t=0}^T\right) \\
&\propto \left(\frac{1}{\sigma_i}\right)^{T+2} \exp\left(-\frac{1}{2}\frac{\sum_{t=1}^T \left(i(t)-\mu_i-a_i\left(i(t-1)-\mu_i\right)\right)^2}{\sigma_i^2}\right) \\
&\propto \left(\sigma_i^2\right)^{-\frac{T}{2}-1} \exp\left(-\frac{\frac{1}{2}\sum_{t=1}^T \left(i(t)-\mu_i-a_i\left(i(t-1)-\mu_i\right)\right)^2}{\sigma_i^2}\right) \\
&\propto \frac{\left(\frac{1}{2}\sum_{t=1}^T \left(i(t)-\mu_i-a_i\left(i(t-1)-\mu_i\right)\right)^2\right)^{\frac{T}{2}+1}}{\Gamma\left(\frac{T}{2}+1\right)} \left(\sigma_i^2\right)^{-\frac{T}{2}-1} \exp\left(-\frac{\frac{1}{2}\sum_{t=1}^T \left(i(t)-\mu_i-a_i\left(i(t-1)-\mu_i\right)\right)^2}{\sigma_i^2}\right),
\end{aligned} \tag{A.2}$$

given that $a_i \in (-1, 1)$. Equation (A.2) is in fact the pdf of an inverse gamma distribution such that

$$\sigma_i^2 \,\middle|\, \mu_i, a_i, \{i(t)\}_{t=0}^T \sim \mathrm{IG}\left(\frac{T}{2}+1, \frac{1}{2}\sum_{t=1}^T \left(i(t)-\mu_i-a_i\left(i(t-1)-\mu_i\right)\right)^2\right).$$

We can use the same rationale for a_i as the one used for μ_i:

$$\begin{aligned}
&\pi\left(a_i \,\middle|\, \mu_i, \sigma_i^2, \{i(t)\}_{t=0}^T\right) \\
&\propto \exp\left(-\frac{1}{2}\frac{\sum_{t=1}^T \left(i(t)-\mu_i-a_i\left(i(t-1)-\mu_i\right)\right)^2}{\sigma_i^2}\right)\mathbf{1}_{\{a_i\in(-1,1)\}} \\
&\propto \exp\left(-\frac{1}{2}\frac{\sum_{t=1}^T \left(a_i\left(i(t-1)-\mu_i\right)-\left(i(t)-\mu_i\right)\right)^2}{\sigma_i^2}\right)\mathbf{1}_{\{a_i\in(-1,1)\}} \\
&\propto \exp\left(-\frac{1}{2}\frac{a_i^2\sum_{t=1}^T \left(i(t-1)-\mu_i\right)^2-2a_i\sum_{t=1}^T\left(i(t-1)-\mu_i\right)\left(i(t)-\mu_i\right)+\sum_{t=1}^T\left(i(t)-\mu_i\right)^2}{\sigma_i^2}\right)\mathbf{1}_{\{a_i\in(-1,1)\}} \\
&\propto \frac{1}{\sqrt{\frac{\sigma_i^2}{\sum_{t=1}^T (i(t-1)-\mu_i)^2}}} \exp\left(-\frac{1}{2}\frac{\left(a_i-\frac{\sum_{t=1}^T (i(t-1)-\mu_i)(i(t)-\mu_i)}{\sum_{t=1}^T (i(t-1)-\mu_i)^2}\right)^2}{\frac{\sigma_i^2}{\sum_{t=1}^T (i(t-1)-\mu_i)^2}}\right)\mathbf{1}_{\{a_i\in(-1,1)\}},
\end{aligned} \tag{A.3}$$

which means that the posterior distribution of a_i is normally distributed, truncated to $(-1, 1)$. In fact, from Equation (A.3), we have that

$$a_i \,\middle|\, \mu_i, \sigma_i^2, \{i(t)\}_{t=0}^T \sim \mathcal{N}_{(-1,1)}\left(\frac{\sum_{t=1}^T \left(i(t-1)-\mu_i\right)\left(i(t)-\mu_i\right)}{\sum_{t=1}^T \left(i(t-1)-\mu_i\right)^2}, \frac{\sigma_i^2}{\sum_{t=1}^T \left(i(t-1)-\mu_i\right)^2}\right).$$

Finally, for i_0, we have that:

$$\begin{aligned}
&\pi\left(i_0 \,\middle|\, \mu_i, \sigma_i^2, a_i, \{i(t)\}_{t=1}^T\right) \\
&\propto \exp\left(-\frac{1}{2}\frac{\left(i(1)-\mu_i-a_i\left(i_0-\mu_i\right)\right)^2}{\sigma_i^2}\right)
\end{aligned}$$

$$\propto \exp\left(-\frac{1}{2}\frac{\left(a_i\left(i_0-\mu_i\right)-\left(i(1)-\mu_i\right)\right)^2}{\sigma_i^2}\right) \propto \frac{1}{\sqrt{\frac{\sigma_i^2}{a_i^2}}}\exp\left(-\frac{1}{2}\frac{\left(i_0-\frac{(i(1)-\mu_i(1-a_i))}{a_i}\right)^2}{\frac{\sigma_i^2}{a_i^2}}\right), \tag{A.4}$$

which is proportional to a normal distribution. Indeed, from Equation (A.4), we can conclude that

$$i_0 \,\Big|\, \mu_i, \sigma_i, a_i, \{i(t)\}_{t=1}^T \sim \mathcal{N}\left(\frac{i(1)-\mu_i(1-a_i)}{a_i}, \frac{\sigma_i^2}{a_i^2}\right).$$

SM.A.2 Proof of Proposition 2

Taking into consideration the long-term interest rate dynamics and the non-informative prior, we obtain the following posterior joint dynamics:

$$\begin{aligned}
&p\left(\{c(t)\}_{t=1}^T, \Theta_c \,\Big|\, \{i(t), y(t)\}_{t=1}^T, \Theta_y\right)\\
&= \mathcal{L}\left(\{c(t)\}_{t=1}^T \,\Big|\, \{i(t), y(t)\}_{t=1}^T, \Theta_y, \Theta_c\right)\pi\left(\Theta_c \,\Big|\, \{i(t), y(t)\}_{t=1}^T, \Theta_y\right)\\
&= \mathcal{L}\left(\{c(t)\}_{t=1}^T \,\Big|\, \{i(t), y(t)\}_{t=1}^T, \Theta_y, \Theta_c\right)\pi\left(\Theta_c\right)\\
&\propto \left(\frac{1}{\sigma_c}\right)^{T+2}\exp\left(-\frac{1}{2}\frac{\sum_{t=1}^T\left(\log\left(c(t)-c_m(t)\right)-\mu_c^*-a_c\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)-y_c z_y(t)\right)^2}{\sigma_d^2}\right)\\
&\quad\times\exp\left(-\frac{1}{2}\left(\frac{c_{m,0}}{10}\right)^2\right)\exp\left(-\frac{1}{2}\left(\frac{c_0}{10}\right)^2\right)\mathbf{1}_{\{a_c\in(-1,1)\}}
\end{aligned}$$

because the prior on Θ_c is independent of the other priors. Again, based on Bayes' theorem, we can find the full conditional density of the model's parameters. As a matter of fact, we can find the density of μ_c^* by ignoring the terms not in μ_c^*:

$$\begin{aligned}
&\pi\left(\mu_c^* \,\Big|\, \sigma_c^2, a_c, y_c, c_{m,0}, c_0, \{i(t), y(t), c(t)\}_{t=0}^T\right)\\
&\propto \exp\left(-\frac{1}{2}\frac{\sum_{t=1}^T\left(\log\left(c(t)-c_m(t)\right)-\mu_c^*-a_c\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)-y_c z_y(t)\right)^2}{\sigma_c^2}\right)\\
&\propto \exp\left(-\frac{1}{2}\frac{\sum_{t=1}^T\left(\mu_c^*(1-a_c)-\left(\log\left(c(t)-c_m(t)\right)-a_c\log\left(c(t-1)-c_m(t-1)\right)-y_c z_y(t)\right)\right)^2}{\sigma_c^2}\right)\\
&\propto \exp\left(-\frac{1}{2}\frac{T\left(\mu_c^*(1-a_c)\right)^2-2\mu_c^*\sum_{t=1}^T(1-a_c)\left(\log\left(c(t)-c_m(t)\right)-a_c\log\left(c(t-1)-c_m(t-1)\right)-y_c z_y(t)\right)}{\sigma_c^2}\right)\\
&\propto \frac{1}{\sqrt{\frac{\sigma_c^2}{T(1-a_c)^2}}}\exp\left(-\frac{1}{2}\frac{\left(\mu_c^*-\frac{\sum_{t=1}^T\left(\log\left(c(t)-c_m(t)\right)-a_c\log\left(c(t-1)-c_m(t-1)\right)-y_c z_y(t)\right)}{T(1-a_c)}\right)^2}{\frac{\sigma_c^2}{T(1-a_c)^2}}\right)
\end{aligned} \tag{A.5}$$

given that $a_c \in (-1, 1)$. Indeed, Equation (A.5) is proportional to a Gaussian pdf:

$$\mu_c^* \,\Big|\, \sigma_c^2, a_c, y_c, c_{m,0}, c_0, \{i(t), y(t), c(t)\}_{t=0}^T \sim \mathcal{N}\left(\frac{\sum_{t=1}^T \left(\log\left(c(t) - c_m(t)\right) - a_c \log\left(c(t-1) - c_m(t-1)\right) - y_c z_y(t)\right)}{T(1-a_c)}, \frac{\sigma_d^2}{T(1-a_c)^2} \right).$$

The strategy used to obtain a sample of the variance parameter is, again, very similar to what has been done previously:

$$\begin{aligned}
&\pi\left(\sigma_c^2 \,\Big|\, \mu_c^*, a_c, y_c, c_{m,0}, c_0, \{i(t), y(t), c(t)\}_{t=0}^T\right) \\
&\propto \left(\frac{1}{\sigma_c}\right)^{T+2} \exp\left(-\frac{1}{2} \frac{\sum_{t=1}^T \left(\log\left(c(t) - c_m(t)\right) - \mu_c^* - a_c\left(\log\left(c(t-1) - c_m(t-1)\right) - \mu_c^*\right) - y_c z_y(t)\right)^2}{\sigma_d^2} \right) \\
&\propto \left(\sigma_c^2\right)^{-\frac{T}{2}-1} \exp\left(-\frac{\frac{1}{2}\sum_{t=1}^T \left(\log\left(c(t) - c_m(t)\right) - \mu_c^* - a_c\left(\log\left(c(t-1) - c_m(t-1)\right) - \mu_c^*\right) - y_c z_y(t)\right)^2}{\sigma_d^2} \right) \\
&\propto \frac{\left(\frac{1}{2}\sum_{t=1}^T \left(\log\left(c(t) - c_m(t)\right) - \mu_c^* - a_c\left(\log\left(c(t-1) - c_m(t-1)\right) - \mu_c^*\right) - y_c z_y(t)\right)^2\right)^{\frac{T}{2}+1}}{\Gamma\left(\frac{T}{2}+1\right)} \left(\sigma_c^2\right)^{-\frac{T}{2}-1} \\
&\quad \times \exp\left(-\frac{\frac{1}{2}\sum_{t=1}^T \left(\log\left(c(t) - c_m(t)\right) - \mu_c^* - a_c\left(\log\left(c(t-1) - c_m(t-1)\right) - \mu_c^*\right) - y_c z_y(t)\right)^2}{\sigma_c^2} \right),
\end{aligned} \tag{A.6}$$

given that $a_c \in (-1, 1)$. Then, the posterior distribution of σ_c^2 is given by an inverse gamma once more:

$$\sigma_c^2 \,\Big|\, \mu_c^*, a_c, y_c, c_{m,0}, c_0, \{i(t), y(t), c(t)\}_{t=0}^T \sim \mathrm{IG}\left(\frac{T}{2} + 1, \frac{1}{2}\sum_{t=1}^T \left(\log\left(c(t) - c_m(t)\right) - \mu_c^* - a_c\left(\log\left(c(t-1) - c_m(t-1)\right) - \mu_c^*\right) - y_c z_y(t)\right)^2 \right).$$

To obtain the full conditional posterior density of a_c, we apply the following rationale:

$$\begin{aligned}
&\pi\left(a_c \,\Big|\, \mu_c^*, \sigma_c^2, y_c, c_{m,0}, c_0, \{i(t), y(t), c(t)\}_{t=0}^T\right) \\
&\propto \exp\left(-\frac{1}{2} \frac{\sum_{t=1}^T \left(\log\left(c(t) - c_m(t)\right) - \mu_c^* - a_c\left(\log\left(c(t-1) - c_m(t-1)\right) - \mu_c^*\right) - y_c z_y(t)\right)^2}{\sigma_c^2} \right) \mathbf{1}_{(-1,1)}(a_c) \\
&\propto \exp\left(-\frac{1}{2} \frac{\sum_{t=1}^T \left(a_c\left(\log\left(c(t-1) - c_m(t-1)\right) - \mu_c^*\right) - \left(\log\left(c(t) - c_m(t)\right) - \mu_c^* - y_c z_y(t)\right)\right)^2}{\sigma_c^2} \right) \mathbf{1}_{(-1,1)}(a_c)
\end{aligned}$$

$$\propto \exp\left(-\frac{1}{2}\frac{\begin{pmatrix}\sum_{t=1}^{T} a_c^2 \left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)^2 \\ -2a_c \sum_{t=1}^{T}\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)\left(\log\left(c(t)-c_m(t)\right)-\mu_c^*-y_c z_y(t)\right)\end{pmatrix}}{\sigma_c^2}\right)\mathbf{1}_{\{a_c\in(-1,1)\}}$$

$$\propto \frac{1}{\sqrt{\frac{\sigma_c^2}{\sum_{t=1}^{T}\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)^2}}}\exp\left(-\frac{1}{2}\frac{\left(a_c-\frac{\sum_{t=1}^{T}\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)\left(\log\left(c(t)-c_m(t)\right)-\mu_c^*-y_c z_y(t)\right)}{\sum_{t=1}^{T}\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)^2}\right)^2}{\frac{\sigma_c^2}{\sum_{t=1}^{T}\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)^2}}\right)\mathbf{1}_{\{a_c\in(-1,1)\}} \quad \text{(A.7)}$$

Equation (A.7) is the pdf of a truncated Gaussian distribution:

$$a_c \,\middle|\, \mu_c^*, \sigma_c^2, y_c, c_{m,0}, c_0, \{i(t), y(t), c(t)\}_{t=0}^{T} \sim \mathcal{N}_{(-1,1)}\left(\frac{\sum_{t=1}^{T}\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)\left(\log\left(c(t)-c_m(t)\right)-\mu_c^*-y_c z_y(t)\right)}{\sum_{t=1}^{T}\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)^2}, \frac{\sigma_c^2}{\sum_{t=1}^{T}\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)^2}\right).$$

The full conditional distribution of y_c—the parameter dealing with the sensitivity to the dividend yield—is proportional to

$$\pi\left(y_c \,\middle|\, \mu_c^*, \sigma_c^2, a_c, c_{m,0}, c_0, \{i(t), y(t), c(t)\}_{t=0}^{T}\right)$$

$$\propto \exp\left(-\frac{1}{2}\frac{\sum_{t=1}^{T}\left(\log\left(c(t)-c_m(t)\right)-\mu_c^*-a_c\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)-y_c z_y(t)\right)^2}{\sigma_c^2}\right)$$

$$\propto \exp\left(-\frac{1}{2}\frac{\sum_{t=1}^{T}\left(y_c z_y(t)-\left(\log\left(c(t)-c_m(t)\right)-\mu_c^*-a_c\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)\right)\right)^2}{\sigma_c^2}\right)$$

$$\propto \exp\left(-\frac{1}{2}\frac{y_c^2\sum_{t=1}^{T} z_y^2(t)-2y_c\sum_{t=1}^{T} z_y(t)\left(\log\left(c(t)-c_m(t)\right)-\mu_c^*-a_c\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)\right)}{\sigma_c^2}\right)$$

$$\propto \frac{1}{\sqrt{\frac{\sigma_c^2}{\sum_{t=1}^{T} z_y^2(t)}}}\exp\left(-\frac{1}{2}\frac{\left(y_c-\frac{\sum_{t=1}^{T} z_y(t)\left(\log\left(c(t)-c_m(t)\right)-\mu_c^*-a_c\left(\log\left(c(t-1)-c_m(t-1)\right)-\mu_c^*\right)\right)}{\sum_{t=1}^{T} z_y^2(t)}\right)^2}{\frac{\sigma_c^2}{\sum_{t=1}^{T} z_y^2(t)}}\right) \quad \text{(A.8)}$$

given that $a_c \in (-1, 1)$, which yields the following posterior distribution for y_c:

$$y_c \,\Big|\, \mu_c^*, \sigma_c^2, a_c, c_{m,0}, c_0, \{i(t), y(t), c(t)\}_{t=0}^T$$
$$\sim \mathcal{N}\left(\frac{\sum_{t=1}^T z_y(t) \left(\log\left(c(t) - c_m(t)\right) - \mu_c^* - a_c \left(\log\left(c(t-1) - c_m(t-1)\right) - \mu_c^*\right)\right)}{\sum_{t=1}^T z_y^2(t)}, \frac{\sigma_c^2}{\sum_{t=1}^T z_y^2(t)} \right).$$

SM.B More Results Using the Non-informative Prior

Figure SM.1 shows the density estimates of the parameters for the non-informative prior distributions. The posterior histograms of the simulated parameters are complemented by moment-matched Gaussian distributions.

Figure SM.2 shows the trace plots for all the model parameters under scrutiny.

Figure SM.3 shows the funnels of doubt in the case of the non-informative prior. Table SM.1 presents the descriptive statistics of the four economic variables under study for the non-informative case. Figure SM.4 exhibits densities for the four different economic variables for a time horizon of 50 years.

Figure SM.5 and Table SM.2 resemble Figure 8 and Table 3, respectively, but for the non-informative prior. One could yield the exact same conclusion that we have found with the subjective prior by investigating the results obtained with the non-informative prior.

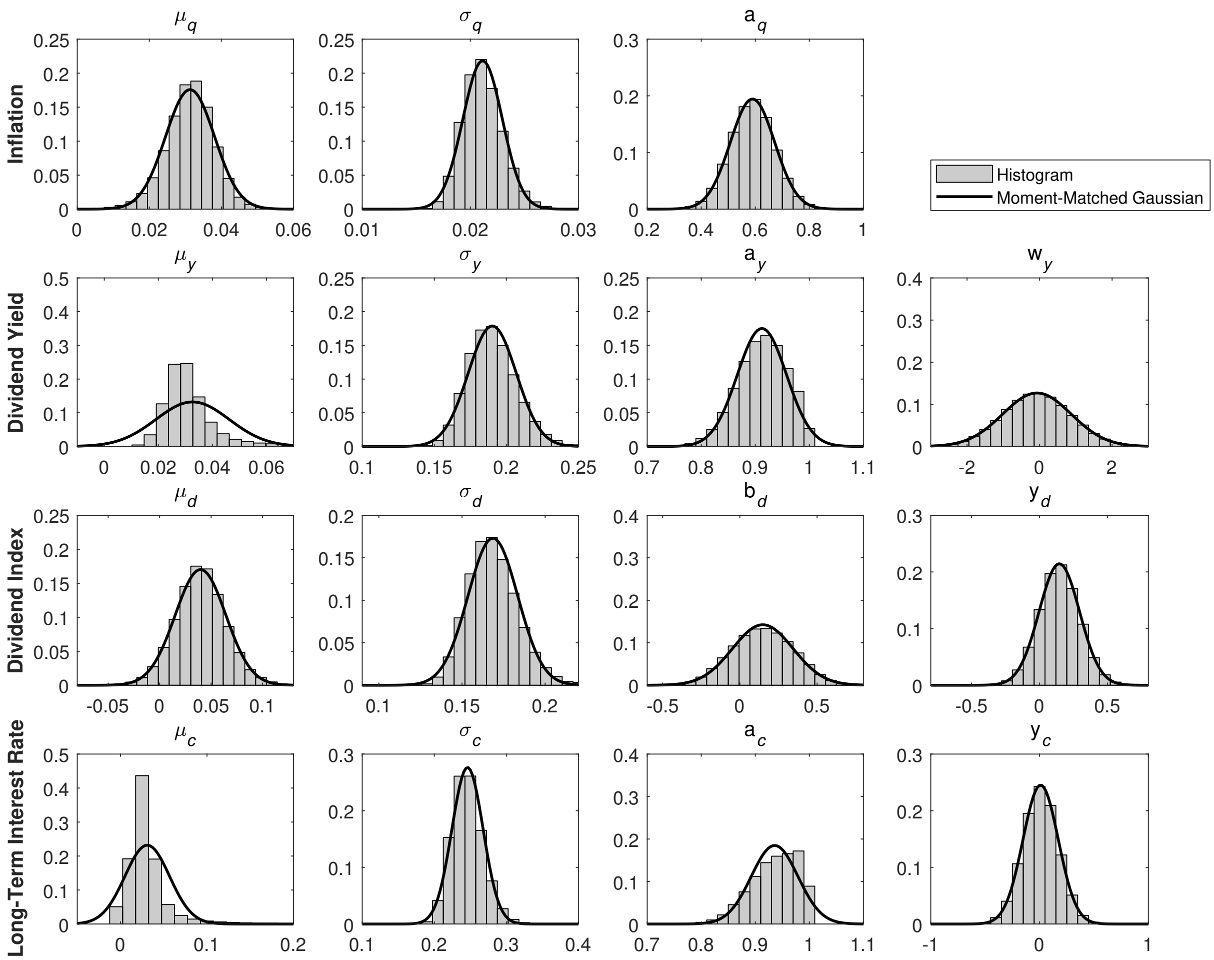

Figure SM.1: **Posterior Distributions for Model Parameter: Non-informative Prior.**
This figure exhibits the (marginal) posterior distribution for each parameter obtained using the Markov chain Monte Carlo methodology explained in Section 5. For each parameter, we obtain a sample of 100,000 values from which we construct a histogram. In this study, we focus on post-World War II data, i.e., data for the year 1945 and after. In addition to determining the histogram, we obtain the moment-matched Gaussian distribution (solid line). The first column of Table 1 reports the values of such means and standard deviations needed to match the moment of the normal distributions.

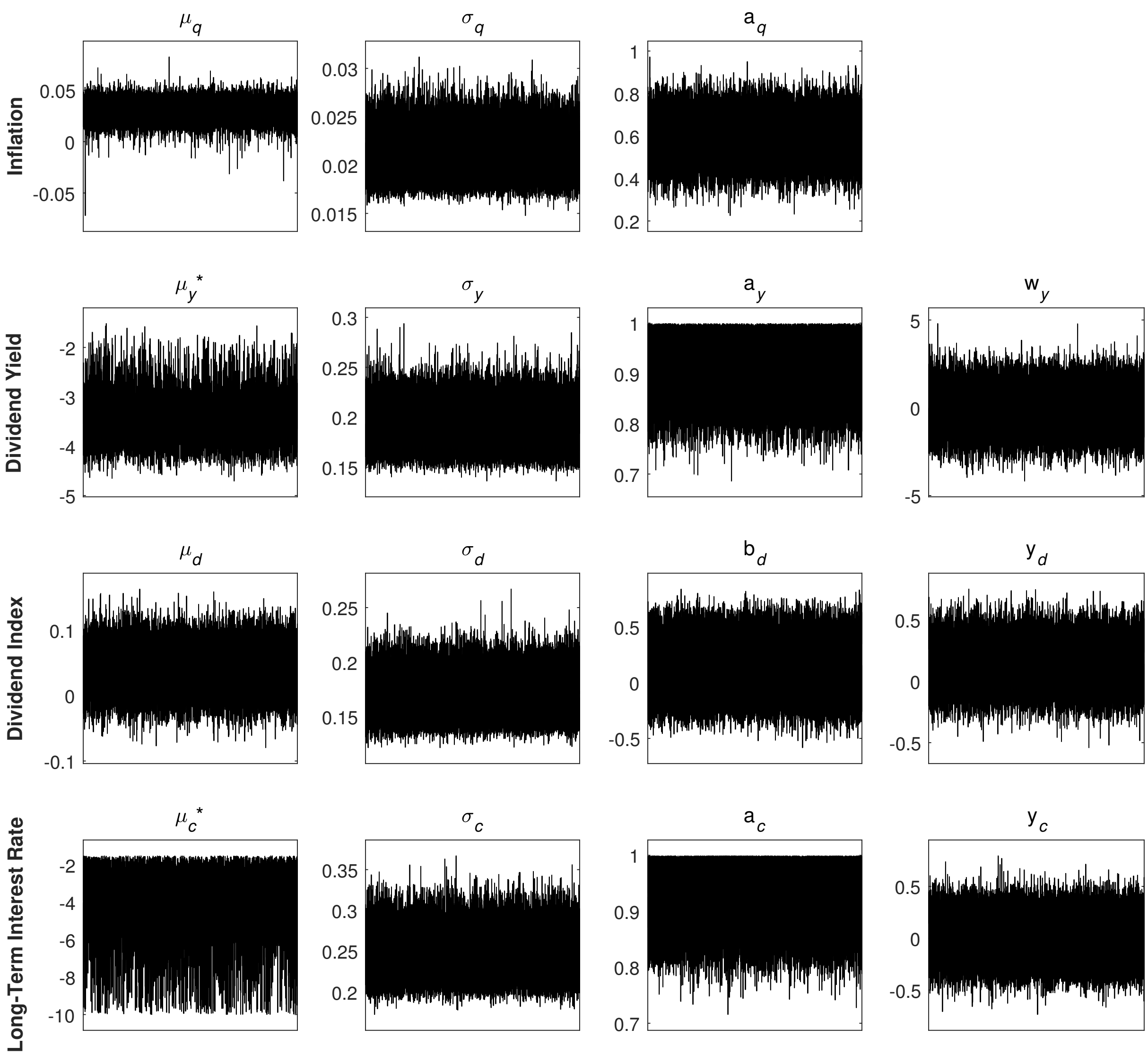

Figure SM.2: **Trace Plots for Model Parameters: Non-informative Prior.**
This figure exhibits the trace plot of the draws from the posterior distribution for each parameter obtained using the Markov chain Monte Carlo methodology explained in Section 5. For each parameter, we obtain a sample of 100,000 values. In this study, we focus on post-World War II data, i.e., data for the year 1945 and after.

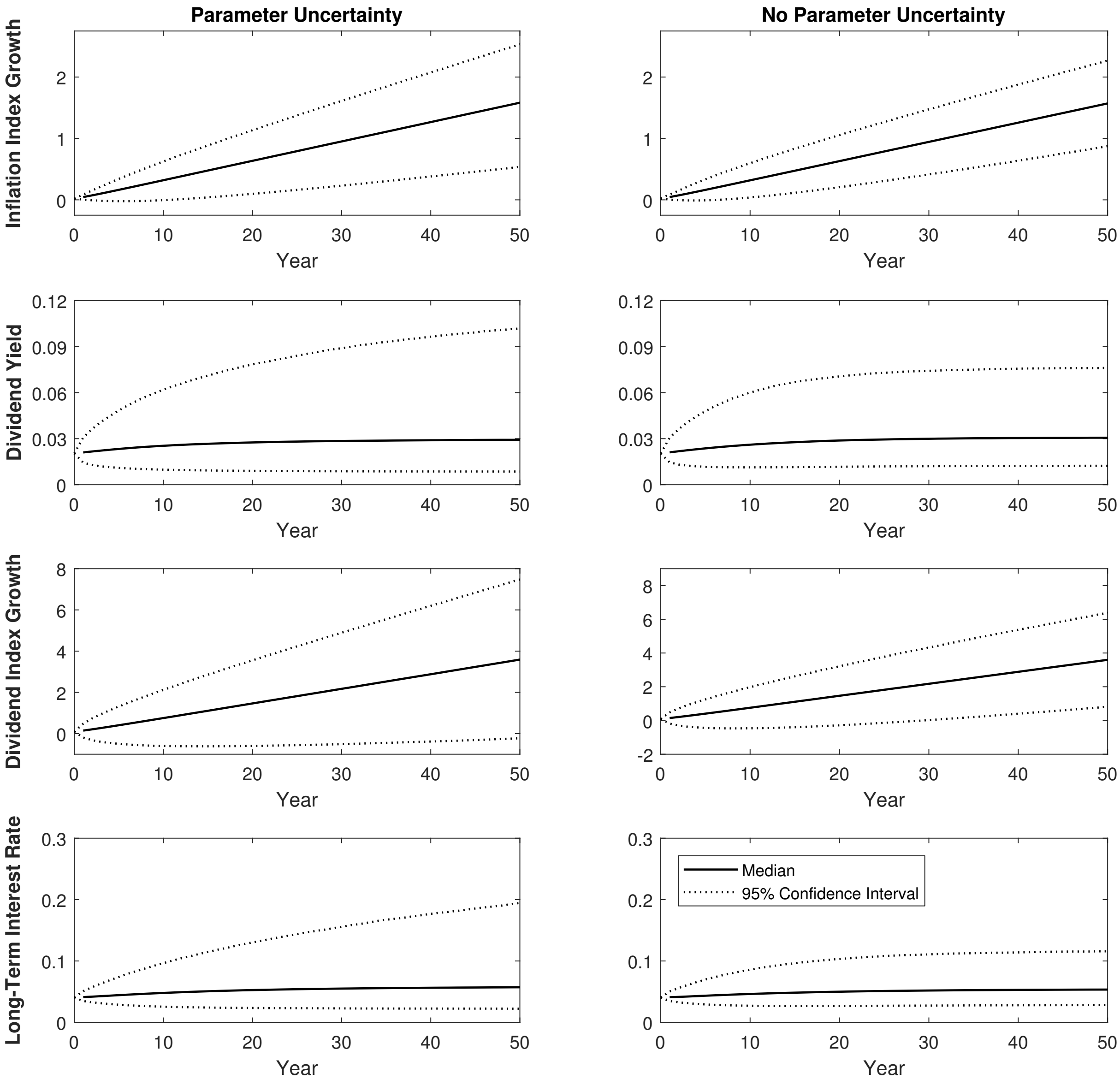

Figure SM.3: **Funnels of Doubt for the Four Economic Variables: Non-informative Prior.**
This figure shows funnels of doubt with (left column) and without (right column) parameter uncertainty, and for each of the four economic variables under study: the total inflation growth, the dividend yield, the total dividend growth, and the long-term interest rate. These plots present the median (solid) as well as the 95% confidence interval (dashed line) for each year. The leftmost figures present the funnels of doubt in the case of parameter uncertainty based on the Markov chain Monte Carlo methodology explained in Section 5. Forr the rightmost figures, we use no parameter uncertainty: instead of using the posterior sets of parameters, we use the average value of each parameters, as given in Table 1.

Table SM.1: **Descriptive Statistics and Quantiles of Economic Variables over Different Time Horizons: Non-informative Prior.**

	Parameter Uncertainty		**No Parameter Uncertainty**	
	Average	Standard Deviation	Average	Standard Deviation
Total Inflation Index Growth				
5 Years	0.1625	0.0920	0.1626	0.0871
10 Years	0.3176	0.1591	0.3180	0.1426
15 Years	0.4742	0.2133	0.4742	0.1833
25 Years	0.7873	0.3058	0.7876	0.2451
50 Years	1.5701	0.5039	1.5704	0.3557
100 Years	3.1353	0.8667	3.1352	0.5087
Dividend Yield				
5 Years	0.0249	0.0097	0.0253	0.0094
10 Years	0.0280	0.0137	0.0285	0.0127
15 Years	0.0301	0.0166	0.0306	0.0145
25 Years	0.0327	0.0214	0.0328	0.0160
50 Years	0.0358	0.0322	0.0341	0.0167
100 Years	0.0387	0.0577	0.0342	0.0168
Total Dividend Index Growth				
5 Years	0.4067	0.4567	0.4058	0.4299
10 Years	0.7563	0.6851	0.7552	0.6224
15 Years	1.1114	0.8745	1.1095	0.7689
25 Years	1.8226	1.2063	1.8202	1.0001
50 Years	3.5980	1.9370	3.5968	1.4228
100 Years	7.1520	3.2881	7.1492	2.0168
Long-Term Interest Rate				
5 Years	0.0461	0.0117	0.0449	0.0104
10 Years	0.0512	0.0187	0.0488	0.0151
15 Years	0.0553	0.0247	0.0516	0.0181
25 Years	0.0612	0.0345	0.0551	0.0212
50 Years	0.0687	0.0506	0.0580	0.0229
100 Years	0.0738	0.0632	0.0587	0.0233

This table presents descriptive statistics with and without parameter uncertainty and for each of the four economic variables under study: the total inflation index growth, the dividend yield, the total dividend index growth, and the long-term interest rate. These statistics are calculated for different time horizons: 5, 10, 15, 25, 50 and 100 years. In the case without parameter uncertainty, instead of using the posterior sets of parameters, we use the average value of each parameter as given in Table 1 to generate 1,000,000 observations for each year.

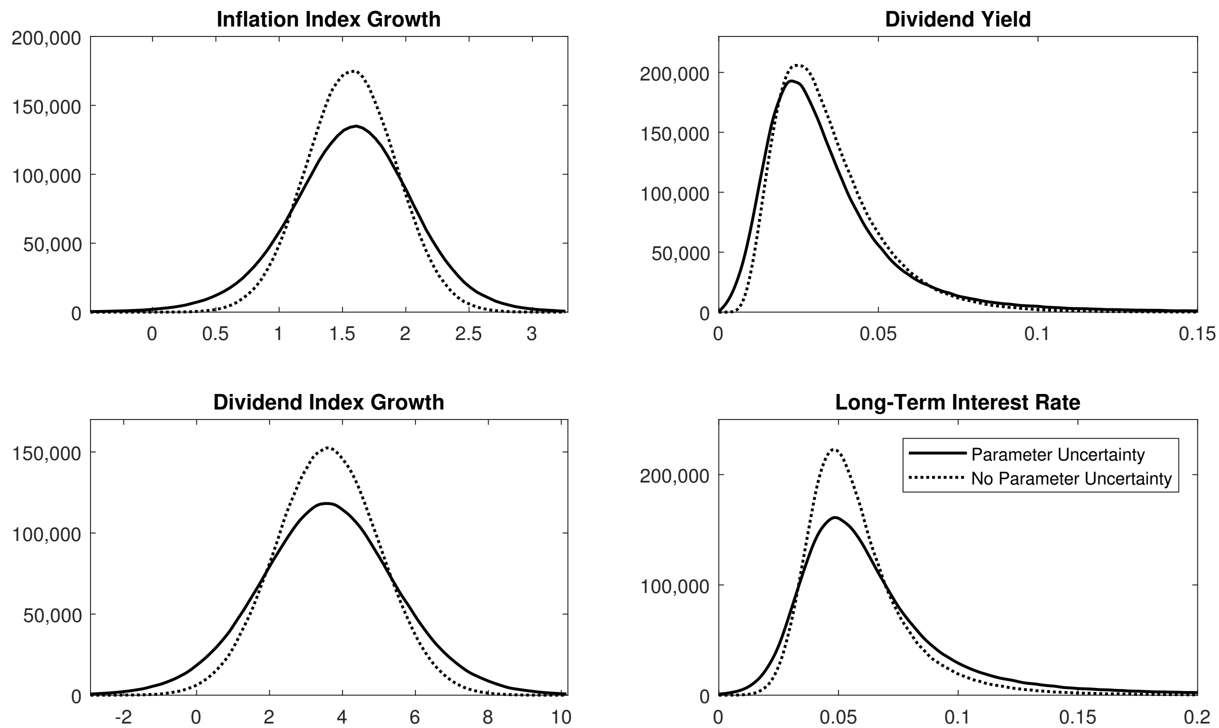

Figure SM.4: **Kernel Smoothed Densities of the Economic Variables for a Period of 50 Years: Non-informative Prior.**
This figure shows densities with (left column) and without (right column) parameter uncertainty, and for each of the four economic variables under study: the total inflation growth, the dividend yield, the total dividend growth, and the long-term interest rate. The leftmost figures present the densities in the case of parameter uncertainty based on the Markov chain Monte Carlo methodology explained in Section 5. For the rightmost figures, we use no parameter uncertainty: instead of using the posterior sets of parameters, we use the average value of each parameter, as given in Table 1.

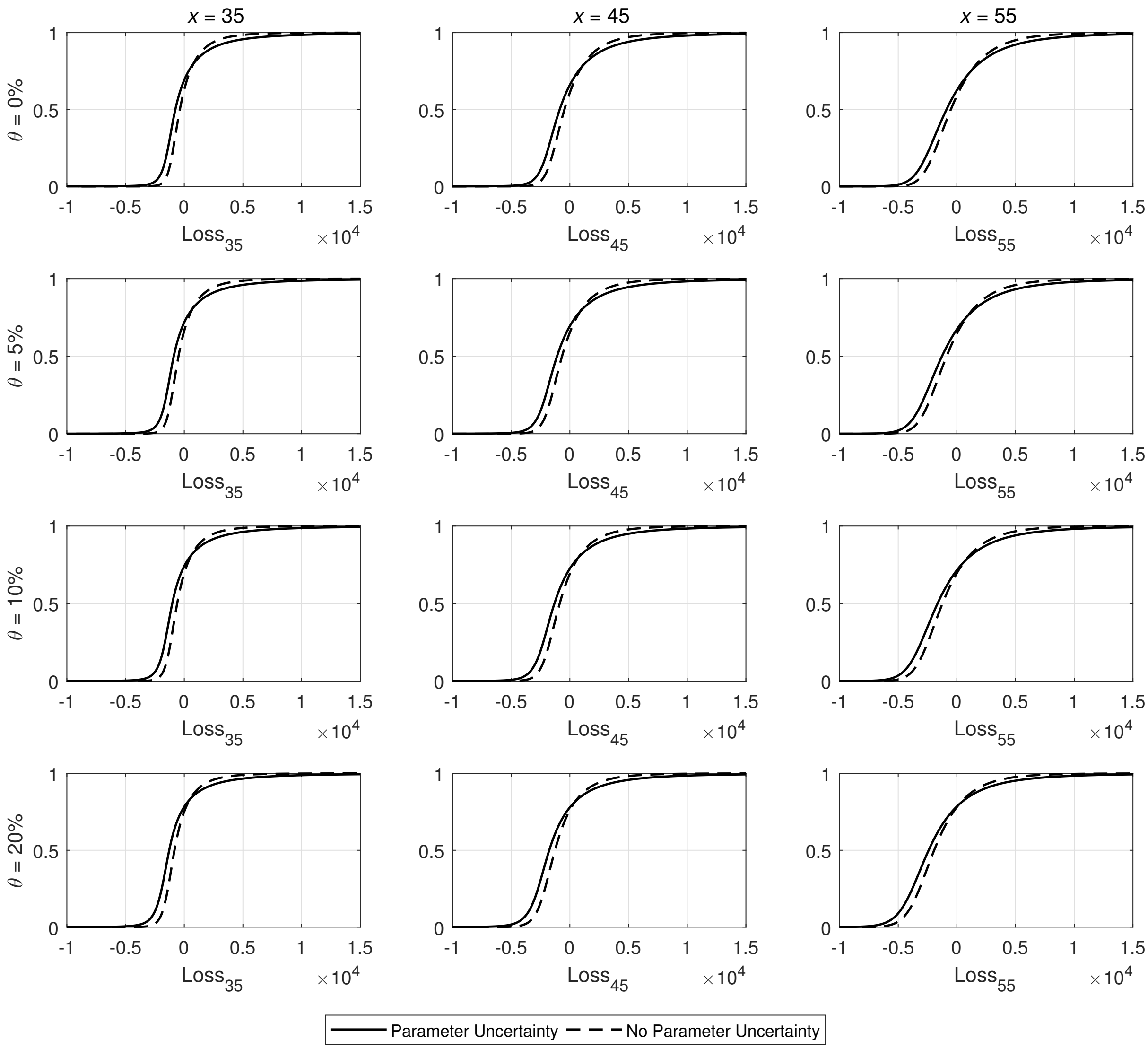

Figure SM.5: **Cumulative Distribution Functions of the Loss at Issue as a Function of Different Initial Ages and Levels of Loading for a Portfolio of 100 Annuities: Non-informative Prior.**
This figure shows cumulative distribution functions for different initial ages (35, 45, and 55 years old) and levels of loading (0, 5, 10, and 20%). Two cases are shown in this figure: parameter uncertainty (solid line) and no parameter uncertainty (dashed line). For the parameter uncertainty case, we use the Markov chain Monte Carlo methodology explained in Section 5. Under no parameter uncertainty, we use the average value of each parameter, as given in Table 1. Associated with each of these paths, we generate the indicator variables $\mathcal{L}_{t,j}$ and $\mathcal{D}_{t,j}$ for 100 lives using the Gompertz model fitter to US data (Pflaumer, 2011). Then, based on these simulations, we obtain the loss at issue under each scenario to obtain a sample of losses, Loss_x.

Table SM.2: **Descriptive Statistics and Quantiles of Loss at Issue for a Portfolio of 100 Annuities: Non-informative Prior.**

	Parameter Uncertainty				No Parameter Uncertainty			
	Premium $\pi(1+\theta)$	Standard Deviation	$Q(0.95)$	CTE(0.95)	Premium $\pi(1+\theta)$	Standard Deviation	$Q(0.95)$	CTE(0.95)
$\boldsymbol{x = 35}$								
$\theta = 0\%$	194.40	3241.40	4455.24	9797.11	166.31	1547.37	2817.62	6580.91
$\theta = 5\%$	204.12	3220.03	4245.52	9550.53	174.63	1536.73	2660.88	6356.40
$\theta = 10\%$	213.84	3199.69	4037.86	9305.00	182.94	1526.94	2504.46	6125.62
$\theta = 20\%$	233.29	3162.53	3622.78	8817.72	199.58	1510.03	2197.29	5671.73
$\boldsymbol{x = 45}$								
$\theta = 0\%$	385.96	3682.90	5636.60	11189.28	347.40	2085.66	3860.32	8127.89
$\theta = 5\%$	405.26	3653.74	5315.11	10823.18	364.76	2069.53	3600.12	7781.35
$\theta = 10\%$	424.56	3626.24	4995.00	10459.08	382.13	2054.90	3341.89	7432.60
$\theta = 20\%$	463.15	3576.52	4359.82	9737.16	416.87	2030.26	2824.81	6740.38
$\boldsymbol{x = 55}$								
$\theta = 0\%$	973.31	3973.55	6667.52	11939.16	911.46	2663.70	4951.50	9345.95
$\theta = 5\%$	1021.98	3942.67	6189.01	11421.85	957.03	2644.38	4529.48	8842.14
$\theta = 10\%$	1070.64	3913.91	5712.79	10907.21	1002.60	2627.26	4109.59	8335.64
$\theta = 20\%$	1167.97	3862.88	4774.67	9885.40	1093.75	2599.79	3275.86	7349.97

This table summarizes descriptive statistics and risk measures of the loss at issue (per annuity sold) for different initial ages (35, 45, and 55 years old) and levels of loading (0, 5, 10, and 20%). Two cases are used in this table: parameter uncertainty (leftmost columns) and no parameter uncertainty (rightmost columns). For the parameter uncertainty case, we use the Markov chain Monte Carlo methodology explained in Section 5 to obtain 100,000 sets of posterior parameters based on the non-informative prior (see Section 6.2). Then, for each set, we generate 10 paths which yields a final sample of 1,000,000 observations for each variable and for each year. Under no parameter uncertainty, we use the average value of each parameter, as given in Table 1, to generate 1,000,000 observations on each year. Associated with each of these paths, we generate the indicator variables $\mathcal{L}_{t,j}$ and $\mathcal{D}_{t,j}$ for 100 lives using the Gompertz model fitted to US data (Pflaumer, 2011). Then, based on these simulations, we obtain the loss at issue under each scenario to determine a sample of this random variable. The table shows, for both cases, the annual premium paid by the annuitant, the standard deviation of the losses at issue, the 95$^{\text{th}}$ quantile of the loss distribution $Q(0.95)$, and the conditional tail expectation (CTE) at the 95% level.