BOOK REVIEWS

WAVELETS, VIBRATIONS AND SCALINGS
(CRM Monograph Series 9)

By Yves Meyer: 133 pp., US$29.00, ISBN 0 8218 0685 8
(American Mathematical Society, 1997).

GENERALIZED WAVELETS AND HYPERGROUPS

By K. Trimèche: 349 pp., £81.00, ISBN 0 90 5699 080 2
(Gordon and Breach, 1997).

These two books show the wide range of areas now lying within the scope of wavelet analysis. Meyer discusses several deep results in function theory on $\mathbf{R}^n$, where the roots of wavelet analysis lie, while Trimèche describes extensions of wavelet theory to a diverse range of abstract group-theoretic settings.

Before discussing the books, it might help to sketch the background of the theory of wavelets. Modern work in wavelet analysis began with the study of the Continuous Wavelet Transform (CWT). Let $\Psi \in L^2(\mathbf{R}^n)$, and assume that $\int_{\mathbf{R}^n} \Psi(x)\, dx = 0$, and that $\int_0^\infty |\hat{\Psi}(t\xi)|^2\, t^{-1}\, dt = 1$ for $\xi \neq 0$. In this last equality, $\hat{\Psi}$ is the Fourier transform of Ψ. The CWT of $f \in L^2(\mathbf{R}^n)$, using Ψ as the *analysing wavelet*, is denoted by $\mathscr{W}f$ and is defined by

$$\mathscr{W}f(a,b) = \int_{\mathbf{R}^n} f(t) \frac{1}{a^n} \overline{\Psi\left(\frac{t-b}{a}\right)}\, dt, \quad b \in \mathbf{R}^n,\ a > 0. \tag{1}$$

In the integrand in (1), the analysing wavelet is *translated* by $b \in \mathbf{R}^n$ and *dilated* by $a \in \mathbf{R}_+$. The CWT has an inversion formula:

$$f(x) = \int_0^\infty \int_{\mathbf{R}^n} \mathscr{W}f(a,b) \frac{1}{a^n} \Psi\left(\frac{x-b}{a}\right) db\, \frac{da}{a}. \tag{2}$$

The most celebrated results in wavelet theory—the ones which lend themselves most readily to computations and applications—concern orthonormal wavelet bases. For simplicity, we shall here restrict ourselves to $\mathbf{R}$ (although Meyer treats the case $\mathbf{R}^n$ as well). If $\psi \in L^2(\mathbf{R})$ generates a set $\{2^{j/2} \psi(2^j x - k)\}_{j,k \in \mathbf{Z}}$ that is an orthonormal basis of $L^2(\mathbf{R})$, then this basis is called a *wavelet basis*. Some of the most important wavelet bases are the ones discovered by Meyer and Lemarié, for which ψ belongs to the Schwartz class $\mathscr{S}(\mathbf{R})$, and the ones discovered by Daubechies, for which ψ belongs to $C_c^m(\mathbf{R})$ for some $m = 0, 1, 2, \ldots$. Let $\psi_{j,k}(x)$ denote $2^{-j} \psi(2^{-j} x - k)$, and let $\langle f, \psi_{j,k} \rangle := \int_{\mathbf{R}} f(x) \overline{\psi_{j,k}(x)}\, dx$. Then $f \in L^2(\mathbf{R})$ can be expanded in the wavelet series

$$f(x) = \sum_{j,k \in \mathbf{Z}} \langle f, \psi_{j,k} \rangle \psi(2^{-j} x - k). \tag{3}$$

More important, the wavelet coefficients $\langle f, \psi_{j,k} \rangle$ are well-defined for $f \in L^p(\mathbf{R})$, $1 < p < \infty$, as well as many other function spaces, and (3) remains valid. Similarly, for

$\Psi \in \mathscr{S}(\mathbf{R}^n)$, the CWT in (1) remains well-defined over a wide range of function spaces.

We now turn to a more detailed description of Meyer's book. The first half of the book describes some of Meyer's new results on wavelet-based characterizations of local Hölder exponents, and other quantities which Meyer calls *scaling exponents*. In Chapters 1–3, Meyer discusses the calculation of the local Hölder exponent and of a new type of scaling exponent. A function $f(x)$ on $\mathbf{R}^n$ is called Lip α at x_0 if there is a constant $K > 0$ and a polynomial $p_{x_0}(x)$ of degree $\lfloor \alpha \rfloor$ such that $|f(x) - p_{x_0}(x)| \leqslant K|x - x_0|^\alpha$. The *local Hölder exponent* $H(f; x_0)$ of a function f at x_0 is defined by $H(f; x_0) = \sup\{\alpha: f \text{ is Lip } \alpha \text{ at } x_0\}$; the new scaling exponent $\beta(f; x_0)$ is defined by $\beta(f; x_0) = \sup\{s \in \mathbf{R}: f \in \Gamma^s(x_0)\}$, where $\Gamma^s(x_0) = \bigcup_{s' \in \mathbf{R}} C_{x_0}^{s,s'}$. Here, each $C_{x_0}^{s,s'}$ is one of the *two-microlocal spaces* used in the study of non-linear PDEs. Meyer shows that these scaling exponents can be computed either from size estimates on the values of a CWT of f or from size estimates on its wavelet coefficients. He also shows that the computation of $H(f; x_0)$ from such size estimates requires an additional condition. Namely, that f is a continuous function satisfying $|f(x+y) - f(x)| \leqslant \omega(|y|)$, where $\omega: [0, \infty) \to [0, \infty)$ is a continuous increasing function for which $\omega(h) = O((\log 1/h)^{-m})$ for some $m \geqslant 1$.

The book concludes with two chapters that describe a new type of wavelet bases. The elements of these bases have their supports and the supports of their Fourier transforms essentially concentrated in rectangles that partition the time-frequency plane in a manner which provides a much finer division near the zero-frequency axis than is the case for standard wavelet bases. These new wavelet bases provide convergent wavelet expansions—not available with other wavelet bases—for the functions $|x|^\gamma$ when $0 < \gamma < 1$. Meyer also uses them to characterize the spaces $S_\gamma := \{f: f \text{ continuous and Lip}\gamma \text{ and } |f(x)| \leqslant C\}$ using size estimates on wavelet coefficients. These results are a perfect illustration of the flexibility and power of wavelet analysis: *the ability it provides to custom design new wavelet bases that are better adapted to the problem at hand.*

In addition to these general theoretical results, Meyer's book is also sprinkled throughout with a fascinating collection of examples and counter-examples. For instance, in Chapter 1, Meyer carefully describes the construction of the Bolzano function: a fractal function that for each $x_0 \in [0, 1]$ has a cusp singularity at x_0.

Meyer's book is suitable for professional researchers in function theory, or as a text for an advanced graduate seminar. It contains no exercises, but Meyer writes in a compressed yet lucid style which invites the reader's participation.

While Meyer's book describes some deep results in a well-established body of work, Trimèche's book is concerned with providing the foundations for building a generalized wavelet analysis suited to a variety of group-theoretic settings. He describes numerwous generalizations of the CWT within group-theoretic settings, such as hypergroups, Gelfand pairs and semisimple Lie groups. Besides making use of generalized Fourier transforms, Trimèche also examines generalized Radon transforms on hypergroups and their inversion formulas via generalized wavelets.

Rather than describing all of the various group-theoretic settings considered by Trimèche, we shall instead concentrate on one representative case: hypergroups. For those who are interested in reading Trimèche's book and who are not well-acquainted with the theory of hypergroups, [1] and [2] are strongly recommended as prerequisites. A hypergroup is a locally compact Hausdorff space X for which a notion of convolution is defined, and for which there are generalized translation operators T_x for each $x \in X$. These translation operators act on compactly supported continuous

functions f via $T_x f(y) = \int_X f(t) \, d(\delta_x * \delta_y)(t)$, where δ_x, δ_y are the Dirac measures at x, y. In addition, there is a continuous involution $x \mapsto x^-$ from X onto itself, which satisfies $(x^-)^- = x$ for all x. For commutative hypergroups (where $T_x \circ T_y = T_y \circ T_x$), as well as for compact or discrete hypergroups, there is a *Haar measure* m that satisfies $\int_X T_x f(y^-) \, dm(y) = \int_X f(y) \, dm(y)$. There also exists a unique *Plancherel measure* π on the dual space $\hat{X}$ of hermitian characters that satisfies $\int_X f(x)\overline{g(x)} \, dx = \int_{\hat{X}} \hat{f}(\chi)\overline{\hat{g}(\chi)} \, d\pi(\chi)$. Hypergroups include all locally compact abelian groups, and include the *Chébli–Trimèche hypergroups* arising from certain singular Sturm–Liouville differential operators.

In order to define a wavelet transform that generalizes (1), it is necessary to generalize the translation and dilation of an analysing wavelet. For a hypergroup, the translation operators serve as the generalizations of translation. Trimèche shows that, for most of the important hypergroups, there is a function $g \in L^2(X, m)$ that satisfies, for some positive constant C_g,

$$C_g = \int_0^\infty |\hat{g}(a\chi)|^2 \, a^{-1} \, da$$

for almost all $\chi \in \operatorname{supp} \pi$. He calls this function g a *generalized wavelet*. Finally, Trimèche defines the *dilation* of the wavelet g by $a \in \mathbf{R}_+$ as the function g_a satisfying $\widehat{g_a}(\chi) = \hat{g}(a\chi)$ a.e. in $\operatorname{supp} \pi$. One problem for future research is, as Trimèche points out, that closed formulas for g_a are known for only a few specific cases. Based on the dilations g_a and the translations T_x, Trimèche defines a *generalized wavelet transform* $\Phi_g f$, of $f \in L^2(X, m)$, as

$$\Phi_g f(a, b) = \int_X f(x)\overline{T_b g_a(x)} \, dm(x), \quad a \in \mathbf{R}_+, \, b \in X. \tag{4}$$

He then proves an inversion formula

$$f(x) = \frac{1}{C_g} \int_0^\infty \int_X \Phi_g f(a, b) \, T_b g_a(x) \, dm(b) \frac{da}{a} \tag{5}$$

under various hypotheses.

Having established the existence of a CWT and discussed its inversion for hypergroups, Trimèche moves on to the same ideas in other group-theoretic settings. He establishes the existence of a CWT and its inverse for Gelfand pairs, semisimple Lie groups and Chébli–Trimèche hypergroups.

Trimèche describes the foundations of a generalized wavelet analysis. What is needed now, and perhaps will be supplied by future research, are applications of these ideas. The classical CWT, defined in (1), provides a means of characterizing various function-theoretic properties (Lip α, L^p-norm, scaling exponents), and has applications to operator theory. Are there analogous results in the group-theoretic settings that Trimèche describes?

Trimèche's book should be considered a guide to the literature, rather than a self-contained reference. Many proofs of foundational results are omitted. References are always supplied, but access to a good library is essential for those readers who are not already well-acquainted with hypergroups, Lie groups or Gelfand pairs. Parts of the book could serve as outlines for advanced graduate seminars, or the book could serve as background for researchers in abstract harmonic analysis who are looking to extend applications of wavelet theory.

References

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University of Wisconsin, Eau Claire   James S. Walker

ERDÖS ON GRAPHS: HIS LEGACY OF UNSOLVED PROBLEMS

By Fan Chung and Ron Graham: 142 pp., US$30.00, isbn 1 56881 079 2
(A. K. Peters, 1998).

Paul Erdös was one of the world's greatest mathematicians. He worked in many areas of mathematics, but the area in which he published about half of his papers, about 700 altogether, was graph theory. The book under review is a very fitting tribute. His peripatetic life-style, constantly on the move, and maintaining close contact with mathematicians all over the world, his warmth and his sense of humour, endeared him to very many, and earned him the affectionate sobriquet 'Uncle Paul'. The present healthy state of graph theory owes much not just to his own numerous discoveries, but to the enormous number of problems he posed—some with a cash prize offered for the solution. Sometimes the problems were easy to solve; the £10 a colleague and I shared was for solving one of these. But many led to the development of new techniques for making substantial advances. And, of course, many remain unsolved.

Fan Chung and Ron Graham, both excellent mathematicians, have gathered together problems into six chapters: Ramsey theory; Extremal graph theory; Colouring, packing and covering; Random graphs and graph enumeration; Hypergraphs; and Infinite graphs.

Each chapter sets the scene well, giving a full introduction and the background to the main areas of interest in the topic. Many of the main conjectures are included, not just those due to Erdös, and many of the main results, including recent ones, are also given. Each chapter can be read independently of the others, and could serve as an interesting introduction for a non-specialist.

It seems very likely that this book will serve to stimulate further activity. Some of the problems will be solved, but many of the well-known problems seem likely to remain unsolved for many years. An example is the Erdös–Faber–Lovász conjecture from 1972 that 'any simple hypergraph H on n vertices has chromatic index at most n'. At first, Erdös was sure it must be trivial, but quickly realized it was harder than he thought. In the past twenty-five years, many mathematicians have tried to prove it without much success.

The book has an interesting Preface and an opening section 'Remembering Uncle Paul', and concludes with three 'Erdös stories' written by Andrew Vázsonyi, a boyhood friend of Erdös. These almost seem to bring Erdös back to life, and will show those who never met Erdös how much he was loved. There are so many 'Erdös stories' that it is very likely that many will go unrecorded, which will be a shame since mathematics lives as much through the people who do it as it does through the theorems they prove.

When I started as a mathematician, there was a fashion to criticize Erdös, or the 'Hungarian School of Mathematics', for concentrating on problems, in the belief that he (or the school) did not develop mathematical theories, or did not concentrate on mathematical structures. I always found it hard to see what the critics meant. Part of Erdös' own response, taken from his last paper on problems, written shortly before his death, was as follows.

> Problems have always been an essential part of my mathematical life. A well chosen problem can isolate an essential difficulty in a particular area, serving as a benchmark against which progress in this area can be measured. An innocent looking problem often gives no hint as to its true nature. It might be like a 'marshmallow', serving as a tasty tidbit supplying a few moments of fleeting enjoyment. Or it might be like an 'acorn', requiring deep and subtle new insights from which a mighty oak can develop.

If anyone still seriously doubts Erdös' interest in mathematical structures, I suggest that they read the chapter on random graphs and graphical enumeration.

This book is not just a wonderful tribute to Paul Erdös. It contains so many problems to stimulate the imagination of mathematicians as they browse through it, that it will be a source of problems and an inspiration to others for many years to come.

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ARITHMETICAL SIMILARITIES: PRIME DECOMPOSITION AND FINITE
GROUP THEORY
(Oxford Mathematical Monographs)

By NORBERT KLINGEN: 274 pp., £55.00, ISBN 0 19 853598 8
(Clarendon Press, 1998).

The title of this book, *Arithmetical similarities*, refers to coincidences in the way primes (that is, prime ideals) decompose in different extensions of a given algebraic number field. For abelian extensions of number fields, such decompositions may be understood using class field theory. This book examines the extent to which the theory for abelian extensions remains valid for nonabelian extensions of algebraic number fields. It develops the theory of several notions of equivalence of number fields based on prime decompositions, including the apparently weak concept of Kronecker equivalence of number fields introduced by Jehne, and also the much stronger notion of arithmetical equivalence. These concepts may be formulated in terms of certain properties of Galois groups, and many of the results in the book involve a nontrivial interplay between group theory and algebraic number theory. The results reported in the book were proved by a joint effort of number theorists and group theorists, and the author is to be congratulated on providing this very accessible account.

The *Kronecker set* $D(K|k)$ of an algebraic extension $K|k$ of number fields is defined as the set of all primes p of k for which there exists a prime $\mathscr{P}$ of K lying over p having residue degree 1. Two finite extensions L and K of k are said to be *Kronecker*

equivalent over k if their Kronecker sets $D(L \mid k)$ and $D(K \mid k)$ differ by a finite number of primes. The stronger notion of arithmetical equivalence is defined in terms of the *decomposition types* $A_{K \mid k}(p)$ in K of primes p of k: $A_{K \mid k}(p) = (f_1, \ldots, f_r)$ is the sequence in non-decreasing order of the residue degrees $f_i = f(\mathscr{P}_i \mid p)$ of all primes $\mathscr{P}_i$ in K lying above p. Two finite extensions $L \mid k$ and $K \mid k$ are said to be *arithmetically equivalent* over k if $A_{L \mid k}(p) = A_{K \mid k}(p)$ for all but a finite number of primes p of k. This is equivalent to having the same zeta function. Each of these concepts admits an equivalent group theoretic formulation. If $N \mid k$ is a Galois extension containing L and K, $G = G(N \mid k)$ is the Galois group of $N \mid k$, and $U = G(N \mid L)$, $V = G(N \mid K)$, then L and K are Kronecker equivalent, or arithmetically equivalent, over k, if and only if the transitive permutation characters for G induced by the subgroups U and V have the same zeros, or are equal, respectively.

Several chapters of the book explore these notions of equivalence, using both number theoretic and group theoretic methods. For example, the Kronecker equivalence class $\mathscr{K}$ corresponding to a finite extension $K \mid k$ may be infinite. This is the case whenever either $K \mid k$ has a nontrivial automorphism of odd order or of order 8, or $K \mid k$ has a subgroup of automorphisms isomorphic to the quaternion group Q_8. On the other hand, it is possible for $\mathscr{K}$ to consist solely of conjugates of the field K, and in this case $\mathscr{K}$ is called *Kroneckerian*. It turns out that every quadratic extension is Kroneckerian. Jehne began this study of quadratic extensions in 1977, and reduced the proof that they are Kroneckerian to a problem concerning maximal subgroups of finite nonabelian simple groups. Several mathematicians, including Klingen, Brandl and Saxl, made contributions culminating in a complete proof in 1986 which required the full force of the finite simple group classification.

The book begins with a review of the basic theory of prime decompositions in algebraic number field extensions, and of certain associated actions of the Galois groups. It contains sufficient background in both number theory and group theory to be accessible to mathematicians and graduate students with an interest in either area.

The final chapter gives a very useful discussion of several 'generalisations and refinements of the theory developed in the preceding chapters, as well as [...] results from related areas which use the same methods or lead to similar group theoretic problems'. It may be regarded as a guide to the literature, and provides numerous suggestions for further work.

University of Western Australia CHERYL E. PRAEGER

TOPICS IN CLASSICAL AUTOMORPHIC FORMS
(Graduate Studies in Mathematics 17)

By HENRYK IWANIEC: 259 pp., US$39.00,
ISBN 0 8218 0777 3 (American Mathematical Society, 1997).

NON-VANISHING OF L-FUNCTIONS AND APPLICATIONS
(Progress in Mathematics 157)

By M. RAM MURTY and V. KUMAR MURTY: 196 pp., SFr.68.00,
ISBN 3 7643 5801 7 (Birkhäuser, 1997).

SPECTRAL THEORY OF THE RIEMANN ZETA-FUNCTION
(Cambridge Tracts in Mathematics 127)

By YOICHI MOTOHASHI: 228 pp., £29.95, ISBN 0 521 44520 5
(Cambridge University Press, 1997).

The three books under review together cover a lot of ground in the theory of automorphic forms and L-functions. In order of increasing difficulty and specialisation, the book by Iwaniec is an excellent graduate text, that by M. R. and V. K. Murty is a research monograph awarded the Ferran Sunyer i Balaguer 1996 prize, and that by Motohashi is a detailed and original treatment based on the author's research.

The book by Iwaniec provides the graduate student and the researcher wishing to acquire the basics on automorphic forms with a beautifully written and self-contained treatment of the classical modular and automorphic forms, Kloosterman sums, Hecke operators, automorphic L-functions, cusp forms and Eisenstein series, spherical functions, theta functions and convolution L-functions. The book is based on Iwaniec's lecture notes from a graduate course given in 1994/95 at Rutger's University, and it has obviously profited from being already tried and tested on graduate students. The emphasis is on highlighting the interaction between different ideas and methods. Some material towards the end of the book is presented in survey form, but all the basic material is given in full detail.

The books by M. R. and V. K. Murty and by Motohashi are, by contrast, much more specialised and directed at the researcher in number theory, although great trouble has been taken in both cases to make the treatments self-contained. A prerequisite for reading either of these two books, which are excellent research monographs in their own right, should be the mastering of the book by Iwaniec. There is some overlap of material between all the books, but it is only in Iwaniec's book that the basic theory is adequately covered with a broad perspective. This book is a great deal more than an introductory text, however, as Iwaniec develops the theory in a way not to be found in other texts. Of special note is his in-depth treatment of theta functions and representations of quadratic forms. It is not possible in the space allotted for this review to do justice to the value of this book. All three books

can be seen as a tribute to the power of analytic methods in the arithmetic study of L-functions. For example, deep results on the distribution of primes in arithmetic progression are closely linked to non-vanishing properties of certain L-functions on the line $\mathrm{Re}(s) = 1$.

The book by M. R. and V. K. Murty is based on original papers by the authors, and concerns results on the non-vanishing of a general L-function, with consequences for Dirichlet L-functions, Artin L-functions and L-functions derived from modular forms. These L-functions are of fundamental importance in the study of prime numbers, in the theory of elliptic curves and in the theory of automorphic functions, with many important results being linked to their non-vanishing properties. The first chapters deal with the Artin L-functions and the Chebotarev density theorem of which the prime number theorem is a special case. The authors then turn to the study of the L-functions associated to modular forms relating the Sato–Tate conjecture to the existence of the analytic continuation of certain L-functions to $\mathrm{Re}(s) \geqslant 1$ and their non-vanishing there. They also study conjectures concerning the non-vanishing of the Dirichlet L-functions at $s = \frac{1}{2}$ by averaging techniques and weighted sums. They show, for example, that for a positive proportion of Dirichlet characters modulo a sufficiently large prime, the associated Dirichlet L-function does not vanish in the interval $\frac{1}{2} \leqslant s < 1$. They study the non-vanishing of quadratic twists of modular L-functions, and show, for example, that for a holomorphic modular newform of weight 2 there is a quadratic character such that the corresponding twisted L-function does not vanish at the central critical point. This once again relies on averaging techniques.

The book described above requires heavier machinery than that by Motohashi. Yet, although the treatment of Motohashi claims to be elementary, this is a reflection more of the analytic techniques used than of the depth of the methods developed. Indeed, Motohashi's book requires a more sophisticated appreciation of analytic techniques than does that by M. R. and V. K. Murty. Motohashi deals with the spectral theory of the Riemann zeta function, starting with a full exposition of the spectral resolution of the non-Euclidean Laplacian and of trace formulae. The author then turns to automorphic L-functions and their relation to explicit formulae for power moments. The climax of the book is the treatment of the power moments of the zeta values,

$$\int_0^T |\zeta(\tfrac{1}{2} + iu)|^{2k}\, du, \quad k = 1, 2, 3, \dots,$$

especially that of the fourth power moment ($k = 2$). Motohashi shows how to derive an explicit formula for the fourth power moment in terms of the values at $s = \frac{1}{2}$ of Hecke L-functions attached to eigenforms, thus relating the zeta function to a whole family of automorphic L-functions. The main point of the book is to bring out how the Riemann zeta function alone links together families of more general L-functions. The very technical nature of the book is considerably alleviated by the section of notes at the end of each chapter where some historical and conceptual background is given.

Université des Sciences et Technologies de Lille   PAULA COHEN

LOCAL COHOMOLOGY: AN ALGEBRAIC INTRODUCTION WITH GEOMETRIC APPLICATIONS
(Cambridge Studies in Advanced Mathematics 60)

By M. P. BRODMANN and R. Y. SHARP: 416 pp., £45.00 (US$69.95),
ISBN 0 521 37286 0 (Cambridge University Press, 1998).

Local cohomology grew out of Serre's sheaf cohomology, introduced in the mid-1950s, and was first systematised in R. Hartshorne's notes of Grothendieck's 1961 Harvard seminar (compare [1]). These notes consisted of a blend of algebraic geometry and local algebra, and soon local cohomology grew to be an indispensable tool in commutative algebra (and algebraic geometry), reflected in its appearance in sketched form in chapters or appendices of texts in this field. The present book is the first to be devoted solely to the topic and, as such, it fills a real gap, especially for beginners. Fortunately for all, Brodmann and Sharp have produced an excellent book: it is clearly, carefully and enthusiastically written; it covers all important aspects and main uses of the subject; and it gives a thorough and well-rounded appreciation of the topic's geometric and algebraic interrelationships. The book contains plenty of exercises (many of which introduce the reader to a wealth of further algebraic and geometric topics), with hints supplied for the more difficult questions.

In brief, then, the book opens with local cohomology functors (in the setting of commutative algebra) being introduced as the right derived functors of torsion functors; the link with ideal transforms (and their derived functors) is established, and the theory is shown to have geometric significance when applied to the ring of regular functions on an irreducible affine variety over an algebraically closed field. In fact, it is at this point that the reader is introduced to a basic running example, used repeatedly to illustrate aspects of the material and to tie the discussion together, namely Hartshorne's famous example of an irreducible surface in $\mathbb{C}^4$ that is not a complete intersection. (Regrettably, this characterisation is not made completely explicit in the book, as far as I could see; compare [2].)

The theory gets off the ground with a treatment of the Mayer–Vietoris sequence, used as an inductive tool and put to work immediately in a vanishing theorem involving arithmetic rank. Indeed, the fundamental vanishing theorems of Grothendieck and of Lichtenbaum and Hartshorne are the subjects of the next few chapters (with Serre's affineness criterion as an application), as are finiteness and Artinian behaviour (the former focusing on annihilator properties, due to Faltings). The behaviour of local cohomology under base change, and links with Čech complexes and Koszul complexes, are also considered. The 'affine' theory is rounded off with a treatment of Matlis duality and local duality, together with applications.

Of course, a graded theory of local cohomology is needed to treat applications in projective geometry, and here this book is especially welcome in that it provides for the first time a full, careful and detailed discussion of graded local cohomology, together with a wealth of important applications, especially in geometry. First, the basics are dealt with thoroughly, with the machinery of the first part reworked in the graded case. Then applications are made to projective varieties and to Castelnuovo regularity (and so to syzygies), especially to questions of bounding the regularity by Hilbert coefficients. The final chapter treats in detail links with sheaf cohomology; but before that, the penultimate two chapters give the book an up-to-the-minute zing

with, respectively, a beautiful algebraic application—namely, L. T. Hoa's theorem on the asymptotic behaviour of reduction numbers of ideals—and beautiful geometric applications to connectedness theorems, such as those of Grothendieck, Barth, Fulton-Hansen and Faltings, and also Zariski's Main Theorem.

Some of the closing chapters appear technically dense now and then, and an earlier introduction of a geometric interpretation or of geometric language would perhaps have leavened the treatment there a little. I also have a few regrets that there was not a bit more discussion or commentary on certain topics. Thus the historical background to ideal transforms in Nagata's work on Hilbert's 14th problem would surely have been worth sketching, given the ring- and sheaf-theoretic treatment already present in the book. Hartshorne's example, and other examples discussed in the book, involve glueing, and some explicit comment on the algebraic and geometric aspects of this (here present just below the surface of the text) would have been of interest. Finally, Hartshorne's example is not formally connected in codimension one, and so is not locally a complete intersection, by Hartshorne's main theorem (see [2]). This relationship, which underpins the example's many pathologies as exhibited in the book, is touched on repeatedly, but in a somewhat tangential way: a few lines of discussion to draw all this together would have been useful.

However, compared to the cornucopia of superb interdisciplinary mathematics treated superbly well in this book—which provides a vital and, some would say, long-overdue service—these are minor cavils. I am sure that this will be a standard text and reference book for years to come.

(Finally, given the book's declared intention to demonstrate the crucial role played by local cohomology, the more anarchic reader may take a perverse (or perhaps childish) pleasure in replacing cohomological proofs by ones involving proper (that is, old-fashioned) algebra—as in, for example, 19.2.7.)

References

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ABELIAN VARIETIES WITH COMPLEX MULTIPLICATION AND
MODULAR FUNCTIONS
(Princeton Mathematical Series 46)

By GORO SHIMURA: 217 pp. US$55.00 (£39.50), ISBN 0 691 01656 9
(Princeton University Press, 1997).

The book under review encompasses both a reworking of a large portion of the famous 1961 work of its author and Y. Taniyama [1], and a sequel treating progress made since then in the subject. It is a beautifully written, self-contained and complete treatment of a subject of which G. Shimura is a founding master, and is a fundamental reference for any researcher or student of the arithmetic theory of abelian varieties and modular functions, and in particular of its applications to class field theory.

Chapters I, II and III, and most of Chapter IV, consist of a considerably revised version of corresponding material in [1] and cover the basic theory of abelian varieties, complex multiplication (CM), reduction of abelian varieties modulo a prime ideal, and the construction of certain class fields. For example, a main result of Chapter IV concerns the construction of unramified class fields over CM-fields by composing with the field of moduli of a polarised abelian variety with CM by the reflex CM-field (Main Theorem 1, page 112). A second main result of Chapter IV uses the points of finite order on such polarised abelian varieties to construct other class fields over CM-fields (Main Theorem 2, page 118). In the elliptic curve case, these important results reduce to an outcome of Kronecker's theory, that the abelian extensions of imaginary quadratic fields are generated by special values of certain elliptic or elliptic modular functions with singular moduli.

In Chapter V, the zeta function of an abelian variety with CM is defined and shown to be a product of Hecke L-functions. In Chapter VI, which uses many publications by the author post-dating [1], the discussion passes to families of polarised abelian varieties and their 'moduli varieties'. A remarkably succinct account is given of these varieties, of their relation to modular forms and functions, and of the existence of canonical models for them. Another new feature of the present book is the closing Chapter VII, which gives an extensive account of theta functions and the projective embeddings of polarised abelian varieties that can be given using these functions. Results of the author apply tools developed in the book to treat algebraic relations between invariants defined, using periods of holomorphic 1-forms on polarised abelian varieties with CM.

Other important modern treatments of the work of Shimura and of Shimura and Taniyama have been developed using a less classical style than that favoured in the book under review. Their impact in no way makes the present book outdated, as it provides the fundamental tools and intuition behind all the modern approaches.

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Université des Sciences et Technologies de Lille   PAULA COHEN

MIXED MOTIVES
(Mathematical Surveys and Monographs 57)

This monograph is a systematic development of ideas which the author originally outlined in a lecture delivered at a conference on algebraic K-theory and number theory, held at Johns Hopkins University in April 1990. It is, in my opinion, a very important book. This is because it lays the foundations for the modern (motivic) approach to cohomological phenomena in algebraic and arithmetic geometry. I remind the reader, by the way, that many of the most important results in this area are cohomological (for example, Poincaré duality, Riemann–Roch theorems, the Weil conjectures).

Some years ago, Grothendieck suggested that algebraic geometers should search for a 'universal' cohomology theory for algebraic varieties. He suggested a menu of plausible axioms for this cohomological motif, and explained how it should be related to all the known cohomology theories (Betti, de Rham, Hodge, étale, crystalline, and so on). The key point, said Grothendieck, was that this cohomology theory should take its values not in the category of abelian groups, but rather in a more general abelian category—the category of 'mixed motives'.

To cut a long story short, this category has yet to be constructed. Were it to be constructed, one would have for each scheme S an abelian tensor category (with a duality involution) of mixed motives over S, denoted by $\mathcal{MM}_S$. For each smooth S-scheme, there would be the motive of X, $M(X)$, in $\mathcal{MM}_S$, having a Kunneth formula for products, being connected with a rich cohomology theory related by Chern classes to algebraic K-theory, connected to Chow groups, zeta functions and many other important facets of algebraic geometry. Even in the presence of $\mathcal{MM}_S$ one could perform much of its cohomological mathematics in the associated triangulated tensor category given by the derived category, $\mathcal{DM}(S)$, whose objects are made from bounded complexes in $\mathcal{MM}_S$.

What the author has done is to construct a category with all the anticipated properties of $\mathcal{DM}(S)$. In addition to establishing most of the desired properties for his category, Levine also shows that it coincides with the very successful motivic category of Voevodsky when S is the spectrum of a perfect field admitting the resolution of singularities. (In this connection, we should recall that Spivakovsky, in Toronto, has been claiming to be able to resolve singularities in characteristic p for almost a decade.)

In conclusion, I have to admit that this volume will constitute difficult reading for many. This is particularly true for those of us who long ago had to slog through homological algebra and category theory done the old-fashioned way. However, I believe that it is not serving the best interests of the next generation of mathematics or mathematicians to ignore the motivic message. We must go out of our way to ensure (here is where the impossibly hard work comes in) that our libraries acquire books like this, and then (here is where further hard work comes in) we should 'encourage' our best PhD students to read them!

University of Southampton VICTOR SNAITH

CHARACTERS AND BLOCKS OF FINITE GROUPS
(London Mathematical Society Lecture Note Series 250)

By GABRIEL NAVARRO: 287 pp., £24.95 (US$39.95, LMS Members' price £19.45), ISBN 0 521 59513 4 (Cambridge University Press, 1998).

Modular representation theory deals with homomorphisms of a finite group G into the general linear group $\mathrm{GL}(n, F)$ of some degree n over a field F of prime characteristic p (and over related rings). When p divides the order of G, there is a rich interplay between representation theory and local group theory (the part of finite group theory starting with Sylow's theorems, and then moving on to Alperin's fusion theorem, control of transfer and criteria for the existence of normal p-complements). One of the highlights of this interplay is Glauberman's Z^*-theorem, asserting that

certain elements of order 2 in G are contained in the centre of G provided that G has no normal subgroup of odd order. This result has played a major role in various classification theorems for finite simple groups.

Navarro has put the Z^*-theorem in the centre of his book. Assuming only a modest background of ordinary (that is, complex) representation theory (all of which can be found in [**2**]), and occasionally some basic facts about algebraic number fields, the author proceeds quickly to R. Brauer's three main theorems on blocks, which are then applied to prove the Z^*-theorem. Having achieved that, the author moves on to other topics, like modular representations of p-solvable groups and of groups with a Sylow p-subgroup of order p. These topics all demonstrate well the power of the methods.

The author's approach is quite classical; but many of the original proofs have been polished, and some new developments have been incorporated. (I myself have sometimes used a similar approach to the subject in my lectures, following lecture notes of Dade [**1**].) Of course, the material selected for the book covers only a small part of modular representation theory. For example, vertices, sources and the Green correspondence are not mentioned at all. (These tools might have been useful for the chapter on groups with a cyclic Sylow p-subgroup.)

Since its beginnings in the pioneering work of R. Brauer about 50 years ago, the subject has grown considerably, and now has interesting connections with areas like topology, cohomology, number theory, algebraic geometry and coding theory. More importantly, many of today's conjectures (like Alperin's weight conjecture and its extensions by Dade, or Broué's dreams about derived equivalent blocks) indicate clearly that major parts of the theory are still waiting to be discovered.

Navarro's book gives a good introduction to this exciting subject. I enjoyed reading it, and I shall recommend it to my students.

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University of Jena   Burkhard Külshammer

PROFINITE GROUPS
(London Mathematical Society Monographs N.S. 19)

By John S. Wilson: 284 pp., £60.00, isbn 0 19 850082 3
(Clarendon Press, 1998).

The idea of subsuming an infinite family of congruences into a single equation over the p-adic integers goes back to Hensel at the beginning of this century; about half-way through the century, mathematicians such as Weil began to systematise problems about number fields by expressing them in terms of the absolute Galois group. Thus profinite groups, and in particular their cohomology, have been a central object of study for at least half a century. However, in the world of group theory they received for a long time only sporadic attention.

Recent years have seen an upsurge of activity in the group-theoretic study of profinite groups, from diverse points of view: to the questions motivated by number theory have been added many others which present themselves naturally to the eyes of an algebraist. These include questions about global structure (for example, describing the subgroups of a free profinite group), and questions about infinite families of finite groups, which can sometimes be encapsulated in the properties of a profinite group (such as the classification of finite p-groups of fixed coclass). The subject is in a hearty growth phase, and is full of attractive and challenging problems. Up to now there has, however, been no systematic and comprehensive text to which one could refer the aspiring research student. The author of the book under review has heroically stepped in to fill the gap.

The first four chapters (together with a Chapter 0) introduce profinite groups and develop their main properties. The material is mostly quite elementary, and carefully explained from first principles. More specialised topics occupy the later chapters. The author warns that 'the pace accelerates gently', but the inexperienced reader need have no qualms: everything is clearly spelled out, with meticulous attention to detail and a resolute avoidance of hand-waving arguments (these are particularly tempting in this subject, where many things are proved by considering the finite case and 'taking inverse limits'). This was a good decision; old hands will comfortably skim over familiar material, while learners will be grateful for the author's efforts.

However, this is much more than a book for beginners. For researchers, it will be the standard reference on all the most important topics in profinite group theory, most of which have not hitherto been systematically set down and developed in such depth. This is especially true of the cohomology theory; with the chapters setting up the necessary theory of profinite modules and completed group algebras, this occupies nearly half the book. The author has put in a lot of work to present a self-contained account which, while starting at the very beginning, culminates with the proofs of deep results, such as Lazard's theorem on the Poincaré duality of p-adic analytic pro-p groups. (Anyone who has tried to follow Lazard's original proof of this result will appreciate the relative simplicity of Wilson's approach.)

Other subjects covered in depth include free and projective groups, groups of finite rank, and finitely presented groups, all topics of current research activity. The author puts his own stamp on each of them; the reader familiar with more specialised books such as [1] and [2] will find plenty of interest here, including the first appearance in book form of Wilson's own recent results on the Golod–Shafarevich inequality.

This is going to be an extremely useful book. It should be in every departmental library; and anyone doing research on profinite groups, or in a related area, needs to have it on the bookshelf.

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Oxford University   Dan Segal

GROUPS ACTING ON HYPERBOLIC SPACES: HARMONIC ANALYSIS
AND NUMBER THEORY
(Springer Monographs in Mathematics)

By Jürgen Elstrodt, Fritz Grunewald and Jens Mennicke: 524 pp., £57.50,
ISBN 3 540 62745 6 (Springer, 1998).

Among discrete subgroups of Lie groups, Kleinian groups (discrete subgroups of
$PSL(2, \mathbb{C})$) acting on hyperbolic space, and Fuchsian groups (discrete subgroups
of $PSL(2, \mathbb{R})$) acting on the hyperbolic plane, are of particular interest. Although
Kleinian and Fuchsian groups are closely related, there are a number of important
differences. Kleinian groups cannot be continuously deformed (by Mostow rigidity),
and the associated hyperbolic 3-manifolds do not carry a complex structure; on the
other hand, because of the higher dimensionality, Kleinian groups give rise to more
complicated and richer geometric and analytic phenomena. In the context of
harmonic analysis on symmetric spaces, Kleinian and Fuchsian groups also play a
special rôle. While in the Fuchsian case there are already a number of treatments, the
book under review is the first monograph on the spectral theory of Kleinian groups,
including Selberg's trace formula.

The study of Kleinian groups (or of Fuchsian groups) is particularly attractive
because it involves learning many diverse branches of mathematics. This may be
illustrated very well by listing the contents of the book under review. The first chapter
is about *hyperbolic geometry* (of hyperbolic 3-space). Different models are presented,
and also the point of view of Lie groups and symmetric spaces is considered. The
second chapter treats the basic theory of *Kleinian groups*, including classification of
parabolic elements and finiteness questions. Also discussed are Poincaré's theorem of
fundamental polyhedra and the question of volumes of hyperbolic 3-manifolds. The
third chapter is about *automorphic functions*. Poincaré series and Eisenstein series are
introduced, the Fourier expansion of Eisenstein series in cusps being fundamental for
this theory. The method of point-pair invariants is discussed, and an explicit formula
for the Selberg transform is deduced. Chapters 4–6 are the heart of the book. They
treat, following the lines of the already classical approach of Maass, Roelcke and
Selberg, the *spectral theory* of the Laplace operator for cofinite Kleinian groups which
culminates in Selberg's trace formula. The resolvent kernel is of Hilbert–Schmidt type
only for cocompact Kleinian groups. The case of cofinite non-compact Kleinian
groups is a good bit more difficult. Their continuous spectrum is treated, as usual,
through the meromorphic continuation of Eisenstein series; the authors follow Colin
de Verdière's approach through pseudo-Laplacians. The theory of eigenpackets is
used (and explained) as well. As in the cocompact case, the discrete spectrum has
non-negative eigenvalues with finite multiplicities, but not much is known about the
size of discrete spectrum of cofinite non-compact Kleinian groups. A number of
applications of trace formula are proved, for instance the prime geodesic theorem and
the fact that the Selberg zeta function is an entire function of order 3. Chapter 7 and
Chapter 10 are about *arithmetic Kleinian groups*. Bianchi groups $PSL(2, \mathcal{O})$ (in
Chapter 7) are the natural analogues of the modular group $PSL(2, \mathbb{Z})$; the authors
also discuss their interesting group theoretic structure. Chapter 10 contains a general
construction of arithmetic Kleinian groups. A list of all thirty-two Kleinian groups
which are Coxeter groups with four generators is given, some of them being non-
arithmetic. The subject treated in Chapter 8 could be called *Kleinian modular forms*.

General results of the spectral theory of Kleinian groups are applied to the groups $PSL(2, \mathcal{O})$ (and their Eisenstein series). This technique has been very successful in number theory. The authors obtain, among other results, an analogue of Kronecker's limit formula, results on non-vanishing of L-functions and, with the help of zeta functions, Weyl's asymptotic law on the distribution of eigenvalues. In Chapter 9 the theory of (arithmetic) Kleinian groups is used to derive classical results on *integral binary Hermitian forms*.

The book under review is very rich, very accessible, and most carefully written. The text is enriched by many original approaches and applications, fruit of the authors' long experience with the subject. This valuable book will be exceedingly useful to all who wish to learn the theory of Kleinian groups in connection with harmonic analysis and number theory.

For the reader's convenience, I have added below some references to recent related literature.

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Université de Neuchâtel   PAUL SCHMUTZ SCHALLER

INTRODUCTION TO GEOMETRIC PROBABILITY

By DANIEL A. KLAIN and GIAN-CARLO ROTA: 178 pp., £12.95 (US$19.95),
ISBN 0 521 59654 8 (Cambridge University Press, 1997).

Geometric probability theory begins, as does this book, with Buffon's needle problem. Few readers will need reminding that its solution says that, if a needle of length L is dropped on a grid of parallel lines in a plane at distance L apart, then the probability that it meets one of the lines is $2/\pi$. From a modern perspective, this answer depends on the existence of a measure on the family of lines in the plane which is invariant under isometries.

This idea generalizes in several directions. The *affine Grassmannian* $\mathrm{Graff}(n, k)$ consists of the family of all k-dimensional flats (affine subspaces) in n-dimensional euclidean space $\mathbf{R}^n$, endowed with its natural invariant measure λ_k^n induced by the group E_n of isometries of $\mathbf{R}^n$. If $k \geqslant 1$, then a k-flat meets a compact set K if and only if it meets the convex hull of K; thus it is natural to confine one's attention to compact convex sets. If K is such a set, then the measure of the k-flats which meet K is clearly proportional to the mean $(n-k)$-volume of the projection of K on the $(n-k)$-dimensional *linear* subspaces of $\mathbf{R}^n$, and thus (in some sense) measures the $(n-k)$-dimensional size of K. In one normalization, this is the *quermassintegral* $W_k(K)$ of K introduced by Minkowski (the definition accounts for the name). Nowadays, an alternative normalization introduced by the reviewer, giving the *intrinsic volume* $V_{n-k}(K)$, is often preferred; one advantage is that $V_r(K)$ is independent of the dimension of the ambient space of K.

The intrinsic volumes V_r (denoted in this book by μ_r) are *valuations*, in that, if K_1

and K_2 are compact convex sets whose union is also convex, then $V_r(K_1 \cup K_2) + V_r(K_1 \cap K_2) = V_r(K_1) + V_r(K_2)$. The abstract theory of valuations was extensively investigated by Hadwiger, one of whose crowning achievements was the characterization of linear combinations of intrinsic volumes as valuations on compact convex sets which are isometry invariant, and continuous with respect to the *Hausdorff metric* $\wp$, given by

$$\wp(K_1, K_2) := \inf\{\rho \geqslant 0 \mid K_1 \subseteq K_2 + \rho B, K_2 \subseteq K_1 + \rho B\},$$

with B the unit ball. This characterization leads directly to many formulae of integral geometry. As one simple example, we have the projection result above. As another, if K_1 and K_2 are compact convex sets, then the measure of those $g \in E_n$ such that $K_1 \cap gK_2 \neq \emptyset$ is given by an expression $\sum_{r=0}^n \beta_r V_r(K_1) V_{n-r}(K_2)$, with constants β_r which are easy to calculate.

Functions ϕ which satisfy the more general *inclusion–exclusion principle*, namely

$$\phi(K) = \sum_i \phi(K_i) - \sum_{i<j} \phi(K_i \cap K_j) + \sum_{i<j<k} \phi(K_i \cap K_j \cap K_k) - \cdots$$

whenever $K = K_1 \cup \cdots \cup K_m$, can be extended to the family Polycon(n) of polyconvex sets, namely finite unions of compact convex sets in $\mathbf{R}^n$. Not all valuations satisfy the inclusion–exclusion principle, but continuous valuations do (since the inclusion–exclusion principle holds for valuations on polytopes). Thus many results in the book are stated for Polycon(n).

At the core of the book lies a new proof by the first author of Hadwiger's theorem, which in turn follows from his new characterization of volume. This is embedded in a fine brief introduction to geometric probability in general. The initial approach is unusual—through parallelotopes—which helps gradually to reinforce the reader's intuition, since the intrinsic volumes have obvious meanings here. There is then a diversion through the analogous valuations on simplicial complexes, a discussion of Polycon(n), and a survey of invariant measures on Grassmannians. Finally, after the kinematic formulae are proved, the corresponding theory on the sphere is considered. To date, in fact, there are more questions than answers here, and the analogue to Hadwiger's characterization theorem is open for spheres of dimension three or more.

Let me end by giving an enthusiastic welcome to this inexpensive little book, which will surely interest more than geometers alone.

University College, London   PETER MCMULLEN

THEORY OF DEGREES WITH APPLICATIONS TO BIFURCATIONS AND DIFFERENTIAL EQUATIONS
(Canadian Mathematical Society Series of Monographs and Advanced Texts)

By WIESLAW KRAWCEWICZ and JIANHONG WU: 374 pp., £70.00,
ISBN 0 471 15740 6 (John Wiley & Sons, 1997).

For an open bounded set $\Omega \subset \mathbb{R}^n$ and a continuous function $f: \overline{\Omega} \to \mathbb{R}^n$, the Brouwer degree of f relative to Ω (and 0) is defined provided that $f(x) \neq 0$ for all $x \in \partial\Omega$. Roughly, it is an algebraic count of the number of solutions inside Ω of the equation $f(x) = 0$. Along with its extension to Leray–Schauder degree in infinite dimensional spaces, it provides a well-known tool in the study of nonlinear problems. In particular, it enables many problems in ordinary and partial differential equations

(ODEs and PDEs) to be reduced to the (often difficult) problem of determining *a priori* bounds for (a family of) solutions.

The book under review aims to provide a unified treatment of the 'classical' theory (as mentioned above) and the equivariant degree theory for maps that are invariant under some group action, for example S^1, which is at an early stage of development. The definition in this text is different from but related to the definition given by Ize, Massabo and Vignoli in several papers from 1989 to 1993; see, for example, [6].

The authors 'intend to address those readers who are applications oriented and have basic knowledge of functional analysis, general topology, and differential equations'. However, the reader is expected to know rather more than this, as knowledge of differential geometry (manifolds, vector bundles, homotopy theory) and algebra (Lie groups and their representations) is essential in the study of equivariant degree, though much less is needed for the 'classical' theory. What is used is clearly stated either in Chapter 1 or as needed.

Chapters 2–4 cover Brouwer degree for continuous maps f in $\mathbb{R}^n$, its extension to the Leray–Schauder degree for maps of the form $f = I - F$ where F is compact, in infinite dimensional spaces, and to 'Nussbaum–Sadovskii' degree when F is condensing. First, the authors present the normalization, additivity and homotopy properties as axioms, and further properties are deduced from these. Uniqueness of the degree under these axioms is not discussed, but reference could have been made to the literature on this, for example, the paper by Amann and Weiss [1]. Brouwer degree is then constructed by the well-known analytic approach as may be found in well-known books by Schwartz, Lloyd, Deimling and Zeidler, and elsewhere. Another recent book, by Fonseca and Gangbo [4], also gives the classical theory, but has an entirely different second part dealing with functions in Sobolev spaces. A lengthy version of the proof of Borsuk's theorem due to Gromes [5] (who is not cited) is given; a succinct version is given in Deimling's book [2].

Leray–Schauder degree is treated in the standard way by using the Schauder approximation of a compact map by a finite dimensional map. (The step of going from $\mathbb{R}^n$ to a finite dimensional vector space is left to the reader.) 'Nussbaum–Sadovskii' degree is handled by means of a bijection theorem proving an equivalence between the compact and condensing cases. The degree is extended to unbounded sets Ω when $f^{-1}(0)$ is compact. Applications are given to some ODEs involving terms that may have quadratic growth, and to ODEs with nonlinear boundary conditions, and to existence of periodic solutions of some neutral functional differential equations (FDEs). Chapter 5 proves a nice account of local and global bifurcation results (Krasnosel'skii, Rabinowitz), utilizing the concept of complementing maps. (Some mention of related work by Fitzpatrick, Ize, Massabo, Pejsachowicz, Vignoli and others could have been usefully added here; see, for example, [3].)

Chapters 6–8 constitute the main new feature of this text, the equivariant degree theory, to which the authors have made major contributions. Chapter 6 treats S^1-equivariant degree. This is a sequence of integers, each integer measuring orbits of a certain type. Necessary results from representation theory are reviewed before the definition is given via the idea of an equivariant normal approximation of a map (of the previously studied types). It is shown that this degree theory satisfies the standard properties of the earlier degree theories. Applications are given in Chapter 7 to global Hopf bifurcation for some neutral FDEs, including a problem of a lossless transmission line. In the final chapter, the authors define G-degree, the equivariant

degree theory based on the equivariant fixed point index due to Dold. (Definition 8.3.3 contains a misprint.)

Generally, the book is well presented, but occasionally formulae are badly split, and use of the definite article is poor. There are a number of cases of wrong cross-references, which is a problem that we do not expect in the days of LaTeX. Also, some misprints have led to problems: for example, Exercises 2.2.6 and 4.3.1 are wrong as stated. The exercises are often extensions of the theory to other contexts, and references to the literature would be useful. The end-of-chapter bibliographical notes are often rather general, and more specific attributions would have enhanced the book.

Because of the technical nature of some of the material, a symbol index is sorely missed. It is not possible to be a grasshopper in reading this book.

The book under review is a very useful addition to the literature, especially because of the part on equivariant degree theory. It is an advanced work, and readers are expected to have wide knowledge and be able to fill gaps by themselves.

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University of Glasgow   JEFFREY WEBB

ALGORITHMIC GEOMETRY

By JEAN-DANIEL BOISSONNAT and MARIETTE YVINEC (translated by HERVÉ BRÖNNIMANN): 519 pp., £24.95 (US$39.95), ISBN 0 521 56529 4 (Cambridge University Press, 1998).

In very many practical applications of processing data by computer, such as graphics, medical imaging (CAT and NMR scans) and computer-aided design (CAD), the efficient solution of a geometric problem lies at the heart of the technique. This book is devoted to the theory of algorithms to solve such problems, much of which is comparatively recent. Before going into what the book does cover, let me first say what it does not. The reader will not find here actual implementations of algorithms. And those problems which are soluble only by brute force are omitted—unfortunately, these include some important topics such as determination of volume, to give only one instance. In other words, we have here the theory of geometric algorithms which are polynomial in their input data; in this context, it is natural to take the dimension as fixed.

The book is split into five parts, each further subdivided into chapters. The first, on algorithmic tools, provides a general introduction which is of interest beyond geometry. It treats the basic notions of complexity of algorithms and problems, optimality and (polynomial) equivalence, the theory of data structures, deterministic methods, and various randomized techniques. The last topic needs a little more

explanation. In implementing some algorithms, one may make random choices at various stages. Surprisingly, perhaps, such methods are often very efficient, and indeed occasionally optimal. As one might expect, however, the illustrations of the algorithms are mainly applications to geometry.

The remaining parts fall into a common pattern. There is an initial chapter on the mathematical background. However, any general abstract theory here is covered fairly rapidly; only those aspects of the subject with future applications to algorithms in mind are treated in detail. So, to give just one example, in the chapter on polytopes the reviewer's Upper Bound Theorem is dealt with in a qualitative way: the maximum number of facets of d-polytopes with n vertices is $O(n^{\lfloor d/2 \rfloor})$. Of course, this is appropriate, since even an exact bound fed into one part of the complexity analysis of an algorithm will usually result only in an order estimate for the algorithm as a whole.

This background chapter is then followed by two (except for convex hulls, where there are three) chapters on algorithms as applied to the subject of the part. Although the authors claim in the preface that they develop the theory as far as possible in a dimensionally independent way, in practice these applications are mostly to small dimensional problems. Indeed, some of the planar algorithms have no analogues in higher dimensions. The main exceptions concern convex hull algorithms and linear programming, both of which are treated in full generality.

Let me end by briefly listing the topics discussed in the four parts. The initial subject of convex hulls (of a given finite set of points) is self-explanatory; various iterative methods of finding the convex hull (on-line, dynamic and by shelling) are explored in depth. Randomized linear programming also occurs here. Triangulations are those of a finite point-set with vertices in the set; the algorithms here are 2- and 3-dimensional. Arrangements are of several kinds: hyperplanes in space, line segments in the plane, and triangles in ordinary space. Voronoi diagrams are partitions of space determined by proximity to finitely many points; the metric here need not be euclidean, and the space is even allowed to be hyperbolic. Associated with a Voronoi diagram is a 'dual' Delaunay triangulation, which is also of some importance.

The book is well written (and the translation is, with rare exceptions, felicitous), covers a wealth of material, is copiously illustrated, and has a comprehensive bibliography. Especially in view of its modest price, the book would be a welcome addition to the shelves of anyone interested in algorithmic geometry.

University College, London   PETER MCMULLEN

PROJECTIVE GEOMETRIES OVER FINITE FIELDS

By J. W. P. HIRSCHFELD: 555 pp., £65.00, ISBN 0 19 850295 8
(Clarendon Press, 1998).

Seven years ago, James Hirschfeld completed his ambitious project of giving a self-contained and comprehensive account of projective spaces over finite fields. The three volumes comprising this work, all in the Oxford Mathematical Monographs Series, are *Projective geometries over finite fields* (1979), *Finite projective spaces of three dimensions* (1986) and *General Galois geometries* (1991). The first volume provides introductory material on finite fields and on general projective spaces, before

studying the line and the plane in detail. The second volume is devoted to spaces of three dimensions, and the third, written jointly with J. A. Thas, to spaces of general dimension.

These monographs have become established as the standard reference work for researchers in finite geometry. They are also invaluable for researchers in closely related areas such as coding theory, group theory and statistical designs. In coding theory in particular, it is becoming increasingly recognised that many problems may be posed most naturally in a geometrical setting, and that many optimal codes are equivalent to natural structures in finite projective spaces.

Dr Hirschfeld has now written a second edition of the first volume of the trilogy. This is, in fact, a complete reworking, taking account of many new results proved since 1979. It is much more attractively typeset than the first edition, and so better complements the second and third volumes. The number of references has increased from around 800 to over 3000. As before, the volume is concisely but clearly written, and contains a wealth of interesting material.

The book is largely concerned with combinatorial properties of subsets of projective spaces, but other recurring themes are properties and characterisations of algebraic varieties and group-theoretical properties of configurations.

The first chapter is a survey of relevant results about finite fields. The second chapter introduces the projective spaces $PG(n, q)$ and algebraic varieties, as well as some connections with groups and codes. The next three chapters are concerned with fundamental general properties of $PG(n, q)$. After a short chapter on the projective line $PG(1, q)$, the remainder of the book, a further eight chapters, gives a detailed study of the plane $PG(2, q)$, covering such topics as conics, quadrics, ovals, arcs, cubic curves and blocking sets. The chapters on arcs and on blocking sets are the most heavily revised from the first edition, so as to include important new results. The final chapter gives a detailed analysis of each of the planes for $q \leqslant 13$.

The author is not planning second editions of the successor volumes, and so the new trilogy, comprising the 1998, 1986 and 1991 volumes, looks set to be the standard reference work on projective spaces over finite fields for many years to come. The publishers are currently offering the three-volume set at just over half-price. Snap it up!

University of Salford   RAY HILL

HYPERBOLIC MANIFOLDS AND KLEINIAN GROUPS
(Oxford Mathematical Monographs)

By KATSUHIKO MATSUZAKI and MASAHIKO TANIGUCHI: 253 pp., £60.00,
ISBN 0 19 850062 9 (Clarendon Press, 1998).

The study of Kleinian groups and 3-dimensional hyperbolic manifolds lies at the intersection of differential geometry, 3-manifold topology, complex analysis, number theory, dynamics and combinatorial group theory. The field was born around the turn of the century in the work of Poincaré, and has undergone two major and distinct revolutions, one in the 1960s with the complex analytic work of Ahlfors and Bers, and one in the 1970s and 1980s with the geometric work of Thurston and the dynamical work of Sullivan. One of the most interesting aspects of the field is the

interplay between the action of a Kleinian group on the Riemann sphere and its action on hyperbolic 3-space.

To date, a number of books exploring various facets of Kleinian groups have been written at various points over the life of the field, including works by Ford [**3**], Lehner [**6**], Kra [**4**], Beardon [**1**], Krushkal, Apanasov and Gusevskii [**5**], Maskit [**7**], Thurston [**10**], Ratcliffe [**8**] and Benedetti and Petronio [**2**], as well as the lecture notes of Thurston [**9**].

The purpose of the book under review is to give an overview of the study of Kleinian groups and hyperbolic 3-manifolds. It covers the basics of the action of Kleinian groups on the Riemann sphere and on hyperbolic 3-space, including a chapter on ends of hyperbolic 3-manifolds; a detailed discussion of algebraic and geometric convergence and of deformation spaces of Kleinian groups; and finiteness theorems for Kleinian groups. One feature of the book that I found particularly nice is that the authors provide sketches of the proofs of the major results, such as the Ahlfors finiteness theorem, the Sullivan rigidity theorem, and the double limit and uniformization theorems of Thurston. They convey an enormous amount of information efficiently. Overall, the book is well written and well organized. It is an expanded and revised version of the authors' 1993 book, and has been reasonably well translated from the Japanese. I believe that this book will be a useful and valuable reference for researchers in the field for years to come.

There is one point of concern that I have, which arises primarily because the book is a survey of existing results and does not go into great depth on all topics. On the whole, the authors do not do a thorough job of providing references to the sources in the published literature in which the results they discuss first appear. I would have found more detailed referencing of results to be useful.

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FIBREWISE HOMOTOPY THEORY
(Springer Monographs in Mathematics)

By MICHAEL CRABB and IOAN JAMES: 341 pp., £49.50, ISBN 1 85233 014 7
(Springer, 1998).

The category of spaces over a fixed space B is quite old. The term adopted here is fibrewise space, which does not imply any uniformity of the fibres. The aim of this

monograph is to present an expository survey of fibrewise homotopy theory as it stands today.

The book is in two parts. Part I consists largely of definitions and elementary results that set up the basic theory. Chapter 1 adapts concepts from ordinary homotopy theory, such as products, cofibrations, homotopy and mapping spaces, to the fibrewise context. For example, a finite covering space is renamed a fibrewise compact discrete space. The appropriate definitions are not always obvious, and need some care; for instance, a fibrewise constant map $X \to Y$ is not the same as a map that is constant on each fibre.

Chapter 2 treats the pointed (based) theory. A pointed fibrewise space is a space X over B with a given section; unlike ordinary homotopy theory, its behaviour can vary greatly with the choice of section. Topics include fibrewise versions of smash products, Puppe sequences, the Freudenthal suspension theorem and Whitehead products.

Chapter 3 covers Lusternik–Schnirelmann category and variants, and continues with the fibrewise James construction. The question of which sphere bundles support fibrewise H-space structures is discussed, with concrete examples.

Part II is billed as an introduction to the stable theory, and contains four chapters. Chapter 1 sets up the fibrewise (graded) stable homotopy category. The Euler class $\gamma(\xi)$ of a vector bundle ξ is the inclusion $B \times S^0 \to \xi_B^+$.

Chapter 2 treats fixed point theory, following Dold. It begins by developing the machinery of Euclidean and absolute neighbourhood retracts (ENRs and ANRs). Given a manifold M and a suitable map $f: U \to M$, where U is open in M, the Lefschetz–Hopf index $\tilde{L}_B(f, U)$ is a stable map $B \times S^0 \to U_{+B}$. This reduces to the transfer $B_+ \to M_+$ when f is the identity. These ideas are used to prove the Adams Conjecture, following Becker–Gottlieb. The main ingredient is finding a p-local stable equivalence $\xi_B^+ \to (\psi^l \xi)_B^+$ for any virtual complex vector bundle ξ over B, where p does not divide l. This is easy when ξ is a line bundle or a sum of line bundles.

The next topic is fibrewise duality. The dual of X is an object X^* equipped with maps $i: B \times S^0 \to X^* \wedge X$ and $e: X^* \wedge X \to B \times S^0$ that satisfy axioms familiar from linear algebra. This generalizes Spanier–Whitehead duality. The object X is invertible if i and e are isomorphisms. When B is a point, this allows only spheres; but in general, X can be any sphere bundle over B and the definition has real content.

Chapter 3 treats manifolds. A fibrewise manifold is modelled on open sets $W \subset B \times E$, where E is a vector space; thus any open subset is again a fibrewise manifold. The foundations of fibrewise differential topology are developed. By the Pontrjagin–Thom construction, a fibrewise smooth map $f: M \to N$ gives rise to the Gysin or Umkehr stable map $f^!: N_B^+ \to M_B^{\nu(f)}$ of Thom spaces, or more generally $N_B^\xi \to M_B^{f^* \xi \oplus \nu(f)}$ for any virtual vector bundle ξ over N. Following Atiyah, $M_B^{-\tau_B(M)}$ is recognized as the dual of M_{+B}, and the map $f^!$ is dual to f_+. Sullivan's result on the Euler class of a flat vector bundle is an application.

Miller showed that the unitary group $U(n)$ splits stably. The claim is that this is best viewed as a fibrewise result. There is an extended discussion of the work on configuration spaces by Snaith, Cohen and many others. The space $C^k(M)$ is the space of distinct k-tuples in a manifold M; as k varies, we obtain $C(M)$. More generally, we have $C^k(M; Y)$, which attaches to each of the k points a label in the pointed space Y, and hence $C(M; Y)$. When Y is connected, the Pontrjagin–Thom construction gives a homotopy equivalence for $C(M; Y)$. The special cases D^1, D^n and S^1 of M yield the models for $\Omega \Sigma Y$ by James, $\Omega^n \Sigma^n Y$ by May, and the free loop space

on ΣY. Further, the natural filtration of $C(M; Y)$ splits stably. This is all done for B a point; everything can be done fibrewise too, including the fibrewise *EHP* sequence.

Chapter 4 very briefly introduces the fibrewise stable homology category, in which homologically equivalent spaces over B are, in effect, identified. There is one for each multiplicative generalized homology theory. After setting up the foundations, this chapter mentions about 12 major topics, all in 20 pages.

This is a reference work rather than a textbook, but it is only partly based on previously published material. It is not entirely self-contained; for many details and some results, the reader is referred elsewhere. Nor is it exhaustive, as the field is not yet mature. The notation often seems overly elaborate; there are relative versions everywhere. Categorical notation and language are generally avoided.

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