THE KISSING NUMBERS OF CONVEX BODIES —A BRIEF SURVEY

Dedicated to Professor Hlawka on the occasion of his 80th birthday

C. ZONG

1. *The thirteen spheres problem*

Nearly four hundred years ago, the *cubic close-packing* of equal spheres in R^3 was discovered by Kepler [**21**], in which each sphere touches 12 others. In 1694, Gregory and Newton discussed the following *thirteen spheres problem*. *Can a rigid material sphere be brought into contact with* 13 *other such spheres of the same size*? Gregory believed 'yes', while Newton thought 'no'.

Let B be the three-dimensional unit sphere centred at the origin o, and let $B+x$ be one which touches B at its boundary. It is easy to calculate that the smallest cone containing $B+x$ and taking o as its vertex intersects the boundary of B at a cap of surface area $2\left(1-\frac{\sqrt{3}}{2}\right)\pi$. By comparing this surface area with the surface area of the unit sphere, one might immediately see that the largest number of the nonoverlapping unit spheres which can be brought into contact with a fixed one is less than or equal to 14. However, this is not enough to solve the thirteen spheres problem.

In Kepler's example, the kissing configuration is stable. In other words, none of the 12 spheres which are in contact with the fixed one can be moved around. However, the configuration of twelve unit spheres kissing a fixed sphere at the vertices of a regular icosahedron is unstable: *each of the twelve spheres can move around freely in a small area*! This fact, in some sense, illustrates the complexity of the thirteen spheres problem.

Although the thirteen spheres problem is very natural and very simple sounding, its solution was first achieved only in 1874 by Hoppe [**18**]. *The largest number of nonoverlapping unit spheres which can be brought into contact with a fixed one is* 12. Newton was right! Later, several simpler proofs for this assertion were given by Günter [**16**], Schütte and van der Waerden [**28**] and Leech [**22**]. However, none of them is trivial. Schütte and van der Waerden's proof involves an argument of graph theory, reduction and complicated computation, while Leech's applies *Euler's theorem* on polyhedra, and heavy computation as well.

There is an immediate connection between the thirteen spheres problem and the *cap packing problem*, or the *Tammes problem* (see [**7**]). *What is the largest diameter* d_m *of m equal caps that can be placed on the surface of a unit sphere without overlap*? More precisely, $d_{13} \geqslant 1$ will imply Gregory's belief, and $d_{13} < 1$ will be in favour of Newton. The truth is $d_{13} < 1$!

Received 19 September 1996; revised 24 January 1997.

1991 *Mathematics Subject Classification* 11H31, 52C17.

This work is supported by a Research Fellowship of The Royal Society.

Bull. London Math. Soc. 30 (1998) 1–10

2. *The kissing numbers of convex bodies, general bounds*

As usual, denote by K a d-dimensional convex body with boundary $\partial(K)$ and interior $\operatorname{int}(K)$, and by Λ a d-dimensional lattice.

KISSING NUMBERS. *The translative kissing number $N(K)$ of K is the largest number of nonoverlapping translates of K which can be brought into contact with K at its boundary. The lattice kissing number $N^*(K)$ of K is the similar number when the translates are taken from any lattice packing of K.*

The translative kissing number of a convex body is different from its *Newton number*. The latter considers the 'congruences of K' instead of the 'translates of K'. However, some authors use *Hadwiger number* for our term translative kissing number.

Let X be a convex set, and let $D(X)$ be the difference set of X, $D(X) = \{x-y: x, y \in X\}$. Then a simple argument involving convexity yields the following. *For two points z_1 and z_2, $(X+z_1) \cap (X+z_2) = \varnothing$ if and only if $(\frac{1}{2}D(X)+z_1) \cap (\frac{1}{2}D(X)+z_2) = \varnothing$.* Therefore, for every convex body K,

$$N(K) = N(D(K)) \quad \text{and} \quad N^*(K) = N^*(D(K)).$$

Since $D(K)$ is centrally symmetric, to search for bounds for $N(K)$ and $N^*(K)$, it is sufficient to deal with only the centrally symmetric convex bodies.

One hundred years ago, in the forerunner to his study of the geometry of numbers, Minkowski [**24**] found the following.

THEOREM 1. *For every d-dimensional convex body K,*

$$N^*(K) \leqslant 3^d - 1, \tag{1}$$

where equality holds if and only if K is a parallelepiped. In addition, if K is strictly convex, then

$$N^*(K) \leqslant 2(2^d - 1).$$

Minkowski's proof is based on convexity and elementary number theory. Without loss of generality, we assume that K is centrally symmetric and $N^*(K)$ is attained in the lattice packing $K+\mathscr{Z}^d$, where $\mathscr{Z}^d$ indicates the d-dimensional *integer lattice*. Then $N^*(K)$ is the number of points in the set $\partial(2K) \cap \mathscr{Z}^d$. If $x = (x^1, x^2, \ldots, x^d)$ and $y = (y^1, y^2, \ldots, y^d)$ are two different points of this set, then

$$x^i - y^i \equiv 0 \bmod 3$$

cannot hold simultaneously for all $i = 1, 2, \ldots, d$. Otherwise, by convexity, we can obtain

$$\tfrac{1}{3}(x - y + o) \in \operatorname{int}(2K) \cap \mathscr{Z}^d,$$

which contradicts the assumption that $K+\mathscr{Z}^d$ is a lattice packing. Then (1) follows easily. This is the key idea of his proof.

To generalize Minkowski's result from $N^*(K)$ to $N(K)$, in 1957 Hadwiger [**17**] obtained the following.

THEOREM 1*. *For every d-dimensional convex body K,*

$$N(K) \leqslant 3^d - 1, \tag{2}$$

where equality holds if and only if K is a parallelepiped.

Hadwiger's proof idea is brilliant—simple and effective. Let K be centrally symmetric, and let $K+z_1, K+z_2, \ldots, K+z_{N(K)}$ be $N(K)$ nonoverlapping translates of K which touch K at its boundary. Then, by convexity and symmetry, one has $K \subset 3K$ and

$$K+z_i \subset 3K, \quad i = 1, 2, \ldots, N(K).$$

Therefore

$$(N(K)+1)\, v(K) \leqslant v(3K) = 3^d v(K),$$

where $v(K)$, as usual, indicates the volume of K. Then (2) follows.

A few years later, applying a similar idea, Groemer [**12**] presented a more detailed proof of this result.

On the other hand, it is very natural to ask the following.

PROBLEM 1. What are the best lower bounds of $N(K)$ and $N^*(K)$ for all d-dimensional convex bodies? For which convex bodies can these bounds be attained?

In 1953, Swinnerton-Dyer [**29**] proved a result which implies the following.

THEOREM 2. *For every d-dimensional convex body K,*

$$N^*(K) \geqslant d(d+1).$$

More precisely, Swinnerton-Dyer discovered the following. *In every densest lattice packing of a d-dimensional convex body, each translate touches at least $d(d+1)$ others.* His proof is comparatively complicated. As a counterpart of Swinnerton-Dyer's result, in the sense of Baire category, Gruber [**13**] proved that: '*Most*' *d-dimensional convex bodies have not more than $2d^2$ neighbours in any of their densest lattice packings.*

In 1961, Grünbaum [**15**] proposed the following.

CONJECTURE 1. (a) *For every d-dimensional simplex S, $N(S) = d(d+1)$.*

(*b*) *For every even number m, $d(d+1) \leqslant m \leqslant 3^d - 1$, there is a d-dimensional convex body K such that $N(K) = m$.*

Contradicting the first part of this conjecture, Zong [**34**] was able to prove the following.

THEOREM 3. *For a tetrahedron T, $18 = N^*(T) \leqslant N(T) \leqslant 19$. For a d-dimensional simplex S, $N(S) \geqslant N^*(S) \geqslant d(d+1) + 6\left[\frac{d}{3}\right]$.*

Combining this theorem with the main result of Hoylman [**19**], one can immediately obtain the following rather counter-intuitive phenomenon. *In the densest lattice tetrahedra packings, every tetrahedron touches* 14 *others. On the other hand, there is a lattice tetrahedra packing with much smaller density, in which every one touches* 18 *others.*

Indeed, Theorem 3 makes Problem 1 more challenging and more interesting. Going further in this direction, applying *Dvoretzky's theorem* about the spherical sections of a centrally symmetric convex body in high dimensions (see [**37**]) and induction, Zong [**36**] obtained the following conditional result. *If $d(d+1)$ is the best lower bound of $N(K)$ for all d-dimensional convex bodies, then for any positive number ε there will be a convex body B^* such that $\delta^H(B, B^*) < \varepsilon$ and $N^*(B^*) = d(d+1)$, where B is a d-dimensional unit sphere and δ^H is the Hausdorff metric.* Comparing this assertion with Theorems 4 and 6 in Section 3, one might easily observe their strange implication.

3. *The kissing numbers of spheres*

Similar to the *packing density problem* (see Rogers [**27**]), spheres are the most interesting cases for kissing numbers too. For convenience, we denote by B^d the d-dimensional unit sphere. In 1965, Wyner [**31**] found the following.

THEOREM 4.

$$N(B^d) \geqslant 2^{0.2075d(1+o(1))}. \tag{3}$$

The idea used to obtain this bound is simple. Let $B^d+z_1, B^d+z_2, \ldots, B^d+z_{N(B^d)}$ be $N(B^d)$ nonoverlapping unit spheres which touch B^d at its boundary. Then

$$z_i \in \partial(2B^d), \quad i = 1, 2, \ldots, N(B^d),$$

and

$$\partial(2B^d) \subset \bigcup_{i=1}^{N(B^d)} (\operatorname{int}(2B^d)+z_i).$$

Consequently, one obtains

$$\bigcup_{i=1}^{N(B^d)} (\partial(2B^d) \cap (\operatorname{int}(2B^d)+z_i)) = \partial(2B^d)$$

and

$$\sum_{i=1}^{N(B^d)} s(\partial(2B^d) \cap (\operatorname{int}(2B^d)+z_i)) \geqslant s(\partial(2B^d)), \tag{4}$$

where $s(X)$ indicates the $(d-1)$-dimensional measure of X. By detailed computation, (3) follows from (4).

From (4) it is easy to see that, as a lower bound, (3) is far from the best. For an upper bound for $N(B^d)$, applying an idea of Blichfeldt [**3**], in 1955 Rankin [**26**] obtained the following.

THEOREM 5.

$$N(B^d) \ll \pi^{1/2} d^{3/2} 2^{(d-1)/2},$$

where $f(d) \ll g(d)$ means $f(d) \leqslant g(d)$ when d is large.

The idea of the proof is splendid, indeed. Roughly speaking, it runs as follows. Let $B^d+z_1, B^d+z_2, \ldots, B^d+z_{N(B^d)}$ be $N(B^d)$ nonoverlapping unit spheres which touch B^d at its boundary. By projecting from o to B^d+z_i, we obtain $N(B^d)$ caps, say $C_1, C_2, \ldots,$

$C_{N(B^d)}$, of *geodesic radius* $\pi/6$, which pack on the surface of B^d. Now we proceed to deal with this cap packing. First, enlarge these caps C_i homothetically to caps C_i^* of geodesic radius $\pi/4$. These new caps perhaps do not form a packing in $\partial(B^d)$. Second, attach a suitable 'mass' $\delta_i(x)$ to every point x of C_i^*, such that

$$\delta(x) = \sum_{i=1}^{N(B^d)} \delta_i(x) \leqslant 1. \tag{5}$$

Let $M_i = \int_{C_i^*} \delta_i(x)\, ds$. Then by (5) one obtains

$$\sum_{i=1}^{N(B^d)} M_i = \int_{\partial(B^d)} \delta(x)\, ds \leqslant s(B^d),$$

and consequently

$$N(B^d) \leqslant \frac{s(B^d)}{M_1}.$$

By some detailed computation, Rankin's result follows.

Let $A(d,\phi)$ be the largest number of points in $\partial(B^d)$ such that the minimum *geodesic distance* among them is ϕ. Evidently, $N(B^d) = A(d,\pi/3)$. Let $F_d(\alpha)$ be the *Schläfli function* defined by

$$F_d(\alpha) = \frac{2^d U}{d!\,\omega_d},$$

where U is the area of a *regular spherical simplex* in $\partial(B^d)$ of angle 2α, and ω_d is the surface area of the d-dimensional unit ball. In 1978, to verify a conjecture of Coxeter [**6**], Böröczky [**4**] proved the following.

Theorem 5*.

$$A(d,\phi) \leqslant \frac{2F_{d-1}(\alpha)}{F_d(\alpha)},$$

where $\sec(2\alpha) = \sec(\phi) + d - 2$.

By setting $\phi = \pi/3$ in this theorem, one obtains

$$\log_2 N(B^d) \ll \frac{d}{2}.$$

This bound is of more or less the same order as Theorem 5.

During the last three decades, the most significant progress concerning the kissing numbers of spheres has been achieved mainly by applying coding theory and linear programming.

Let C be a *spherical code*, in other words, a finite subset of $\partial(B^d)$. Let $\delta_t(x)$ be the *Dirac delta function* defined by

$$\delta_t(x) = \begin{cases} 0 & x \neq t, \\ \infty & x = t, \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta_t(x)\, dx = 1. \tag{6}$$

Clearly, the Dirac delta function is not a proper function in the usual sense, and a more formal definition may be given as the limit of a suitable sequence of analytical functions satisfying (6).

For $-1 \leqslant t \leqslant 1$, let $p(t)$ be the number of ordered pairs $c, c' \in C$ such that $\langle c, c' \rangle = t$, where $\langle x, y \rangle$ is the *inner product* of x and y, and define

$$F(t) = \frac{1}{\operatorname{card}\{C\}} \sum_{x:\, p(x) \neq 0} \delta_x(t)\, p(x).$$

Then a simple combinatorical argument yields

$$\int_{-1}^{1} F(t)\, dt = \operatorname{card}\{C\}.$$

In addition, denoting by $P_i^{\alpha,\alpha}(t)$ the *Jacobi polynomial* (see [**1**]) with $i \geqslant 0$, we have

$$\int_{-1}^{1} F(t)\, P_i^{\alpha,\alpha}(t)\, dt = \frac{1}{\operatorname{card}\{C\}} \sum_{c, c' \in C} P_i^{\alpha,\alpha}(\langle c, c' \rangle) \geqslant 0,$$

where $P_i^{\alpha,\alpha}(\langle x, y \rangle)$ is positive definite.

If there are $N(B^d)$ nonoverlapping unit spheres touching B^d at its boundary, then the $N(B^d)$ touching points form a spherical code with $F(t) = 0$ for $\frac{1}{2} < t < 1$. Therefore an upper bound to $N(B^d)$ will be given by the optimal solution to the following linear programming problem. *Choose the $F(t)$ so as to maximize $\int_{-1}^{1/2} F(t)\, dt$ subject to the constraints $F(t) \geqslant 0$ for $-1 \leqslant t \leqslant \frac{1}{2}$ and*

$$\int_{-1}^{1/2} F(t)\, P_i^{\alpha,\alpha}(t)\, dt \geqslant -P_i^{\alpha,\alpha}(1)$$

for $i = 0, 1, \ldots$.

This argument leads to the following fundamental lemma (see [**8**], [**20**] or [**25**]).

LEMMA 1. *Assume $d \geqslant 3$. If $f(t)$ is a real polynomial which satisfies*
(1) *$f(t) \leqslant 0$ for $-1 \leqslant t \leqslant \frac{1}{2}$, and*
(2) *the coefficients in the expansion of $f(t)$ in terms of Jacobi polynomials*

$$f(t) = \sum_{i=0}^{k} f_i\, P_i^{\alpha,\alpha}(t),$$

where $\alpha = (d-3)/2$, satisfy $f_0 > 0, f_1 \geqslant 0, \ldots, f_k \geqslant 0$, then

$$N(B^d) \leqslant \frac{f(1)}{f_0}.$$

Based on this lemma, with the help of *harmonic polynomials* and linear programming, Rankin's upper bound has been improved by Levenštein [**23**] and by Kabatjanski and Levenštein [**20**] to the following.

THEOREM 5**.

$$N(B^d) \leqslant 2^{0.401d(1+o(1))}.$$

Applying Lemma 1 with skillful choice of $f(t)$, Levenštein [**23**] and Odlyzko and Sloane [**25**] independently found the following.

THEOREM 6.

$$N^*(B^8) = N(B^8) = 240, \quad N^*(B^{24}) = N(B^{24}) = 196\,560.$$

In addition, Bannai and Sloane [**2**] were able to prove the following.

THEOREM 6*. *There is a unique way* (*up to isometry*) *of arranging* 240 (*or* 196 560) *nonoverlapping unit spheres in* R^8 (*or* R^{24}) *so that they touch another unit sphere.*

Theorems 6 and 6* are very surprising; they are the only exact results about $N(B^d)$ that have been obtained during the last century! Even today, we do not know whether the exact value of $N(B^4)$ is 24 or 25. There are hundreds of articles dealing with sphere packings and coding theory; we shall not try to go into detail here. For more information on this subject, see Conway and Sloane [**5**].

In the 1970s, in a series of papers, Watson was able to determine $N^*(B^d)$ for $d = 4, 5, 6, 7, 8$ and 9 by studying *positive quadratic forms*. Denote by F the family of positive quadratic forms

$$f(x) = \sum_{i,j=1}^{d} a_{ij} x_i x_j,$$

and by $m(f)$ the number of the points $z \in \mathscr{Z}^d \backslash \{o\}$ at which $f(z)$ attains its minimum. It is easy to see that

$$N^*(B^d) = \max_{f \in F} \{m(f)\}.$$

By reduction, one need only deal with finitely many classes of quadratic forms (reduced ones). In this way, Watson [**30**] was able to prove the following.

THEOREM 6**.

d	4	5	6	7	8	9
$N^*(B^d)$	24	40	72	126	240	272

4. *Miscellaneous*

4.1 *Results concerning two-dimensional convex domains.* In 1961, to prove a conjecture of Hadwiger, Grünbaum [**15**] proved the following.

THEOREM 7. *Let D be a two-dimensional convex domain. Then*

$$N(D) = N^*(D) = \begin{cases} 8 & \text{if } K \text{ is a parallelogram,} \\ 6 & \text{otherwise.} \end{cases}$$

Although this assertion itself is simple, its proof is rather technical. It is proved by applying the following basic result.

LEMMA 2. *For each boundary point x of any centrally symmetric convex domain D, there is an inscribed affine regular hexagon of D which takes x as one of its six vertices.*

Let K_1 and K_2 be convex bodies in R^m and R^n, respectively. Denote by $K_1 \oplus K_2$ the *Cartesian product* of K_1 and K_2. Observing the kissing configuration, one can easily deduce

$$N(K_1 \oplus K_2) \geqslant (N(K_1)+1)(N(K_2)+1)-1.$$

Recently, Zong [**35**] proved the following.

THEOREM 7*. *Let D be a two-dimensional convex domain, and let K be a d-dimensional convex body. Then*

$$N(D \oplus K) = \begin{cases} 9N(K)+8 & \textit{if } D \textit{ is a parallelogram}, \\ 7N(K)+6 & \textit{otherwise}. \end{cases}$$

The proof of this result is comparatively complicated. Let δ^C be the *Minkowski metric* given by an m-dimensional centrally symmetric convex body C. If C can be partitioned into $P(C)$ parts $X_1, X_2, \ldots, X_{P(C)}$ such that

$$\delta^C(x, y) < 1$$

whenever both x and y belong to the same part, then by projection (the argument is not simple) one obtains

$$N(C \oplus K) \leqslant P(C)(N(K)+1)-1.$$

Then, to prove Theorem 7*, one is left to partition a centrally symmetric convex domain properly by applying Lemma 2.

Zong [**35**] also proposed the following.

CONJECTURE 2. *There are two convex bodies K_1 and K_2 in high dimensions such that*

$$N(K_1 \oplus K_2) \neq (N(K_1)+1)(N(K_2)+1)-1.$$

For a similar problem concerning density, see Gruber and Lekkerkerker [**14**].

4.2 *Difference between $N(K)$ and $N^*(K)$.* The first convex body K for which $N(K) \neq N^*(K)$ was found by Watson [**30**] in 1971; he showed the following.

THEOREM 8.

$$N(B^9) \geqslant 306, \quad N^*(B^9) = 272.$$

For spheres, nine is so far the only known dimension in which this difference exists. Perhaps $N(B^d) = N^*(B^d)$ holds for infinitely many values of d. However, it looks as though $N(B^d) \neq N^*(B^d)$ happens more frequently. This certainly is an interesting problem, but also very hard.

For general convex bodies, Zong [**32**] found the following.

THEOREM 8*. *Whenever $d \geqslant 3$, there exists a d-dimensional convex body K such that $N(K) > N^*(K)$.*

Although the proof is rather complicated, the example is simple. Cutting off two pairs of small opposite corners from a unit cube yields this result.

Following Theorems 8 and 8*, Zong [**32**] proposed the following.

PROBLEM 2. What is the maximum difference between $N(K)$ and $N^*(K)$ for all d-dimensional convex bodies? For which convex bodies can this maximum be attained?

4.3 *Connections with blocking numbers*. Denote by $B(K)$ the *blocking number* of K, the smallest number of nonoverlapping translates of K which touch K and prevent any other from touching K. It is easy to see that the blocking number is a counterpart and a limited case of the kissing number. Similarly, we have $B(K) = B(D(K))$ for every convex body. In [**33**] (see also [**37**]), Zong observed the following rather counter-intuitive phenomenon.

THEOREM 9. *When the dimension number d is large, there are two convex bodies K_1 and K_2 such that*

$$N(K_1) < N(K_2) \quad \textit{but} \quad B(K_1) > B(K_2).$$

As for the bounds of $B(K)$, Zong [**33**] made the following conjecture.

CONJECTURE 3. *For every d-dimensional convex body K*,

$$2d \leqslant B(K) \leqslant 2^d.$$

This conjecture implies *Hadwiger's convering conjecture* (see [**7**]) for centrally symmetric convex bodies.

ACKNOWLEDGEMENTS. I am obliged to Professors Larman and Rogers for helpful discussions, and to the referee for helpful comments.

References

1. M. ABRAMOWITZ and I. A. STEGUN, *Handbook of mathematical functions* (Dover, New York, 1972).
2. E. BANNAI and N. J. A. SLOANE, 'Uniqueness of certain spherical codes', *Canad. J. Math.* 33 (1981) 437–449.
3. H. F. BLICHFELDT, 'The minimum value of quadratic forms and the closest packing of spheres', *Math. Ann.* 101 (1929) 605–608.
4. K. BÖRÖCZKY, 'Packing of spheres in spaces of constant curvature', *Acta Math. Acad. Sci. Hungar.* 32 (1978) 243–261.
5. J. H. CONWAY and N. J. A. SLOANE, *Sphere packings, lattices and groups* (Springer, New York, 1988).
6. H. S. M. COXETER, 'An upper bound for the number of equal nonoverlapping spheres that can touch another of the same size', *Proc. Sympos. Pure Math.* 7 (1963) 53–71.
7. H. T. CROFT, K. J. FALCONER and R. K. GUY, *Unsolved problems in geometry* (Springer, New York, 1988).
8. P. DELSARTE, J. M. GOETHALS and J. J. SEIDEL, 'Spherical codes and designs', *Geom. Dedicata* 6 (1977) 363–388.
9. P. ERDÖS, P. M. GRUBER and J. HAMMER, *Lattice points* (Longman, Essex, 1989).
10. G. FEJES TÓTH and W. KUPERBERG, 'Packing and covering with convex sets', *Handbook of convex geometry* (ed. P. M. Gruber and J. M. Wills, North-Holland, Amsterdam, 1993) 799–860.
11. L. FEJES TÓTH, *Lagerungen in der Ebene, auf der Kugel und im Raum* (Springer, Berlin, 1972).
12. H. GROEMER, 'Abschätzung für die Anzahl der konvexen Körper, die einen konvexen Körper berühren', *Monatsh. Math.* 65 (1960) 74–81.
13. P. M. GRUBER, 'Typical convex bodies have surprisingly few neighbours in densest lattice packings', *Studia Sci. Math. Hungar.* 21 (1986) 163–173.
14. P. M. GRUBER and C. G. LEKKERKERKER, *Geometry of numbers* (2nd edn, North-Holland, Amsterdam, 1987).

15. B. GRÜNBAUM, 'On a conjecture of H. Hadwiger', *Pacific J. Math.* 11 (1961) 215–219.
16. S. GÜNTER, 'Ein stereometrisches Problem', *Grunert Arch.* 57 (1875) 209–215.
17. H. HADWIGER, 'Über Treffanzahlen bei translations gleichen Eikörpern', *Arch. Math.* 8 (1957) 212–213.
18. R. HOPPE, 'Bestimmung der grössten Anzahl gleich grosser Kugeln, whlche sich auf eine Kugel von demselben Radius, wie die übrigen, auflegen lassen', *Grunert Arch.* 56 (1874) 302–313.
19. J. D. HOYLMAN, 'The densest lattice packing of tetrahedra', *Bull. Amer. Math. Soc.* 76 (1970) 135–137.
20. G. A. KABATJANSKI and V. I. LEVENŠTEIN, 'Bounds for packings on a sphere and in space', *Problemy Peredachi Informatsii* 14 (1978) 3–25.
21. J. KEPLER, 'De nive sexangula', *Gesammelte Werke*, Vol. 4 (Beck, Munich, 1911) 259–280.
22. J. LEECH, 'The problem of the thirteen spheres', *Math. Gaz.* 40 (1956) 22–23.
23. V. I. LEVENŠTEIN, 'On bounds for packings in *n*-dimensional Euclidean space', *Dokl. Akad. Nauk SSSR* 245 (1979) 1299–1303.
24. H. MINKOWSKI, *Diophantische Approximationen* (Teubner, Leipzig 1907, 1927; Chelsea, New York, 1957).
25. A. M. ODLYZKO and N. J. A. SLOANE, 'New bounds on the number of unit spheres that can touch a unit sphere in *n* dimensions', *J. Combin. Theory Ser. A* 26 (1979) 210–214.
26. R. A. RANKIN, 'The closest packing of spherical caps in *n* dimensions', *Proc. Glasgow Math. Assoc.* 2 (1955) 139–144.
27. C. A. ROGERS, *Packing and covering* (Cambridge University Press, 1964).
28. K. SCHÜTTE and B. L. VAN DER WAERDEN, 'Das Problem der dreizehn Kugel', *Math. Ann.* 125 (1953) 325–334.
29. H. P. F. SWINNERTON-DYER, 'Extremal lattices of convex bodies', *Proc. Cambridge Philos. Soc.* 49 (1953) 161–162.
30. G. L. WATSON, 'The number of minimum points of a positive quadratic form', *Dissertationes Math.* 84 (1971) 1–42.
31. A. D. WYNER, 'Capabilities of bounded discrepancy decoding', *Bell Systems Tech. J.* 54 (1965) 1061–1122.
32. C. ZONG, 'An example concerning translative kissing numbers of a convex body', *Discrete Comput. Geom.* 12 (1994) 183–188.
33. C. ZONG, 'Some remarks concerning kissing numbers, blocking numbers and covering numbers', *Period. Math. Hungar.* 30 (1995) 233–238.
34. C. ZONG, 'The kissing numbers of tetrahedra', *Discrete Comput. Geom.* 15 (1996) 239–252.
35. C. ZONG, 'The translative kissing numbers of the Cartesian product of two convex bodies, one of which is two-dimensional', *Geom. Dedicata*, to appear.
36. C. ZONG, 'A few remarks on kissing numbers of a convex body', *Anz. Österreich. Akad. Wiss. Math.-Naturwiss. Kl.* 132 (1996) 11–15.
37. C. ZONG, *Strange phenomena in convex and discrete geometry* (Springer, New York, 1996).

Department of Mathematics
University College London
Gower Street
London WC1E 6BT

and

Institute of Mathematics
The Chinese Academy of Sciences
Beijing 100080
P. R. China