BOOK REVIEWS

SPECIAL FUNCTIONS
(Encyclopedia of Mathematics and its Applications 71)

By George E. Andrews, Richard Askey and Ranjan Roy: 664 pp., £55.00 (US$85.00), isbn 0-521-62321-9 (Cambridge University Press, 1999).

As suggested by their name, special functions form a particular and privileged class among all functions that are conceivable. Their study has in many cases paved the way to general ideas and concepts in mathematics. Famous examples are the Γ-function, the ζ-function, and hypergeometric functions studied by Euler and Gauss. As is well known, study of their properties has been a source of inspiration for the development of complex analysis. This is an ongoing process. Nowadays, we know many instances of special functions which may lead to entirely new fields of mathematics. Examples are Ramanujan's mock theta functions, and hypergeometric functions satisfying difference-differential equations associated to Lie algebras.

The present book is a volume in the Encyclopedia of Mathematics and its Applications. In writing such a volume, the authors had the difficult task of making a choice from the overwhelming multitude of subjects which constitute 'special functions'. In principle, their choice is motivated by the class of hypergeometric functions in one variable. Classical functions such as sine, cosine and log are examples, as are Bessel functions and a large array of orthogonal polynomials. The first eight chapters of the book give an introduction to the Γ-function, Gauss' hypergeometric function, Bessel functions and orthogonal polynomials. An interesting feature of these chapters is that some aspects of hypergeometric functions are treated which are not easy to find elsewhere in book form, or which are scattered throughout various specialised books. For example, the Barnes integral approach, evaluation of generalised hypergeometric series at $x = 1$, the arithmetic–geometric mean in a formula for π, and the Wilf–Zeilberger method for mechanical summation of series can be found in Chapters 2 and 3.

If the reader would like to have an introduction to orthogonal polynomials, then a quick and thorough one can be found in Chapter 5, with many examples in Chapter 6 and applications in Chapter 7.

As we read further, we come to a chapter on Selberg's formula, which expresses the integral

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} [x_i^{a_i-1}(1-x)^{b_i-1}]\,|\Delta(x)|^{2\gamma}\,dx_1 \ldots dx_n, \quad \Delta(x) = \prod_{i<j}(x_i - x_j),$$

as a finite product of gamma factors. In the book we find two proofs, one by Aomoto and one by Anderson. The latter found this from his proof of a finite field analogue of the formula. This is an aspect of interest to readers oriented towards number theory. Classical functions such as the Γ-function and Euler's beta function have their finite field counterparts in the form of Gauss sums and Jacobi sums. There is an interesting parallel between the theorems which hold in the two worlds of special functions and number theory. The authors pay attention to this aspect at several places in the book.

Other analogues of classical functions are the q-functions, such as q-gamma functions and q-hypergeometric functions, also known as basic hypergeometric functions. As a matter of philosophy, a q-analogue is formed by replacing Pochhammer symbols $(\alpha)_n$ by their q-versions $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$. Here q is thought of as some number with $|q| < 1$ for convergence purposes. In this way, the hypergeometric series

$$ {}_2F_1(\alpha,\beta,\gamma\,|\,z) = \sum_{n\geqslant 0} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} z^n $$

has

$$ {}_2\phi_1(a,b,c\,|\,q,x) = \sum_{n\geqslant 0} \frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n} x^n $$

as its q-version. A number of classical theorems on hypergeometric functions have q-counterparts. In the final chapters of the book, we find several aspects of q-series such as, for example, their relation to elliptic functions, Jacobi's triple product formula, basic hypergeometric series and their use in the theory of partitions, and last but not least, the Rogers–Ramanujan identities.

As a valuable addition to the book, each chapter ends with a large collection of exercises, some easy, some hard. A number of them deal with interesting and amusing trivia; others guide a reader through additional theory.

To summarise the content of the book, the large collection of exercises and an extensive bibliography make this Encyclopedic volume a worthy entry-point into the world of Special Functions.

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INTEGRABLE SYSTEMS: TWISTORS, LOOP GROUPS, AND RIEMANN SURFACES
(Oxford Graduate Texts in Mathematics 4)

By N. J. HITCHIN, G. B. SEGAL and R. S. WARD: 136 pp., £25.00,
ISBN 0-19-850421-7 (Clarendon Press, Oxford, 1999).

The study of integrable systems has a long history, but despite this there is still no generally agreed definition of an integrable system. Instead, they are characterised by a number of generally recognisable features, and as a result their study is not confined to a narrow area of mathematics but involves the interplay of analytic, algebraic and geometric analysis. The book under review is based on a series of lectures given at an instructional conference for graduate students. Despite comprising only 136 pages, the lectures cover an enormous amount of material, ranging from algebraic geometry and the theory of Riemann surfaces to loop groups, connections, Yang–Mills equations and twistor theory. However, despite this wide range, the book is surprisingly self-contained and readable.

One of the earliest integrable systems to be studied was the Euler top, which describes as asymmetric body rotating about its centre of mass. As Nigel Hitchin points out in the Introduction, this system exhibits three characteristic features of an integrable system: the ability to give explicit solutions, the existence of many conserved quantities, and the presence of algebraic geometry. A second classic example of an integrable system is given by the Korteweg–de Vries (KdV) equation. This non-linear partial differential equation describes the behaviour of water waves in a shallow channel, and was used to explain the solitary wave observed by John Scott Russell in 1834. The existence of soliton solutions which can be superposed despite the non-linear nature of the equations is another feature of integrable systems. The integrable nature of the KdV equation arises from the existence of a Lax pair, that is, the existence of a pair of operators A, B such that the non-linear equation is the consistency condition for solutions of the linear equation

$$\frac{dA}{dt} = [A, B].$$

In the first chapter, Nigel Hitchin considers the case where $A(z)$ is a polynomial-valued matrix. The first two sections give a rapid introduction to Riemann surfaces, while the next two consider vector bundles on Riemann surfaces. This material explains the relationship between a Riemann surface M and the matrix $A(z)$ via the spectral curve $\det(y - A(z)) = 0$, and correspondingly the way that the eigenspace of $A^t(z)$ defines a line bundle over M. The key point about the Lax pair equation is that it preserves the spectrum of $A(z)$ (so the coefficients of the spectral curve are preserved), and this allows one to encode the dynamics in the evolution of the line bundle. One obtains integrable systems by requiring that this evolution is linear, and it is shown that such flows are generated by elements of a cohomology class of M. This construction may be thought of as a generalisation of the description of integrable Hamiltonian systems in which the action variables are constant and the angle variables evolve linearly on an n-dimensional torus.

The second chapter, by Graeme Segal, examines the mathematical structure which underlies the use of the inverse scattering method to solve integrable equations. This again starts by considering the representation in terms of a Lax pair, although in this case A and B are differential operators: for example, in the KdV case, $A = -\partial_x^2 + u$ and $B = 4\partial_x^3 - 6u\partial_x - 3u'$. Provided that u satisfies the KdV equation, the eigenvalues of A are preserved, while the evolution of the corresponding eigenfunctions is generated by B. This leads to a simple rule for the evolution of the asymptotic behaviour or 'scattering data' for these eigenfunctions. If one has a means of reconstructing u from the scattering data, then this gives a way of solving the KdV equation which is a non-linear generalisation of the Fourier transform. The scattering caused by u may be described in terms of a scattering or holonomy matrix g_λ (with $g_\lambda \to 1$ as $\lambda \to \pm\infty$), and thus determines an element of a loop group. Solving the inverse scattering problem turns out to be a Riemann–Hilbert problem for g which can be written as a linear integral equation. The general setting for such problems is loop groups and Grassmannians, and in the final two sections the relationship between integrable systems, restricted Grassmannians and algebraic curves is considered, making contact between this picture and that considered by Hitchin in the previous chapter.

The last chapter, by Richard Ward, discusses integrable systems in terms of twistor theory. Once again the starting point is the Lax pair description, but this time A and B are matrix differential operators depending on a spectral parameter λ.

One may then reformulate the Lax equation as the condition that the covariant derivatives of a connection commute and hence that the curvature vanishes. Many examples of integrable systems can be described in this way; indeed, most well-known examples are reductions of the self-dual Yang–Mills equations. Twistor theory originated in Roger Penrose's attempts to unite general relativity and quantum physics using methods of complex holomorphic geometry. Although progress in achieving this goal is slow, twistors have proved an extremely useful tool in mathematical physics. In particular, solutions of the self-dual Yang–Mills equations correspond to holomorphic vector bundles over (regions of) twistor space CP^3, so again algebraic geometry may be used to solve the integrable system.

The book gives a wide-ranging introduction to a modern approach to integrable systems which explores the relationship between the geometrical and algebraic aspects of the theory. In order to cover this much material in so little space, many topics are presented in a concise form which many graduate students would find challenging. However, all the sections are clearly written, and the approach has the advantage that one can see the key features of the overall structure very clearly. I certainly found the book very stimulating, and it made me want to look at the original papers for further details. I can recommend it to anyone with a background in geometry and an interest in dynamical systems.

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4-MANIFOLDS AND KIRBY CALCULUS
(Graduate Studies in Mathematics 20)

By ROBERT E. GOMPF and ANDRÁS I. STIPSICZ: 558 pp., US$65.00, ISBN 0-8218-0994-6 (American Mathematical Society, Providence, RI, 1999).

The study of 4-dimensional manifolds reaches back to the beginning of the subject of topology, yet it yields some of the most prominent open questions in this general area. It is natural to see what one might call the 'modern era' in 4-manifold theory as beginning in the early 1980s with Freedman's work on topological 4-manifolds, and the use of differential geometic methods, closely allied to particle physics, in the smooth case. (Recall that a smooth manifold is one which is given by a collection of charts glued together by C^∞ maps; for a topological manifold one requires only that the maps are continuous.) This latter development reached a fairly mature form in the mid 1990s following the introduction of the Seiberg–Witten theory. The current position can be summarised roughly as follows. First, thanks to Freedman, we have an almost complete classification of topological 4-manifolds. Secondly, we have fairly strong results about which topological 4-manifolds can support *any* smooth structure. Thirdly, we have invariants for smooth manifolds which are usually not too hard to calculate and whose general properties are quite well understood. Fourthly, we have an enormous stock of examples of smooth 4-manifolds which are *homeomorphic* but not *diffeomorphic*: otherwise said, topological 4-manifolds which carry inequivalent smooth structures. In practice, this means that the manifolds in question have isomorphic cohomology rings but different Seiberg–Witten invariants. The invariants detect phenomena which are both specific to 4 dimensions and which rely essentially on smoothness; the main conclusion of Freedman's work is that the topological theory in 4 dimensions runs parallel to the theory of 'high-dimensional' manifolds which was brought into fairly complete form in the 1960s and 1970s. When we ask questions which go beyond these four areas of progress, we are stuck: we have no idea of the full classification of smooth 4-manifolds, even in the simply-connected case. (A special instance of this is the 4-dimensional smooth Poincaré conjecture.) Probably the feeling among workers in the field is that the Seiberg–Witten invariants detect only the tip of an iceberg, analogous to the Alexander polynomial invariant of knots [**1**], but it seems that some quite new technique is needed to make any real headway.

The book under review provides a unique and comprehensive account of almost all that is known about the topology of 4-manifolds and the existing techniques for studying them. The central theme of the book is the Kirby calculus. This is a procedure for manipulating handle decompositions, and its great virtue is that it allows questions about 4-manifolds to be expressed in terms of diagrams, and hence to a large extent visualised: Kirby calculus is the best way we have of 'seeing' 4-dimensional topology. More precisely, a handle decomposition in any dimenion n represents a manifold as a nested tower of subsets M_i, each subset M_{i+1} being obtained from the previous one, M_i, by attaching 'k-handles', that is, sets of the form $D^k \times D^{n-k}$, using a map from $\partial D^k \times D^{n-k}$ to the boundary of M_i. Here D^k denotes the k-dimensional disc, so its boundary ∂D^k is a sphere, and $\partial D^k \times D^{n-k}$ is a thickened sphere. Handle decomposition is an important tool throughout geometric topology; in some guise it lies at the heart of the high-dimensional theory mentioned above, and in 3 dimensions it amounts to the notion of a Heegard splitting. In the 4-dimensional case, one starts with a 0-handle which is just a 4-disc, and then attaches 1-, 2- and 3-handles, finally capping off with a 4-handle to obtain a closed 4-manifold. The point is that the boundary of the 4-disc is a 3-dimensional sphere, which can be represented as $\mathbf{R}^3$ with a point at infinity, and the handles are specified (roughly speaking) by drawing the attaching sets in ordinary 3-space. The most important questions involve the 2-handles, which are attached along thickened circles, making up a link in 3-space. The authors develop this material very thoroughly, and show how the technique can be used to study an enormous range of concrete examples, covering essentially everything which is known about 4-manifolds.

While the Kirby calculus, a technique from geometric topology, lies at the centre of this book, the authors devote considerable space to other techniques. In particular, they explore at length the topology of complex algebraic surfaces, and more generally of symplectic manifolds. Interactions with complex geometry have always been prominent in 4-manifold theory. (In fact, one might say that geometric topology, as a serious mathematical subject, was born in the 19th century with the realisation that the topology of 2-manifolds played a role in complex analysis and algebraic geometry. From this point of view, the topology of complex *surfaces* is the next case to look at.) Complex variables give a different way of visualising 4 dimensions, and the marriage of the two points of view enhances each. Some of the material in the book, on symplectic manifolds and Lefschetz fibrations, has not appeared before. In other directions, the book contains a substantial discussion of the more standard algebraic topology (bundle theory, and so on) relevant to 4-manifolds. There is also an introduction to the Seiberg–Witten theory which, backed up by some of the technical material in [**2**], for example, should be sufficient to give the reader a working knowledge of this area.

This book is important and valuable in that it both gives a comprehensive and accessible picture of an area which has developed rapidly in the past 20 years and also provides the reader with techniques to begin research in the field. The book is pedagogically very strong, with many examples and exercises (including solutions to selected exercises). The material will not go out of date, and however the field may develop in the future, this will be an important reference for many years to come.

References

1. R. Fintushel and R. Stern, 'Knots, links and 4-manifolds', *Invent. Math.* 134 (1998) 363–400.
2. J. Morgan, *The Seiberg–Witten equations and applications to the topology of smooth four-manifolds*, Math. Notes 44 (Princeton University Press, 1996).

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SPHERE PACKINGS
(Universitext)

By Chuanming Zong: 241 pp., £30.50, isbn 0-387-98794-0 (Springer, New York, 1999).

With the Kepler sphere packing conjecture apparently proven by Hales with help from Ferguson, this is an appropriate time to collect together results about sphere packings and see where we stand. The book under review does exactly that. It includes proofs of all main results, plus others, except in cases where a proof is so long and detailed that its inclusion is not practicable. Open problems and their status are also discussed. What follows, after some definitions, is a brief description of what the reader can expect.

For any convex body K and set X in Euclidean n-dimensional space, the collection of translates $K+X = \{K+x : x \in X\}$ is a translative packing of K if no two translates have overlapping interiors. If l is a positive real number and $I_n = \{x = (x_1, x_2, \ldots, x_n) : |x_i| \leqslant 1/2\}$ is the n-dimensional unit cube, then the density of such a packing is

$$\delta(K, X) = \limsup_{l \to \infty} \frac{\operatorname{card}(X \cap lI_n)\, v(K)}{v(lI_n)},$$

where card$(\cdot)$ denotes a cardinal number and $v(\cdot)$ denotes a volume. The translative packing density of K is

$$\delta(K) = \sup_X \delta(K, X).$$

If the lim sup and supremum are taken over only packings where X is a lattice, then the supremum is denoted by $\delta^*(K)$ and called the lattice packing density of K. The translative kissing number $k(K)$ of K is the maximum number of translates of K with non-overlapping interiors that can be touching the boundary of K. If the translates are restricted to those formed by points from a lattice, then the kissing number is denoted by $k^*(K)$ and called the lattice kissing number. Historically, the problems of most interest have been concerned with finding the values of $\delta(S_n)$, $\delta^*(S_n)$, $k(S_n)$ and $k^*(S_n)$ for the n-dimensional sphere S_n.

Chapter 1 contains proofs of the results $k(S_2) = k^*(S_2) = 6$, $\delta(S_2) = \delta^*(S_2) = \pi/\sqrt{12}$ and $k(S_3) = k^*(S_3) = 12$. Kepler's conjecture that $\delta(S_3) = \pi/\sqrt{18}$ is discussed, including a description of the approach to its proof by the methods of Fejes Tóth and

Hsiang, and then by the method of Hales. Not surprisingly, Hales' complete proof is not given. Chapter 2 presents the connection between quadratic forms and lattice sphere packing, and includes proofs of the results $\delta^*(S_3) = \pi/\sqrt{18}$, $\delta^*(S_4) = \pi^2/16$ and $\delta^*(S_5) = \pi^2/(15\sqrt{2})$. An outline of the ideas involved in establishing the densest lattice sphere packings in dimensions 6, 7 and 8 is presented, but full proofs are omitted. The same is done for the lattice kissing numbers for dimensions 4 to 9. The densest lattice sphere packings for dimensions larger than 8 are unknown, as are the lattice kissing numbers for dimensions larger than 9, with the exception of dimension 24. (Proofs by Bonnai and Sloane that $k^*(S_8) = 240$ and $k^*(S_{24}) = 196\,560$ are given in Chapter 9.)

Chapter 3 is concerned with lower bounds for packing densities $\delta(S_n)$, and includes a proof of the classic result of Minkowski and Hlawka, along with improvements by others to their lower bound. It also includes proofs of the classic work on density of lattice sphere coverings by Rogers and by Rogers, Few and Coxeter. Chapter 4 takes up the interesting idea of the blocking number for a convex body, which was introduced by Zong in 1994. The blocking number $b(K)$ of a convex body K is the smallest number of non-overlapping copies of K that can be touching the boundary of K, but which do not allow any additional non-overlapping copies to touch the boundary of K. Numerous results are proven or just presented in this chapter; proofs of the results $b(S_2) = 4$, $b(S_3) = 6$ and $b(S_4) = 9$ are given. Chapter 5 presents connections between codes and sphere packings, including descriptions of the codes relevant to sphere packings. Chapters 6, 7 and 8 discuss upper bounds for the packing densities $\delta(S_n)$ and kissing numbers $k(S_n)$. These chapters describe the work, including proofs, of many people, such as Blichfeldt, Rankin and Rogers. The remaining chapters deal with important topics related to those already mentioned. Their titles speak for themselves: 10: Multiple sphere packings; 11: Holes in sphere packings; 12: Problems of blocking light rays; and 13: Finite sphere packing.

The fact that this book is very readable and error-free indicates the care taken in its preparation. It is accessible to graduate students or advanced undergraduates. I also appreciated the biographical sketches of some of the mathematicians who have made major contributions to the subject. The bibliography seems complete, and the book will no doubt become a standard reference. It will certainly remain on my desk, and I anticipate reaching for it many times.

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GEOMETRIC NONLINEAR FUNCTIONAL ANALYSIS, Volume 1
(American Mathematical Society Colloquium Publications 48)

By YOAV BENYAMINI and JORAM LINDENSTRAUSS: 488 pp., US$65.00,
ISBN 0-8218-0835-4 (American Mathematical Society, Providence, RI, 2000).

This is a fascinating monograph, containing a wealth of information on the one hand and motivating further research on the other. The motivational success has been already proven by the appearance of results by several young mathematicians who solved problems stemming from its preliminary version. The unifying theme of the book is 'the study of uniformly continuous and, in particular, Lipschitz functions between Banach spaces'. This programme distinguishes it from a number of other texts that understand 'nonlinear functional analysis' as the study of questions related to nonlinear differential equations. By carefully avoiding an overlap with this direction (and so also with texts on general fixed-point theory), the authors have been able to cover many aspects of the development that have not been treated in a similar form before. However, a number of important subareas have not made it to this first volume, so we must wait for the second volume to see an accessible account of them.

It is impossible to give a detailed account of all important points that are treated here, so the following account just highlights some of them. The book starts with uniformly continuous and Lipschitz retractions, extensions and selections. After basic results (Kirszbraun's theorem, Michael's selection theorem, etc.), it moves to detailed study of various related problems, such as approximation by functions with prescribed modulus of continuity, precise estimates of the modulus of continuity of the nearest-point map, and a description of Banach spaces E for which every Lipschitz mapping of a subset of E to E may be extended to the whole space preserving the Lipschitz constant.

A chapter on fixed points is restricted to the connections between fixed-point theory and the geometry of Banach spaces (so it includes only the main classical fixed-point theorems). From this point of view, the main result is a construction, for any non-compact set C, of a Lipschitz map $f: C \to C$ with no (even approximate) fixed points. As a corollary, one obtains, in any infinite-dimensional Banach space, the contractibility of the sphere or existence of a Lipschitz retraction of a ball to its boundary. This is followed by a detailed discussion of challenging problems around the existence of fixed points for non-expansive mappings.

Linearisation of mappings is one of the most important tools in nonlinear analysis. The authors enter this subject via its best understood case of differentiability of convex continuous functions, where for Gâteaux differentiability even a complete description of non-differentiability sets (due to Zajíček) is known, and for Fréchet differentiability the results, although not so complete, also appear to be satisfactory. Another reasonably well understood case is that of differentiability of Lipschitz mappings of the real line to a Banach space, which is closely related to and may serve as a definition of the Radon–Nikodým property (RNP). A chapter is devoted to those results on the RNP that bring it into the present context. In particular, the distinction between differentiability and affine approximation of Lipschitz mappings of the real line to a Banach space X is traced to that between the RNP and the property that the unit sphere of X does not contain ε dyadic trees, thus shedding new light on the examples due to Bourgain and Rosenthal.

One may expect that, analogously to the classical results of Lebesgue and Rademacher, Lipschitz functions between reasonable spaces are differentiable almost everywhere with respect to a suitable notion of null sets. A chapter is therefore devoted to the discussion of various notions of null sets, such as Haar null sets (as defined by Christensen), cube null sets, Gaussian null sets and Aronszajn null sets, and includes the remarkable recent result of Csörnyei that the last three notions coincide. Two arguments leading to a generalisation of Rademacher's theorem to Gâteaux differentiability of Lipschitz mappings between infinite-dimensional spaces are given. It is also shown that the situation is considerably more complicated if we consider Fréchet differentiability of real-valued Lipschitz, or even convex, functions.

As a natural application of linearisation techniques, one should be able to answer the 'Lipschitz isomorphism problem', that is, to show that reasonable (say, separable and reflexive) Lipschitz equivalent Banach spaces are linearly equivalent. Similarly, one should be able to answer the corresponding problems for Lipschitz embeddings and Lipschitz quotients. (The rather recent notions of Lipschitz and uniform quotients are treated here for the first time in book form.) However, known linearisation techniques are not strong enough to provide satisfactory answers (with the exception of Lipschitz embeddings), and indeed many important questions, including the isomorphism problem itself, are still open, although using also other techniques, including those of linear theory, the isomorphism problem has been answered for many classical spaces. The need for better understanding of linearisation and differentiability techniques is illustrated, for instance, by the fact that every Banach space is Lipschitz equivalent to a subset of c_0, by an example of a Lipschitz isomorphism of a separable Hilbert space onto itself whose derivative fails to be an isomorphism at many points, or by an example of non-separable Lipschitz isomorphic spaces that are not linearly isomorphic.

The study of uniform isomorphisms, embeddings and quotients considerably differs from that of Lipschitz ones, because no direct use of differentiability techniques is available. As an illuminating example, one may use the result that the (quasi-normed) spaces that embed uniformly into a Hilbert space are precisely those that are linearly isomorphic to a subspace of the space of measurable functions on some measure space, or that the unit sphere of a very large class of spaces is uniformly isomorphic to the unit sphere in a Hilbert space. On the other hand, for uniform isomorphisms defined on the whole space, ultraproduct techniques lead to Lipschitz isomorphisms, and one may use the results from previous chapters to obtain information on the linear structure of the spaces. Together with other methods (among which at least Gorelik's principle should surely be mentioned), this shows that in many classical situations uniform isomorphism implies linear isomorphism, but that this is far from being true in general: one can even find a Banach space which is uniformly isomorphic to exactly two (linearly different) Banach spaces.

Two chapters are devoted to important questions concerning oscillation of uniformly continuous functions on unit spheres in finite- and infinite-dimensional spaces. The results start from the fundamental theorem of Dvoretzky, and lead to an accessible account of many of the recent major developments, such as the solution of the distortion problem due to Odell and Schlumprecht, or the results of Gowers and Maurey.

Mappings close to isometries are well known from the pioneering study of F. John which led to the introduction of BMO functions. The study presented here indicates which of the natural questions find their answers in general Banach spaces, and which need one to go to finite-dimensional Euclidean spaces. A special chapter is devoted to the well-understood case of global surjective near isometries.

The study of quasi-linear functions

$$\|f(x+y)-f(x)-f(y)\| = O(\|x\|+\|y\|)$$

is related to that of twisted sums of Banach spaces, for which a natural setting is that of quasi-Banach spaces. In particular, quasi-linear functions may be used to construct non-trivial twisted sums of a Hilbert space with itself; these are then used, for example, to construct Banach spaces having quite unexpected properties.

The final chapter is devoted to questions related to Hilbert's fifth problem in the setting of groups modelled on an infinite-dimensional Banach space; the subject is related to the previous themes via the need for uniform continuity of the group operations.

The promised second volume should contain results on Fréchet differentiability of Lipschitz functions on Banach spaces, study Lipschitz maps from a discrete point of view, and explain connections with analytic functions. Surely something to look forward to!

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STRONG SHAPE AND HOMOLOGY
(Springer Monographs in Mathematics)

By SIBE MARDEŠIĆ: 489 pp., £55.00, ISBN 3-540-66198-0 (Springer, Berlin, 2000).

As a postgraduate, I attended a British Mathematical Colloquium, and there met other postgraduates. Of course, one of the first questions asked was: 'What is your research area?' When informed that I was studying the homotopy theory of general spaces (such as compact metric spaces with no restrictions on their local behaviour), a fellow postgraduate from another university asked: 'Why on earth should anyone want to look at the algebraic topology of spaces that are not manifolds or at least CW-complexes?' I was unable to formulate an answer that might have attempted to satisfy him. My answer should have been a question: '*How is a given space specified?*' It may arise as a manifold, but it may also be given as the spectrum of a C*-algebra obtained from some analytic problem, as a space of leaves of a foliation, as an attractor of a dynamical system, or as a fractal given by some iterated function system. In these cases, the apparatus of ordinary algebraic topology can prove powerless! The space as given is extremely unlikely to have a CW-complex structure. Yet experience from the algebraic topology of manifolds and complexes showed that subtle geometric and 'analytic' properties of such a space are detectable via the invariants that modern homotopy and homology theory provide. How can one try to extend the methods of homotopy and homology theory to such spaces?

The question has, of course, a long history, with names such as Čech, Vietoris and Alexandroff involved in the late 1920s in defining homological invariants for compact spaces. Although quite well behaved, Čech homology theory did not yield exact sequences in general, so this made any attempt at calculation much more difficult. The problem was that the definition used the inverse limit, and that did not preserve exactness. In 1940, Steenrod published a paper giving a definition of a different homology construction for metric compacta which did give exact sequences. This was a strong homology in the sense of this book.

The theory of inverse systems was first used by Lefschetz in the 1930s. He discussed their properties and revealed some of the difficulties in their use. A student of his, Christie, published work (1944) on a homotopy theory parallel to Čech homology. For metric compacta, he also introduced a stronger form of his theory which was a type of strong shape theory. Christie's work lay fallow for 24 years until Borsuk (1968) developed shape theory. This was closely related to Christie's Čech-type homotopy theory, and like that suffered from the same weakness as Čech homology, namely lack of exactness. Finally, in about 1973, various shape theorists realised that shape theory could be strengthened by retaining more of the data. All of the Čech-style theories work by using inverse systems of polyhedra to approximate the spaces being studied, but – and it is a big 'but' – these are often inverse systems in the homotopy category of polyhedra. For strong shape, one works not with proHo(Top), that is, with inverse systems in this homotopy category, but with Ho(proTop), a homotopy category of inverse systems of spaces, and therein lies the technical difficulty that makes this book necessary. This category Ho(proTop) has a beautiful interpretation in terms of inverse systems, and homotopy coherent mappings of inverse systems, but to work with homotopy coherence you need to take a lot of combinatorial care of your defining data.

This book takes the reader step-by-step through the detailed treatment of strong shape and the related strong homology. The author has taken a lot of care in his description of coherence, and has succeeded in giving a treatment that is, in many ways, as elementary as it can be. There *are* pages of detailed formulae, but that is because the very essence of homotopy coherence is that, for instance, it keeps information on why two maps are homotopic (by specifying the homotopy) rather than merely asking that a homotopy exist, so those homotopies (and the homotopies between homotopies, etc.) have to be explicitly constructed.

The book is very well organised. It is divided into four parts: I: Coherent homotopy; II: Strong shape; III: Higher derived limits; and IV: Homology groups. The content of the first two of these is as suggested by their titles. The third part may need a little more description. The key weakness with the construction of Čech homology was that it used inverse limits, and they destroy exactness. To measure the lack of exactness of Lim on a given inverse system, one uses its derived functors $\mathrm{Lim}^{(i)}$. Part III contains a well-written, self-contained exposition of the theory of the derived functors of Lim. The material has been collected with great care, and again is 'elementary' and approachable without a lifetime of preparatory reading.

The algebraic results of Part III are used in Part IV to show the dependence of the various strong homology groups on subtle topological properties of the spaces. Some of this material appears here for the first time in a book.

Formulae are explicitly given. This can be daunting if you try to 'dip' into the book, but their motivation and geometric interpretation are clear if the earlier sections on homotopy cohernt systems are tackled first.

Does the book answer that question asked of me? It does not. There is a lot of active research going on in this area, but there are still many problems concerned with linking the data, say, for an iterated function system or dynamical system to the methods used here. There clearly is a great chance of an important link-up, and the author promises that a forthcoming survey article by himself and J. Segal will cover some of these developing links. I look forward to seeing it.

Finally, is there any adverse criticism to be made about the book? To that question I have to reply that although the historical notes at the end of each chapter were useful, I did not always agree with their content. Some very interesting early pieces of work were omitted, and various links were played down. That, however, is a minor criticism. This is a very well-written book, dealing with an important area of topology. The theory *is* hard, but it is tackling very difficult problems at the interface between (algebraic) topology and analysis. It is a very important addition to the literature.

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