An analytic solution for gust–cascade interaction noise including effects of realistic aerofoil geometry – supplementary material

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This supplementary material includes several details of calculations and methods used in the main paper.

S:1. $\mathscr{O}(1)$ solution

We may write the $\mathscr{O}(1)$ problem as

$$\frac{\partial^2 h_0}{\partial \phi^2} + \frac{\partial^2 h_0}{\partial \psi^2} + k^2 w^2 h_0 = 0, \tag{S:1.1a}$$

subject to no discontinuities upstream:

$$\varDelta_0 \left[h_0\right](\phi) = 0, \qquad \phi < 0, \tag{S:1.1b}$$

no-flux across the blade:

$$\varSigma_0 \left[\frac{\partial h_0}{\partial \psi}\right](\phi) = -2\frac{A_n}{\beta_\infty}\mathrm{e}^{\mathrm{i}k\delta\phi}, \qquad 0 < \phi < c_\phi, \tag{S:1.1c}$$

$$\varDelta_0 \left[\frac{\partial h_0}{\partial \psi}\right](\phi) = 0, \qquad 0 < \phi < c_\phi, \tag{S:1.1d}$$

no-flux across the wake:

$$\varDelta_0 \left[\frac{\partial h_0}{\partial \psi}\right](\phi) = 0, \qquad \phi > c_\phi, \tag{S:1.1e}$$

and no pressure jump across the wake:

$$\varDelta_0 \left[h_0\right](\phi) = 2\pi\mathrm{i}P_0\mathrm{e}^{\mathrm{i}k\delta\phi}, \qquad \phi > c_\phi. \tag{S:1.1f}$$

The solution is given in terms of the Fourier transform of the jump in acoustic potential either side of the blade streamline in (3.11). In that equation, $\mathscr{A}_{0,n}$, $\mathscr{B}_{0,n}$ and $\mathscr{C}_{0,n}$ are defined analogously to section S:2.2 with

$$\mathscr{T}_{0,0} = -2\frac{A_n}{\beta_\infty}, \qquad \mathscr{T}_{0,l} = 0, \qquad l \neq 0.$$

The Fourier transform of the acoustic field may be inverted in the downstream region (Glegg 1999, §5.1) to obtain the value of the acoustic potential along the wake for insertion into the boundary conditions (3.8b) and (3.9b). In a strip in the downstream region

$(0 < \psi < s_\phi,\ s_\phi(\phi - c_\phi) < d_\phi \psi)$, we have

$$h_0(\phi,\psi) = \sum_{m=-\infty}^{\infty} \mathscr{H}^-_{0,m} \mathrm{e}^{-\mathrm{i}(\lambda^-_m \phi + \zeta^-_m \psi)}, \tag{S:1.2}$$

$$+ \sum_{\pm} \mathscr{H}^-_{0,\kappa^-_0,\pm} \mathrm{e}^{-\mathrm{i}(\kappa^-_0 \phi \pm \zeta_{\kappa^-_0} \psi)}, \tag{S:1.3}$$

where

$$\mathscr{H}^-_{0,m} = \frac{-\pi \zeta^-_m D^{(2,4)}_0(\lambda^-_m)}{\Delta_\phi \sqrt{k^2 w^2 - f_m^2}},$$

$$\mathscr{H}^-_{0,\kappa^-_0,\pm} = \frac{\pi \mathrm{i} P_0}{2} \cdot \frac{\mathrm{e}^{\mathrm{i}(d_\phi \kappa^-_0 + \sigma'_\phi)} - \mathrm{e}^{\pm \mathrm{i} s_\phi \zeta_{\kappa^-_0}}}{\cos(d_\phi \kappa^-_0 + \sigma'_\phi) - \cos(s_\phi \zeta_{\kappa^-_0})}.$$

Along the lower side of the wake ($\psi = 0^-$) we have

$$h^-_0(\phi) = \sum_{\pm} \mathscr{H}^-_{0,\kappa^-_0,\pm} \mathrm{e}^{-\mathrm{i}(\kappa^-_0 \phi + d_\phi \kappa^-_0 + \sigma'_\phi \pm s_\phi \zeta_{\kappa^-_0})} + \sum_{m=-\infty}^{\infty} \mathscr{H}^-_{0,m} \mathrm{e}^{-\mathrm{i}\lambda^-_m \phi},$$

$$\frac{\partial h^-_0}{\partial \psi}(\phi) = \sum_{\pm} \mp \mathrm{i} \zeta_{\kappa^-_0} \mathscr{H}^-_{0,\kappa^-_0,\pm} \mathrm{e}^{-\mathrm{i}(\kappa^-_0 \phi + d_\phi \kappa^-_0 + \sigma'_\phi \pm s_\phi \zeta_{\kappa^-_0})} - \mathrm{i} \sum_{m=-\infty}^{\infty} \zeta^-_m \mathscr{H}^-_{0,m} \mathrm{e}^{-\mathrm{i}\lambda^-_m \phi},$$

where we have used the quasi-periodicity condition (3.14).

Finally, we define the constants

$$\mathscr{H}^{\pm\prime}_{0,\kappa^-_0} = -\mathrm{i}\left(\kappa^-_0 + k M^2_\infty \delta\right) \mathscr{H}^-_{0,\kappa^-_0,\pm} \mathrm{e}^{-\mathrm{i}(d_\phi \kappa^-_0 + \sigma'_\phi \pm s_\phi \zeta_{\kappa^-_0})}, \tag{S:1.4.a}$$

$$\mathscr{H}'_{0,m} = -\mathrm{i}\left(\lambda^-_m + k M^2_\infty \delta\right) \mathscr{H}^-_{0,m}, \tag{S:1.4.b}$$

so that in the normal velocity continuity condition (2.23) we may insert

$$\frac{\partial^2 h^-}{\partial \phi \partial \psi}(\phi) - \mathrm{i} k M^2_\infty \delta \frac{\partial h^-}{\partial \psi}(\phi) = \sum_{\pm} \mp \mathrm{i} \zeta_{\kappa^-_0} \mathscr{H}^{\pm\prime}_{0,\kappa^-_0} \mathrm{e}^{-\mathrm{i}\kappa^-_0 \phi} - \mathrm{i} \sum_{m=-\infty}^{\infty} \zeta^-_m \mathscr{H}'_{0,m} \mathrm{e}^{-\mathrm{i}\lambda^-_m \phi}, \tag{S:1.5}$$

and in the pressure continuity condition (2.27) we may insert

$$\frac{\partial h^-}{\partial \phi}(\phi) - \mathrm{i} k M^2_\infty \delta h^-(\phi) = \sum_{\pm} \mathscr{H}^{\pm\prime}_{0,\kappa^-_0} \mathrm{e}^{-\mathrm{i}\kappa^-_0 \phi} + \sum_{m=-\infty}^{\infty} \mathscr{H}'_{0,m} \mathrm{e}^{-\mathrm{i}\lambda^-_m \phi}, . \tag{S:1.6}$$

S:2. Solutions to Wiener–Hopf problems

In this section we provide the details of the solutions to the integral equations (3.31a), (3.33a), (3.35a) and (3.37a) subject to the relevant boundary conditions. This corresponds to step V of the road map. The first solution, for $h_{1,\Sigma}$, will include most of the mathematical details and the subsequent solutions for $h_{1,\Delta}$, $h_{1,\Gamma}$ and $h_{1,S}$ will refer to the first solution in many instances since the techniques employed are similar.

We begin by providing some crucial details of the Wiener–Hopf kernel.

S:2.1. *Details of Wiener–Hopf kernel, $j(\gamma)$*

Similarly to Glegg (1999), the multiplicative splitting of the Wiener–Hopf kernel is given by

$$j(\gamma) = \frac{\zeta}{4\pi}\left\{\frac{\sin(\zeta s_\phi)}{\cos(\zeta s_\phi) - \cos(\gamma d_\phi + \sigma'_\phi)}\right\} = J_+(\gamma)J_-(\gamma), \tag{S:2.1}$$

where

$$J_+(\gamma) = \frac{kw\sin(kws_\phi)}{4\pi\left(\cos(kws_\phi) - \cos(\sigma'_\phi)\right)} \cdot \frac{\prod_{n=0}^{\infty}\left(1-\gamma/\theta_n^-\right)}{\prod_{m=-\infty}^{\infty}\left(1-\gamma/\lambda_m^-\right)}\mathrm{e}^{\varPhi},$$

$$J_-(\gamma) = \frac{\prod_{m=0}^{\infty}\left(1-\gamma/\theta_n^+\right)}{\prod_{m=-\infty}^{\infty}\left(1-\gamma/\lambda_m^+\right)}\mathrm{e}^{-\varPhi},$$

and

$$\varPhi = \frac{-\mathrm{i}\gamma}{\pi}\left(s_\phi \log(2\cos(\chi_\phi) + \chi_\phi d_\phi)\right).$$

The zeros of $J_\pm$ are given by the duct modes

$$\theta_n^\pm = \pm\sqrt{k^2w^2 - \left(\frac{n\pi}{s_\phi}\right)^2},$$

and the poles are given by the acoustic modes

$$\lambda_m^\pm = -f_m\sin(\chi_\phi) \pm \cos(\chi_\phi)\sqrt{k^2w^2 - f_m^2}, \qquad f_m = \frac{\sigma'_\phi - 2\pi m}{\varDelta_\phi}. \tag{S:2.2}$$

The derivatives of $J^\pm$ evaluated at the duct modes are

$$J'_\pm(\theta_n^\mp) = \frac{-\theta_n^\mp s_\phi(1+\delta_{n,1})}{4\pi\left(1-(-1)^n\cos(d_\phi\theta_n^\mp + \sigma'_\phi)\right)J_\mp(\theta_n^\mp)}.$$

It may be demonstrated using standard results of the Gamma function (Peake 1992, Appendix B), that the split kernel has asymptotic behaviour

$$J^\pm(\gamma) = \mathscr{O}\left(\gamma^{1/2}\right), \qquad \text{as } |\gamma| \to \infty \text{ in } \mathscr{M}^\pm, \tag{S:2.3}$$

where $\mathscr{M}^\pm$ correspond to the upper and lower half planes with an overlapping strip. During the Wiener–Hopf analysis it is necessary to perform additive splittings of functions involving the Wiener–Hopf kernel. To expedite this processes, we define the auxiliary functions

$$\tilde{J}_\pm(\gamma, \eta^\pm) = \frac{J_-(\gamma)}{\gamma - \eta^\pm}. \tag{S:2.4}$$

Note that $\tilde{J}_\pm$ is not analytic in $\mathscr{M}^\pm$ since it possesses a simple pole at $\gamma = \eta^\pm$. Accordingly, we use pole removal to obtain the additive splittings

$$\left[\tilde{J}_\pm(\gamma,\eta^\pm)\right]_\pm = \frac{J_\pm(\gamma) - J_\pm(\eta^\pm)}{\gamma - \eta^\pm}, \qquad \left[\tilde{J}_\pm(\gamma,\eta^\pm)\right]_\mp = \frac{J_\pm(\eta^\pm)}{\gamma - \eta^\pm}. \tag{S:2.5}$$

We now present the solutions for the Wiener–Hopf equations for the $D_{1,\varSigma}$, $D_{1,\varDelta}$, $D_{1,\varGamma}$ and $D_{1,S}$ contributions.

S:2.2. *Solution for $D_{1,\Sigma}$*

The integral equation and boundary conditions for $\Delta_0\left[h_{1,\Sigma}\right]$ and $D_{1,\Sigma}$ are summarised in (3.31a–d). In a similar way to Glegg (1999), we split this problem into four coupled problems which are amenable to the Wiener–Hopf method. We write

$$\Delta_0\left[h_{1,\Sigma}\right](\phi) = \Delta_0\left[h_{1,\Sigma}^{(1)}\right](\phi) + \Delta_0\left[h_{1,\Sigma}^{(2)}\right](\phi) + \Delta_0\left[h_{1,\Sigma}^{(3)}\right](\phi) + \Delta_0\left[h_{1,\Sigma}^{(4)}\right](\phi), \quad \text{(S:2.6)}$$

and its Fourier transform

$$D_{1,\Sigma}(\gamma) = D_{1,\Sigma}^{(1)}(\gamma) + D_{1,\Sigma}^{(2)}(\gamma) + D_{1,\Sigma}^{(3)}(\gamma) + D_{1,\Sigma}^{(4)}(\gamma), \quad \text{(S:2.7)}$$

where each $\Delta_0\left[h_{1,\Sigma}^{(n)}\right]$ and $D_{1,\Sigma}^{(n)}$ satisfy a semi-infinite integral equation of the form

$$f_{1,\Sigma}^{(n)}(\phi) = -4\pi\int_{-\infty}^{\infty} D_{1,\Sigma}^{(n)}(\gamma)j(\gamma)\mathrm{e}^{-\mathrm{i}\gamma\phi}\,\mathrm{d}\gamma, \quad \text{(S:2.9)}$$

for $n = 1, 2, 3, 4$. The corresponding boundary values are

$$f_{1,\Sigma}^{(1)}(\phi) = \sum_{l=-\infty}^{\infty} \mathscr{c}_{\Sigma,l}\,\mathrm{e}^{-\mathrm{i}\kappa_l^-\phi}, \quad \phi > 0; \quad \text{(S:2.9.a)}$$

$$\Delta_0\left[h_{1,\Sigma}^{(1)}\right](\phi) = 0, \quad \phi < 0; \quad \text{(S:2.9.b)}$$

$$f_{1,\Sigma}^{(2)}(\phi) = 0, \quad \phi < c_\phi; \text{(S:2.10.a)}$$

$$\Delta_0\left[h_{1,\Sigma}^{(1)}\right](\phi) + \Delta_0\left[h_{1,\Sigma}^{(2)}\right](\phi) = 2\pi\mathrm{i}P_{1,\Sigma}^{(2)}\mathrm{e}^{-\mathrm{i}\kappa_0^-\phi}, \quad \phi > c_\phi; \text{(S:2.10.b)}$$

$$f_{1,\Sigma}^{(3)}(\phi) = 0, \quad \phi > 0; \text{(S:2.11.a)}$$

$$\Delta_0\left[h_{1,\Sigma}^{(2)}\right](\phi) + \Delta_0\left[h_{1,\Sigma}^{(3)}\right](\phi) + \Delta_0\left[h_{1,\Sigma}^{(4)}\right](\phi) = 0, \quad \phi < 0; \text{(S:2.11.b)}$$

$$f_{1,\Sigma}^{(4)}(\phi) = 0, \quad \phi < c_\phi; \text{(S:2.12.a)}$$

$$\Delta_0\left[h_{1,\Sigma}^{(3)}\right](\phi) + \Delta_0\left[h_{1,\Sigma}^{(4)}\right](\phi) = 2\pi\mathrm{i}P_{1,\Sigma}^{(4)}\mathrm{e}^{-\mathrm{i}\kappa_0^-\phi}, \quad \phi > c_\phi; \text{(S:2.12.b)}$$

where $P_{1,\Sigma}^{(2)}$ and $P_{1,\Sigma}^{(4)}$ are two constants of integration that will be specified to enforce the Kutta condition. Summing the four above conditions results in the original boundary conditions and, consequently, we may apply the Wiener–Hopf method to each semi-infinite integral equation and sum the resulting contributions to obtain a solution to the original equations.

S:2.2.1. *Solution to first Wiener–Hopf problem – $D_{1,\Sigma}^{(1)}$*

In this section we solve the integral equation (S:2.9) for $n = 1$,

$$f_{1,\Sigma}^{(1)}(\phi) = -4\pi\int_{-\infty}^{\infty} D_{1,\Sigma}^{(1)}(\gamma)j(\gamma)\mathrm{e}^{-\mathrm{i}\gamma\phi}\,\mathrm{d}\gamma, \quad \text{(S:2.13.a)}$$

subject to (S:2.9.a) and (S:2.9.b):

$$f_{1,\Sigma}^{(1)}(\phi) = \sum_{l=-\infty}^{\infty} \mathscr{c}_{\Sigma,l}\,\mathrm{e}^{-\mathrm{i}\kappa_l^-\phi}, \quad \phi > 0; \qquad \Delta_0\left[h_{1,\Sigma}^{(1)}\right](\phi) = 0, \quad \phi < 0. \quad \text{(S:2.13.b)}$$

Taking a Fourier transform of (S:2.13.a) gives

$$F^{(1)}_{1,\Sigma,-}(\gamma) + F^{(1)}_{1,\Sigma,+}(\gamma) = -4\pi D^{(1)}_{1,\Sigma,+}(\gamma) j(\gamma), \tag{S:2.14}$$

where

$$F^{(1)}_{1,\Sigma,-}(\gamma) = \frac{1}{2\pi}\int_{-\infty}^{0} f^{(1)}_{1,\Sigma}(\phi)\mathrm{e}^{\mathrm{i}\gamma\phi}\mathrm{d}\phi, \quad F^{(1)}_{1,\Sigma,+}(\gamma) = \frac{1}{2\pi}\int_{0}^{\infty} f^{(1)}_{1,\Sigma}(\phi)\mathrm{e}^{\mathrm{i}\gamma\phi}\mathrm{d}\phi, \tag{S:2.15.a}$$

$$D^{(1)}_{1,\Sigma,+}(\gamma) = \int_{0}^{\infty} \Delta_0\left[h^{(1)}_{1,\Sigma}\right](\phi)\mathrm{e}^{\mathrm{i}\gamma\phi}\mathrm{d}\phi. \tag{S:2.15.b}$$

We may use (S:2.1) to multiplicatively factorise $j = J_- J_+$ and express the Wiener–Hopf equation (S:2.14) as

$$\frac{F^{(1)}_{1,\Sigma,-}(\gamma)}{4\pi J_-(\gamma)} + \frac{F^{(1)}_{1,\Sigma,+}(\gamma)}{4\pi J_-(\gamma)} = -D^{(1)}_{1,\Sigma,+}(\gamma) J_+(\gamma). \tag{S:2.16}$$

Since we have decomposed the boundary data into a Fourier series, we are able to analytically calculate its half-range Fourier transform. Applying the first relation in (S:2.13.b) results in

$$F^{(1)}_{1,\Sigma,+}(\gamma) = \frac{1}{2\pi\mathrm{i}}\sum_{l=-\infty}^{\infty}\frac{\mathscr{c}_{\Sigma,l}}{\gamma-\kappa_l^-}.$$

The next step in the Wiener–Hopf analysis is to additively factorise the left-hand side of (S:2.16). The first term is already analytic in the lower half plane but the second term is not analytic in either half plane. Therefore, we apply the typical technique of pole removal (Noble 1958) to obtain

$$\frac{F^{(1)}_{1,\Sigma,+}(\gamma)}{J_-(\gamma)} = \underbrace{\frac{1}{2\pi\mathrm{i}}\sum_{l=-\infty}^{\infty}\frac{\mathscr{c}_{\Sigma,l}}{\gamma-\kappa_l^-}\frac{1}{J_-(\kappa_l^-)}}_{+} + \underbrace{\frac{1}{2\pi\mathrm{i}}\sum_{l=-\infty}^{\infty}\frac{\mathscr{c}_{\Sigma,l}}{\gamma-\kappa_l^-}\left\{\frac{1}{J_-(\gamma)} - \frac{1}{J_-(\kappa_l^-)}\right\}}_{-},$$

where the underbrace $\pm$ denotes that the function is analytic in the upper or lower half-plane respectively. Therefore, (S:2.16) becomes

$$\begin{aligned}\frac{F^{(1)}_{1,\Sigma,-}(\gamma)}{4\pi J_-(\gamma)} &+ \frac{1}{8\pi^2\mathrm{i}}\sum_{l=-\infty}^{\infty}\frac{\mathscr{c}_{\Sigma,l}}{\gamma-\kappa_l^-}\left\{\frac{1}{J_-(\gamma)} - \frac{1}{J_-(\kappa_l^-)}\right\} \\ &= -D^{(1)}_{1,\Sigma,+}(\gamma)J_+(\gamma) - \frac{1}{8\pi^2\mathrm{i}}\sum_{l=-\infty}^{\infty}\frac{\mathscr{c}_{\Sigma,l}}{\gamma-\kappa_l^-}\frac{1}{J_-(\kappa_l^-)}.\end{aligned} \tag{S:2.17}$$

We may now apply the standard Wiener–Hopf argument: since the left- and right-hand sides of (S:2.17) are analytic in $\mathscr{M}^{\mp}$ respectively, and they agree on a strip, each side defines the analytic continuation of the other. Therefore, equation (S:2.17) defines an entire function, $E_1(\gamma)$. By appealing to typical arguments that are justified in section S:4, as $|\gamma| \to \infty$ in $\mathscr{M}^-$, the left-hand side of (S:2.17) decays due to (S:4.1.a) and (S:2.3). Similarly, as $|\gamma| \to \infty$ in $\mathscr{M}^+$, the right-hand side of (S:2.17) is bounded due to (S:4.1.b) and (S:2.3). Therefore, $E_1(\gamma)$ is bounded in the entire plane so Liouville's theorem tells us that it must be a constant and, since $E_1(\gamma)$ decays in $\mathscr{M}^-$, this constant must be zero.

Therefore, we may rearrange the right-hand side of (S:2.17) to obtain

$$D^{(1)}_{1,\Sigma,+}(\gamma) = \sum_{l=-\infty}^{\infty} \frac{\mathscr{T}_{1,\Sigma,l}}{(\gamma - \kappa_l^-)} \cdot \frac{1}{J_+(\gamma)}, \tag{S:2.18}$$

where

$$\mathscr{T}_{1,\Sigma,l} = -\frac{\mathscr{c}_{\Sigma,l}}{8\pi^2 \mathrm{i}} \cdot \frac{1}{J_-(\kappa_l^-)}.$$

S:2.2.2. *Solution to second Wiener–Hopf problem* – $D^{(2)}_{1,\Sigma}$

In this section we solve the integral equation (S:2.9) for $n = 2$,

$$f^{(2)}_{1,\Sigma}(\phi) = -4\pi \int_{-\infty}^{\infty} D^{(2)}_{1,\Sigma}(\gamma) j(\gamma) \mathrm{e}^{-\mathrm{i}\gamma\phi} \mathrm{d}\gamma, \tag{S:2.19.a}$$

subject to (S:2.10.a) and (S:2.10.b):

$$f^{(2)}_{1,\Sigma}(\phi) = 0, \quad \phi < c_\phi; \qquad \Delta_0\left[h^{(1)}_{1,\Sigma}\right](\phi) + \Delta_0\left[h^{(2)}_{1,\Sigma}\right](\phi) = 2\pi\mathrm{i} P^{(2)}_{1,\Sigma} \mathrm{e}^{-\mathrm{i}\kappa_0^- \phi}, \quad \phi > c_\phi. \tag{S:2.19.b}$$

Taking the Fourier transform of (S:2.19.a) gives

$$F^{(2)}_{1,\Sigma,+}(\gamma) = -4\pi \left(D^{(2)}_{1,\Sigma,-}(\gamma) + D^{(2)}_{1,\Sigma,+}\right) j(\gamma), \tag{S:2.20}$$

where

$$\begin{aligned} F^{(2)}_{1,\Sigma,+}(\gamma) &= \frac{1}{2\pi} \int_{c_\phi}^{\infty} f^{(2)}_{1,\Sigma}(\phi) \mathrm{e}^{\mathrm{i}\gamma\phi} \mathrm{d}\phi \\ &= \frac{\mathrm{e}^{\mathrm{i}\gamma c_\phi}}{2\pi} \int_0^{\infty} f^{(2)}_{1,\Sigma}(\phi + c_\phi) \mathrm{e}^{\mathrm{i}\phi\gamma} \mathrm{d}\phi = \mathrm{e}^{\mathrm{i}\gamma c_\phi} F^{*(2)}_{1,\Sigma,+}(\gamma), \end{aligned} \tag{S:2.21.a}$$

$$\begin{aligned} D^{(2)}_{1,\Sigma,+}(\gamma) &= \frac{1}{2\pi} \int_{c_\phi}^{\infty} \Delta_0\left[h^{(2)}_{1,\Sigma}\right](\phi) \mathrm{e}^{\mathrm{i}\gamma\phi} \mathrm{d}\phi \\ &= \frac{\mathrm{e}^{\mathrm{i}\gamma c_\phi}}{2\pi} \int_0^{\infty} \Delta_0\left[h^{(2)}_{1,\Sigma}\right](\phi + c_\phi) \mathrm{e}^{\mathrm{i}\phi\gamma} \mathrm{d}\phi = \mathrm{e}^{\mathrm{i}\gamma c_\phi} D^{*(2)}_{1,\Sigma,+}(\gamma), \end{aligned} \tag{S:2.21.b}$$

$$\begin{aligned} D^{(2)}_{1,\Sigma,-}(\gamma) &= \frac{1}{2\pi} \int_{-\infty}^{c_\phi} \Delta_0\left[h^{(2)}_{1,\Sigma}\right](\phi) \mathrm{e}^{\mathrm{i}\gamma\phi} \mathrm{d}\phi \\ &= \frac{\mathrm{e}^{\mathrm{i}\gamma c_\phi}}{2\pi} \int_{-\infty}^{0} \Delta_0\left[h^{(2)}_{1,\Sigma}\right](\phi + c_\phi) \mathrm{e}^{\mathrm{i}\phi\gamma} \mathrm{d}\phi = \mathrm{e}^{\mathrm{i}\gamma c_\phi} D^{*(2)}_{1,\Sigma,-}(\gamma). \end{aligned} \tag{S:2.21.c}$$

Factoring out the $\mathrm{e}^{\mathrm{i}\gamma c_\phi}$ dependence and dividing by J_+ transforms the Wiener–Hopf equation (S:2.20) to

$$\frac{F^{*(2)}_{1,\Sigma,+}(\gamma)}{4\pi J_+(\gamma)} = -\left(D^{*(2)}_{1,\Sigma,-}(\gamma) + D^{*(2)}_{1,\Sigma,+}(\gamma)\right) J_-(\gamma). \tag{S:2.22}$$

We may use the downstream boundary condition for this problem (S:2.19.b) to write

$$D^{*(2)}_{1,\Sigma,+}(\gamma) = -\frac{P^{*(2)}_{1,\Sigma}}{\gamma-\kappa_0^-} - \frac{1}{2\pi}\int_0^\infty \Delta_0\left[h^{(1)}_{1,\Sigma}\right](\phi+c_\phi)\mathrm{e}^{\mathrm{i}\gamma\phi}\mathrm{d}\phi. \tag{S:2.23}$$

where $P^{*(2)}_{1,\Sigma} = P^{(2)}_{1,\Sigma}\mathrm{e}^{-\mathrm{i}\kappa_0^- c_\phi}$. To calculate the remaining integral, we use the inversion formula for the Fourier transform:

$$\Delta_0\left[h^{(1)}_{1,\Sigma}\right](\phi) = \int_{-\infty-\mathrm{i}\tau_1}^{\infty-\mathrm{i}\tau_1} D^{(1)}_{1,\Sigma}(\gamma)\mathrm{e}^{-\mathrm{i}\gamma\phi}\mathrm{d}\gamma,$$

for some sufficiently small $\tau_1 > 0$. By substituting this representation into our desired integral in (S:2.23), we obtain

$$\frac{1}{2\pi}\int_0^\infty \Delta_0\left[h^{(1)}_{1,\Sigma}\right](\phi+c_\phi)\mathrm{e}^{\mathrm{i}\gamma\phi}\mathrm{d}\phi = \frac{1}{2\pi}\int_0^\infty\int_{-\infty-\mathrm{i}\tau_1}^{\infty-\mathrm{i}\tau_1} D^{(1)}_{1,\Sigma}(\gamma_1)\mathrm{e}^{-\mathrm{i}\gamma_1(\phi+c_\phi)}\mathrm{d}\gamma_1\mathrm{e}^{\mathrm{i}\gamma\phi}\mathrm{d}\phi.$$

Rearranging the order of integration and computing the resulting ϕ-integral gives

$$\frac{1}{2\pi}\int_0^\infty \Delta_0\left[h^{(1)}_{1,\Sigma}\right](\phi+c_\phi)\mathrm{e}^{\mathrm{i}\gamma\phi}\mathrm{d}\phi = \frac{1}{2\pi\mathrm{i}}\int_{-\infty-\mathrm{i}\tau_1}^{\infty-\mathrm{i}\tau_1}\frac{D^{(1)}_{1,\Sigma}(\gamma_1)\mathrm{e}^{-\mathrm{i}\gamma_1 c_\phi}}{\gamma_1-\gamma}\mathrm{d}\gamma_1.$$

Inspection of the asymptotic behaviour of (S:2.18) determines that, since $d_\phi < c_\phi$, this integral can be closed in $\mathscr{M}^-$. Since there are no branches in the integrand, the integral will consist of the residues of simple poles; $D^{(1)}_{1,\Sigma}$ possesses simple poles at $\gamma = \kappa_l^-$, θ_n^-. Accordingly, we obtain

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\int_{-\infty-\mathrm{i}\tau_1}^{\infty-\mathrm{i}\tau_1}\frac{1}{J_+(\gamma_1)}\sum_{l=-\infty}^{\infty}\frac{\mathscr{T}_{1,\Sigma,l}}{(\gamma_1-\kappa_l^-)}\cdot\frac{\mathrm{e}^{-\mathrm{i}\gamma_1 c_\phi}}{\gamma_1-\gamma}\mathrm{d}\gamma_1 \\
&= \sum_{l=-\infty}^{\infty}\frac{\mathscr{T}_{1,\Sigma,l}}{J_+(\kappa_l^-)}\cdot\frac{\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\gamma-\kappa_l^-} + \sum_{n=0}^{\infty}\left\{\sum_{l=-\infty}^{\infty}\frac{\mathscr{T}_{1,\Sigma,l}}{(\theta_n^- - \kappa_l^-)}\right\}\cdot\frac{\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{J'_+(\theta_n^-)(\gamma-\theta_n^-)},
\end{aligned}$$

and, therefore,

$$\begin{aligned}
\frac{1}{2\pi\mathrm{i}}\int_{-\infty-\mathrm{i}\tau_1}^{\infty-\mathrm{i}\tau_1}\frac{D^{(1)}_{1,\Sigma}(\gamma_1)\mathrm{e}^{-\mathrm{i}\gamma_1 c_\phi}}{\gamma_1-\gamma}\mathrm{d}\gamma_1 &= \sum_{l=-\infty}^{\infty}\frac{\mathscr{S}_{1,\Sigma,l}\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)(\gamma-\kappa_l^-)} \\
&\quad + \sum_{n=0}^{\infty}\frac{\mathscr{A}_{1,\Sigma,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)(\gamma-\theta_n^-)},
\end{aligned}$$

where the constants $\mathscr{S}_{1,\Sigma,l}$ and $\mathscr{A}_{1,\Sigma,n}$ are defined as

$$\mathscr{S}_{1,\Sigma,l} = \frac{\mathrm{i}(\kappa_0^- - \kappa_l^-)}{J_+(\kappa_l^-)}\mathscr{T}_{1,\Sigma,l}, \qquad \mathscr{A}_{1,\Sigma,n} = \frac{\mathrm{i}(\theta_n^- - \kappa_0^-)}{J'_+(\theta_n^-)}\cdot\sum_{l=-\infty}^{\infty}\frac{\mathscr{T}_{1,\Sigma,l}}{\theta_n^- - \kappa_l^-}.$$

Finally, we combine the integral with (S:2.23) to obtain the expression

$$D^{*(2)}_{1,\Sigma,+}(\gamma) = -\sum_{l=-\infty}^{\infty}\frac{\mathscr{S}_{1,\Sigma,l}\mathrm{e}^{\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)(\gamma-\kappa_l^-)} - \sum_{n=0}^{\infty}\frac{\mathscr{A}_{1,\Sigma,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)(\gamma-\theta_n^-)} - \frac{P^{*(2)}_{1,\Sigma}}{\gamma-\kappa_0^-}. \tag{S:2.24}$$

We may now substitute this representation for $D^{*(2)}_{1,\Sigma,+}$ into the Wiener–Hopf equation (S:2.22) to obtain

$$\frac{F^{*(2)}_{1,\Sigma,+}(\gamma)}{4\pi J_+(\gamma)} = - J_-(\gamma)D^{*(2)}_{1,\Sigma,-}(\gamma) + P^{*(2)}_{1,\Sigma}\tilde{J}_-(\gamma,\kappa_0^-) + \sum_{l=-\infty}^{\infty}\frac{\mathscr{S}_{1,\Sigma,l}\mathrm{e}^{\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)}\tilde{J}_-(\gamma,\kappa_l^-) + \sum_{n=0}^{\infty}\frac{\mathscr{A}_{1,\Sigma,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)}\tilde{J}_-(\gamma,\theta_n^-).$$

where we have used the auxiliary function $\tilde{J}$ introduced in (S:2.4).

The left-hand side of the above equation is analytic in the upper half plane, but there are terms on the right-hand side which are not analytic in either plane. Therefore, we use the additive splitting (S:2.5) to obtain

$$\frac{F^{*(2)}_{1,\Sigma,+}(\gamma)}{4\pi J_+(\gamma)} - \sum_{l=-\infty}^{\infty}\frac{\mathscr{S}_{1,\Sigma,l}\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)}\left[\tilde{J}_-(\gamma,\kappa_l^-)\right]_+ - \sum_{n=0}^{\infty}\frac{\mathscr{A}_{1,\Sigma,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)}\left[\tilde{J}_-(\gamma,\theta_n^-)\right]_+$$
$$-P^{*(2)}_{1,\Sigma}\left[\tilde{J}_-(\gamma,\kappa_0^-)\right]_+ = -J_-(\gamma)D^{*(2)}_{1,\Sigma,-}(\gamma) + \sum_{l=-\infty}^{\infty}\frac{\mathscr{S}_{1,\Sigma,l}\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)}\left[\tilde{J}_-(\gamma,\kappa_l^-)\right]_-$$
$$+\sum_{n=0}^{\infty}\frac{\mathscr{A}_{1,\Sigma,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)}\left[\tilde{J}_-(\gamma,\theta_n^-)\right]_+ + P^{*(2)}_{1,\Sigma}\left[\tilde{J}_-(\gamma,\kappa_0^-)\right]_+. \tag{S:2.25}$$

In a similar way to section S:2.2.1, we now apply the standard Wiener–Hopf argument. In contrast to the previous section where the edge condition constrained the leading-edge pressure singularity, we now apply the unsteady Kutta condition (Ayton *et al.* 2016) which tells us that the intensity of the wake is such that both the velocity and pressure are bounded at the trailing-edge. Practically, this means that $\Delta_0\left[h_{1,\Sigma}\right](c_\phi) \neq 0$ and its derivatives are finite. Consequently, the left-hand side of (S:2.25) is bounded. In a similar way to before, integration by parts can be applied to show that $D^{*(2)}_{1,\Sigma,-}(\gamma) = \mathscr{O}(1/\gamma)$ as $|\gamma| \to \infty$ in $\mathscr{M}^-$. Therefore, in this limit the right-hand side of (S:2.25) tends to a constant. Analytic continuation, Liouville's theorem, and the fact that the left-hand side of (S:2.25) decays in $\mathscr{M}^+$, indicates that this constant must be zero. This allows us to find an appropriate value of $\Delta_0\left[h^{(2)}_{1,\Sigma}\right](c_\phi)$ that satisfies the Kutta condition. If we multiply the left-hand side of (S:2.25) by γ and let $|\gamma| \to \infty$ in $\mathscr{M}^+$, the left-hand side of (S:2.25) indicates that

$$P^{*(2)}_{1,\Sigma} = -\sum_{l=-\infty}^{\infty}\frac{\mathscr{S}_{1,\Sigma,l}\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)}\cdot\frac{J_-(\kappa_l^-)}{J_-(\kappa_0^-)} - \sum_{n=0}^{\infty}\frac{\mathscr{A}_{1,\Sigma,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)}\cdot\frac{J_-(\theta_n^-)}{J_-(\kappa_0^-)}.$$

So that, after substituting in the downstream representation (S:2.24) and the expression for the pressure constant (S:2.26), the right-hand side of (S:2.25) yields

$$D^{(2)}_{1,\Sigma}(\gamma) = -\sum_{l=-\infty}^{\infty}\frac{\mathscr{S}_{1,\Sigma,l}\mathrm{e}^{\mathrm{i}(\gamma-\kappa_l^-)c_\phi}}{\mathrm{i}(\gamma-\kappa_0^-)(\gamma-\kappa_l^-)}\cdot\frac{J_-(\kappa_l^-)}{J_-(\gamma)} - \sum_{n=0}^{\infty}\frac{\mathscr{A}_{1,\Sigma,n}\mathrm{e}^{\mathrm{i}(\gamma-\theta_n^-)c_\phi}}{\mathrm{i}(\gamma-\kappa_0^-)(\gamma-\theta_n^-)}\cdot\frac{J_-(\theta_n^-)}{J_-(\gamma)}. \tag{S:2.26}$$

It should be noted that the only poles of $D^{(2)}_{1,\Sigma}$ in $\mathscr{M}^+$ are at the zeros of J_- where $\gamma = \theta_n^+$.

S:2.2.3. *Solutions to third and fourth Wiener–Hopf problems* – $D^{(3)}_{1,\Sigma}$, $D^{(4)}_{1,\Sigma}$

Since the boundary conditions (S:2.11.b) and (S:2.12.b) for $\Delta_0\left[h^{(3)}_{1,\Sigma}\right]$ and $\Delta_0\left[h^{(4)}_{1,\Sigma}\right]$ are coupled, we must solve for them simultaneously. Taking a Fourier Transform of (S:2.9) and applying the boundary conditions for $n = 3, 4$ gives

$$F^{(3)}_{1,\Sigma,-}(\gamma) = -4\pi\left(D^{(3)}_{1,\Sigma,-}(\gamma) + D^{(3)}_{1,\Sigma,+}(\gamma)\right) j(\gamma), \tag{S:2.27}$$

$$F^{*(4)}_{1,\Sigma,+}(\gamma) = -4\pi\left(D^{*(4)}_{1,\Sigma,-}(\gamma) + D^{*(4)}_{1,\Sigma,+}(\gamma)\right) j(\gamma), \tag{S:2.28}$$

where $F^{(3)}_{1,\Sigma,-}$, $D^{(3)}_{1,\Sigma,-}$ and $D^{(3)}_{1,\Sigma,+}$ are defined in an analogous way to (S:2.15.a) and (S:2.15.b), and $F^{*(4)}_{1,\Sigma,+}$, $D^{*(4)}_{1,\Sigma,+}$ and $D^{*(4)}_{1,\Sigma,-}$ are defined in an analogous way to (S:2.21.a), (S:2.21.b) and (S:2.21.c). Using a similar approach to section S:2.2.2, we may show that the upstream boundary condition (S:2.11.a) is equivalent to

$$D^{(3)}_{1,\Sigma,-}(\gamma) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}\tau_0}^{\infty+\mathrm{i}\tau_0} \frac{D^{(2)}_{1,\Sigma}(\gamma_1) + D^{(4)}_{1,\Sigma}(\gamma_1)}{\gamma_1 - \gamma}\mathrm{d}\gamma_1,$$

for sufficiently small $\tau_0 > 0$. We now assume that the only poles of $D^{(4)}_{1,\Sigma,-}$ in $\mathscr{M}^+$ are at $\gamma = \theta^+_n$. An equivalent assumption was made by Glegg (1999), and it turns out that the same result is achieved by matching upstream and downstream solutions across the inter-blade region (Peake 1993). This assumption, and sufficiently fast decay, allows us to close the above integral in $\mathscr{M}^+$ and evaluate it as

$$D^{(3)}_{1,\Sigma,-}(\gamma) = -\sum_{n=0}^{\infty}\frac{\mathscr{B}_{1,\Sigma,n}}{\gamma - \theta^+_n}, \tag{S:2.29}$$

where $\mathscr{B}_{1,\Sigma,n}$ are the residues of $D^{(2)}_{1,\Sigma,-}(\gamma_1) + D^{(4)}_{1,\Sigma,-}(\gamma_1)$ in $\gamma = \theta^+_n$. The residues of $D^{(4)}_{1,\Sigma,-}$ are currently unknown, but the residues of $D^{(2)}_{1,\Sigma,-}$ are given by

$$D^{(2)}_{r,\Sigma,k} = \frac{-\mathrm{e}^{\mathrm{i}\theta^+_k c_\phi}}{\mathrm{i}(\theta^+_k - \kappa^-_0)J'_-(\theta^+_k)}\left\{\sum_{l=-\infty}^{\infty}\frac{\mathscr{S}_{1,\Sigma,l}\mathrm{e}^{-\mathrm{i}\kappa^-_l\phi_c}}{\theta^+_k - \kappa^-_l}\cdot J_-(\kappa^-_l) + \sum_{n=0}^{\infty}\frac{\mathscr{A}_{1,\Sigma,n}\mathrm{e}^{-\mathrm{i}\theta^-_n c_\phi}}{\theta^+_k - \theta^-_n}\cdot J_-(\theta^-_n)\right\}.$$

We may now substitute (S:2.29) into (S:2.27) and use the factorisations (S:2.1) and (S:2.5) to obtain the Wiener–Hopf equation

$$\frac{F^{(3)}_{1,\Sigma,-}(\gamma)}{4\pi J_-(\gamma)} + \sum_{n=0}^{\infty}\frac{\mathscr{B}_{1,\Sigma,n}}{\gamma - \theta^+_n}\left[\tilde{J}_+(\gamma,\theta^+_n)\right]_- = D^{(3)}_{1,\Sigma,+}(\gamma)J_+(\gamma) - \sum_{n=0}^{\infty}\frac{\mathscr{B}_{1,\Sigma,n}}{\gamma - \theta^+_n}\left[\tilde{J}_+(\gamma,\theta^+_n)\right]_+.$$

We appeal to similar edge conditions as those in section S:2.2.1, and consequently apply the typical Wiener–Hopf argument to obtain

$$D^{(3)}_{1,\Sigma,+}(\gamma) = \sum_{n=0}^{\infty}\frac{\mathscr{B}_{1,\Sigma,n}}{\gamma - \theta^+_n}\cdot\left\{1 - \frac{J_+(\theta^+_n)}{J_+(\gamma)}\right\}.$$

When we combine this expression with the upstream representation (S:2.29) we obtain

$$D^{(3)}_{1,\Sigma}(\gamma) = -\sum_{n=0}^{\infty}\frac{\mathscr{B}_{1,\Sigma,n}}{\gamma - \theta^+_n}\cdot\frac{J_+(\theta^+_n)}{J_+(\gamma)}. \tag{S:2.30}$$

The residues $\mathscr{B}_{1,\Sigma,n}$ still need to be determined; we now move onto the solution for $D^{(4)}_{1,\Sigma}$. In a similar way to section S:2.2.2, we may invert the Fourier transform for the downstream boundary condition (S:2.12.b) to write

$$D^{*(4)}_{1,\Sigma,+}(\gamma) = -\frac{P^{*(4)}_{1,\Sigma}}{\gamma-\kappa_0^-} - \frac{1}{2\pi\mathrm{i}}\int_{-\infty-\mathrm{i}\tau_1}^{\infty+\mathrm{i}\tau_1}\sum_{n=0}^{\infty}\frac{\mathscr{B}_{1,\Sigma,n}}{(\gamma_1-\gamma)(\gamma_1-\theta_n^+)}\left\{1-\frac{J_+(\theta_n^+)}{J_+(\gamma_1)}\right\}\mathrm{e}^{-\mathrm{i}\gamma_1 c_\phi}\mathrm{d}\gamma_1.$$

This integral can be closed in $\mathscr{M}^-$ to obtain

$$D^{*(4)}_{1,\Sigma,+}(\gamma) = -\frac{P^{*(4)}_{1,\Sigma}}{\gamma-\kappa_0^-} - \sum_{n=0}^{\infty}\frac{\mathscr{C}_{1,\Sigma,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)(\gamma-\theta_n^-)}, \tag{S:2.31}$$

where

$$\mathscr{C}_{1,\Sigma,n} = \sum_{k=0}^{\infty}\frac{\mathrm{i}(\theta_n^- - \kappa_0^-)}{(\theta_k^+ - \theta_n^-)}\cdot\frac{J_+(\theta_k^+)}{J_+'(\theta_n^-)}\cdot\mathscr{B}_{1,\Sigma,k}.$$

If we truncate the series, we may write this system of equations in matrix form:

$$\boldsymbol{\mathscr{C}}_{1,\Sigma} = \mathbf{L}\boldsymbol{\mathscr{B}}_{1,\Sigma}, \tag{S:2.32}$$

where

$$\{\mathbf{L}\}_{n,m} = \frac{\mathrm{i}(\theta_n^- - \kappa_0^-)}{(\theta_m^+ - \theta_n^-)}\cdot\frac{J_+(\theta_m^+)}{J_+'(\theta_n^-)}.$$

We now use the factorisations (S:2.1) and (S:2.5) to write the Wiener–Hopf equation (S:2.28) in the form

$$\begin{aligned}\frac{F^{*(4)}_{1,\Sigma,+}(\gamma)}{4\pi J_+(\gamma)} + P^{*(4)}_{1,\Sigma}\left[\tilde{J}_-(\gamma,\kappa_0^-)\right]_+ + \sum_{n=0}^{\infty}\frac{\mathscr{C}_{1,\Sigma,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)}\left[\tilde{J}_-(\gamma,\theta_n^-)\right]_+ \\ = D^{*(4)}_{1,\Sigma,-}(\gamma)J_-(\gamma) - P^{*(4)}_{1,\Sigma}\left[\tilde{J}_-(\gamma,\kappa_0^-)\right]_- - \sum_{n=0}^{\infty}\frac{\mathscr{C}_{1,\Sigma,n}\mathrm{e}^{-\mathrm{i}c_\phi\theta_n^-}}{\mathrm{i}(\theta_n^- - \kappa_0^-)}\left[\tilde{J}_-(\gamma,\theta_n^-)\right]_-.\end{aligned} \tag{S:2.33}$$

A similar argument to section S:2.2.2 and application of the unsteady Kutta condition yields

$$P^{*(4)}_{1,\Sigma} = -\sum_{n=0}^{\infty}\frac{\mathscr{C}_{1,\Sigma,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)}\cdot\frac{J_-(\theta_n^-)}{J_-(\kappa_0^-)}.$$

Therefore, rearranging (S:2.33) and applying the downstream boundary condition (S:2.31) yields

$$D^{(4)}_{1,\Sigma}(\gamma) = -\sum_{n=0}^{\infty}\frac{\mathscr{C}_{1,\Sigma,n}\mathrm{e}^{\mathrm{i}c_\phi(\gamma-\theta_n^-)}}{\mathrm{i}(\theta_n^- - \kappa_0^-)(\gamma-\theta_n^-)}\cdot\frac{J_-(\theta_n^-)}{J_-(\gamma)}. \tag{S:2.34}$$

As expected from our previous assumption, $D^{(4)}_{1,\Sigma}$ only posseses poles in $\mathscr{M}^+$ at $\gamma=\theta_n^+$. We calculate the residues at these points as

$$\mathscr{B}_{1,\Sigma,n} = D_{1,\Sigma,n} - \sum_{m=0}^{\infty}\frac{\mathscr{C}_{1,\Sigma,n}\mathrm{e}^{\mathrm{i}(\theta_n^+-\theta_m^-)c_\phi}}{\mathrm{i}(\theta_n^+ - \kappa_0^-)(\theta_n^+ - \theta_m^-)}\cdot\frac{J_-(\theta_m^-)}{J_-'(\theta_n^+)},$$

or, in matrix form,

$$\boldsymbol{\mathscr{B}}_{1,\Sigma} = \mathbf{D}_{r,\Sigma} + \mathbf{F}\boldsymbol{\mathscr{C}}_{1,\Sigma}, \tag{S:2.35}$$

where

$$\{\mathbf{F}\}_{n,m} = -\frac{\mathrm{e}^{\mathrm{i}(\theta_n^+ - \theta_m^-)c_\phi}}{\mathrm{i}(\theta_n^+ - \kappa_0^-)(\theta_n^+ - \theta_m^-)} \cdot \frac{J_-(\theta_m^-)}{J_-'(\theta_n^+)}.$$

In a similar way to Glegg (1999), the pair of matrix equations (S:2.32) and (S:2.35) may be combined and solved to give expressions for $\mathscr{B}_{1,\Sigma,n}$ and $\mathscr{C}_{1,\Sigma,n}$. The solution for $D_{1,\Sigma}$ is now complete.

S:2.3. *Solution for $D_{1,\Delta}$*

The integral equation and boundary conditions for $\Delta_0\,[h_{1,\Delta}]$ and $D_{1,\Delta}$ are summarised in (3.33a–e). Similarly to section S:2.2, we split this problem into four coupled problems in an analogous way to (S:2.6) and (S:2.7). Consequently, each $\Delta_0\left[h_{1,\Delta}^{(n)}\right]$ and $D_{1,\Delta}^{(n)}$ satisfy a semi-infinite integral equation of the form

$$f_{1,\Delta}^{(n)}(\phi) = 4\pi \int_{-\infty}^{\infty} \left\{-D_{1,\Delta}(\gamma)j(\gamma) + \left(\delta_{n,1}G_\Delta^{(1)}(\gamma) + \delta_{n,2}G_\Delta^{(2)}(\gamma)\right)\cdot k(\gamma)\right\}\mathrm{e}^{-\mathrm{i}\gamma\phi}\mathrm{d}\gamma, \tag{S:2.36}$$

for $n = 1, 2, 3, 4$, where

$$G_\Delta^{(1)}(\gamma) = -\sum_{l=-\infty}^{\infty} \frac{\mathscr{c}_{\Delta,l}}{2\pi\mathrm{i}(\gamma - \kappa_l^-)}, \qquad G_\Delta^{(2)}(\gamma) = \sum_{l=-\infty}^{\infty} \frac{(-1)^l \mathscr{c}_{\Delta,l}}{2\pi\mathrm{i}(\gamma - \kappa_l^-)}\mathrm{e}^{\mathrm{i}(\gamma-\kappa_0^-)c_\phi}.$$

The corresponding boundary values are

$$f_{1,\Delta}^{(1)}(\phi) = 0, \qquad \phi > 0; \tag{S:2.37.a}$$

$$\Delta_0\left[h_{1,\Delta}^{(1)}\right](\phi) = 0, \qquad \phi < 0; \tag{S:2.37.b}$$

$$f_{1,\Delta}^{(2)}(\phi) = 0, \qquad \phi < c_\phi; \tag{S:2.38.a}$$

$$\Delta_0\left[h_{1,\Delta}^{(1)}\right](\phi) + \Delta_0\left[h_{1,\Delta}^{(2)}\right](\phi) = 2\pi\mathrm{i}P_{1,\Delta}^{(2)}\mathrm{e}^{-\mathrm{i}\kappa_0^-\phi}, \quad \phi > c_\phi; \tag{S:2.38.b}$$

$$f_{1,\Delta}^{(3)}(\phi) = 0, \qquad \phi > 0; \tag{S:2.39.a}$$

$$\Delta_0\left[h_{1,\Delta}^{(2)}\right](\phi) + \Delta_0\left[h_{1,\Delta}^{(3)}\right](\phi) + \Delta_0\left[h_{1,\Delta}^{(4)}\right](\phi) = 0, \qquad \phi < 0; \tag{S:2.39.b}$$

$$f_{1,\Delta}^{(4)}(\phi) = 0, \qquad \phi < c_\phi; \tag{S:2.40.a}$$

$$\Delta_0\left[h_{1,\Delta}^{(3)}\right](\phi) + \Delta_0\left[h_{1,\Delta}^{(4)}\right](\phi) = 2\pi\mathrm{i}P_{1,\Delta}^{(4)}\mathrm{e}^{-\mathrm{i}\kappa_0^-\phi}, \quad \phi > c_\phi; \tag{S:2.40.b}$$

where $P_{1,\Delta}^{(2)}$ and $P_{1,\Delta}^{(4)}$ are two constants of integration that will be specified to enforce the Kutta condition. Summing the four above boundary values results in the original integral equation. Consequently, we may apply the Wiener–Hopf method to each individual integral equation and sum the resulting contributions to obtain a solution to the original integral equation.

S:2.3.1. *Solution to first Wiener–Hopf problem* – $D^{(1)}_{1,\Delta}$

Taking a Fourier transform of (S:2.36) for $n = 1$, using the kernel splitting relation (S:2.1), and applying (S:2.37.a) and (S:2.37.b) gives

$$\frac{F^{(1)}_{1,\Delta,-}(\gamma)}{4\pi J_-(\gamma)} = - D^{(1)}_{1,\Delta,+}(\gamma)J_+(\gamma) + G^{(1)}_{\Delta}(\gamma)\frac{k(\gamma)}{J_-(\gamma)}. \tag{S:2.41}$$

Now we note that the only poles of $G^{(1)}_{\Delta}(\gamma)k(\gamma)$ in $\mathscr{M}^-$ are at $\gamma = \kappa^-_l$, λ^-_m. Therefore, we may apply pole removal to remove the singularities at these locations and write

$$\frac{k(\gamma)G^{(1)}_{\Delta}(\gamma)}{J_-(\gamma)} = \underbrace{\sum_{m=-\infty}^{\infty}\frac{k^-_{r,m}G^{(1)}_{\Delta}(\lambda^-_m)}{(\gamma-\lambda^-_m)J_-(\lambda^-_m)} - \sum_{l=-\infty}^{\infty}\frac{k(\kappa^-_l)\mathscr{c}_{\Delta,l}}{2\pi\mathrm{i}(\gamma-\kappa^-_l)J_-(\kappa^-_l)}}_{+}$$
$$+\underbrace{\frac{k(\gamma)G^{(1)}_{\Delta}(\gamma)}{J_-(\gamma)} - \sum_{m=-\infty}^{\infty}\frac{k^-_{r,m}G^{(1)}_{\Delta}(\lambda^-_m)}{(\gamma-\lambda^-_m)J_-(\lambda^-_m)} + \sum_{l=-\infty}^{\infty}\frac{k(\kappa^-_l)\mathscr{c}_{\Delta,l}}{2\pi\mathrm{i}(\gamma-\kappa^-_l)J_-(\kappa^-_l)}}_{-}, \tag{S:2.42}$$

where $k^-_{r,m}$ are the residues of k at $\gamma = \lambda^-_m$. Therefore, (S:2.41) becomes

$$\frac{F^{(1)}_{1,\Delta,-}(\gamma)}{4\pi J_-(\gamma)} - \frac{G^{(1)}_{\Delta}(\gamma)k(\gamma)}{J_-(\gamma)} + \sum_{m=-\infty}^{\infty}\frac{k^-_{r,m}G^{(1)}_{\Delta}(\lambda^-_m)}{(\gamma-\lambda^-_m)J_m(\lambda^-_m)} - \sum_{l=-\infty}^{\infty}\frac{k(\kappa^-_0)\mathscr{c}_{\Delta,l}}{2\pi\mathrm{i}(\gamma-\kappa^-_l)J_-(\kappa^-_l)} =$$
$$-D^{(1)}_{1,\Delta,+}(\gamma)J_+(\gamma) + \sum_{m=-\infty}^{\infty}\frac{k^-_{r,m}G^{(1)}_{\Delta}(\lambda^-_m)}{(\gamma-\lambda^-_m)J_-(\lambda^-_m)} - \sum_{l=-\infty}^{\infty}\frac{k(\kappa^-_l)\mathscr{c}_{\Delta,l}}{2\pi\mathrm{i}(\gamma-\kappa^-_l)J_-(\kappa^-_l)}. \tag{S:2.43}$$

We now appeal to an argument similar to that in section S:2.2.1 to conclude that each side of the above expression is equivalent to a constant that must vanish. Consequently, we obtain the following expression

$$D^{(1)}_{1,\Delta}(\gamma) = \sum_{l=-\infty}^{\infty}\frac{\mathscr{T}_{1,\Delta,l}}{(\gamma-\kappa^-_l)}\cdot\frac{1}{J_+(\gamma)} + \sum_{m=-\infty}^{\infty}\frac{\mathscr{R}_{1,\Delta,m}}{(\gamma-\lambda^-_m)}\cdot\frac{1}{J_+(\gamma)}, \tag{S:2.44}$$

where

$$\mathscr{R}_{1,\Delta,m} = \frac{k^-_{r,m}G^{(1)}_{\Delta}(\lambda^-_m)}{J_-(\lambda^-_m)}, \qquad \mathscr{T}_{1,\Delta,l} = \frac{\mathscr{c}_{\Delta,l}k(\kappa^-_l)}{2\pi\mathrm{i}J_-(\kappa^-_l)}.$$

S:2.3.2. *Solution to second Wiener–Hopf problem* – $D^{(2)}_{1,\Delta}$

Taking a Fourier transform of (S:2.36) for $n = 2$, using the kernel splitting relation (S:2.1), and applying the analogous form of the upstream boundary values (S:2.38.a) gives

$$\frac{F^{*(2)}_{1+}(\gamma)}{4\pi J_+(\gamma)} = -\left(D^{*(2)}_{1,\Delta-}(\gamma) + D^{*(2)}_{1,\Delta+}(\gamma)\right)J_-(\gamma) + G^{*(2)}_{\Delta}(\gamma)\frac{k(\gamma)}{J_+(\gamma)}, \tag{S:2.45}$$

where $F^{*(2)}_{1,\Delta,+}$, $D^{*(2)}_{1,\Delta,-}$ and $D^{*(2)}_{1,\Delta,+}$ are defined analogously to (S:2.21.a), (S:2.21.b) and (S:2.21.c) respectively, and $G^{*(2)}_{\Delta}(\gamma) = G^{(2)}_{\Delta}(\gamma)\mathrm{e}^{-\mathrm{i}c_\phi\gamma}$. Using a similar approach to section

S:2.2.2, we may use the downstream boundary values (S:2.38.b) to write

$$D^{*(2)}_{1,\Delta,+}(\gamma) = -\frac{P^{*(2)}_{1,\Delta}}{\gamma-\kappa_l^-} - \frac{1}{2\pi\mathrm{i}}\int_{-\infty-\mathrm{i}\tau_1}^{\infty-\mathrm{i}\tau_1} \frac{D^{(1)}_{1,\Delta}(\gamma_1)\mathrm{e}^{-\mathrm{i}\gamma_1 c_\phi}}{\gamma_1-\gamma}\mathrm{d}\gamma_1. \tag{S:2.46}$$

This integral can be closed in $\mathscr{M}^-$. Inspection of (S:2.44) determines that the only poles of $D^{(1)}_{1,\Delta}$ in $\mathscr{M}^-$ are at $\gamma = \kappa_l^-$, θ_n^-. By noting the identities

$$\frac{1}{2\pi\mathrm{i}}\int_{-\infty-\mathrm{i}\tau_1}^{\infty-\mathrm{i}\tau_1} \frac{1}{J_+(\gamma_1)}\sum_{l=-\infty}^{\infty}\frac{\mathscr{T}_{1,\Delta,l}}{(\gamma_1-\kappa_l^-)}\cdot\frac{\mathrm{e}^{-\mathrm{i}\gamma_1 c_\phi}}{\gamma_1-\gamma}\mathrm{d}\gamma_1 =$$

$$\sum_{l=-\infty}^{\infty}\frac{\mathscr{T}_{1,\Delta,l}}{J_+(\kappa_l^-)}\cdot\frac{\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\gamma-\kappa_l^-} + \sum_{n=0}^{\infty}\left\{\sum_{l=-\infty}^{\infty}\frac{\mathscr{T}_{1,\Delta,l}}{\theta_n^- - \kappa_l^-}\right\}\cdot\frac{\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{J'_+(\theta_n^-)(\gamma-\theta_n^-)},$$

$$\frac{1}{2\pi\mathrm{i}}\int_{-\infty-\mathrm{i}\tau_1}^{\infty-\mathrm{i}\tau_1} \frac{1}{J_+(\gamma_1)}\sum_{m=-\infty}^{\infty}\frac{\mathscr{R}_{1,\Delta,m}}{\gamma_1-\lambda_m^-}\cdot\frac{\mathrm{e}^{-\mathrm{i}\gamma_1 c_\phi}}{\gamma_1-\gamma}\mathrm{d}\gamma_1 =$$

$$\sum_{n=0}^{\infty}\left\{\sum_{m=-\infty}^{\infty}\frac{\mathscr{R}_{1,\Delta,m}}{\theta_n^- - \lambda_m^-}\right\}\frac{\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{J'(\theta_n^-)(\gamma-\theta_n^-)},$$

we conclude that

$$\frac{1}{2\pi\mathrm{i}}\int_{-\infty-\mathrm{i}\tau_1}^{\infty-\mathrm{i}\tau_1} D^{(1)}_{1,\Delta}(\gamma_1)\frac{\mathrm{e}^{-\mathrm{i}\gamma_1 c_\phi}}{\gamma_1-\gamma}\mathrm{d}\gamma_1 = \sum_{l=-\infty}^{\infty}\frac{\mathscr{S}_l\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)(\gamma-\kappa_l^-)} + \sum_{n=0}^{\infty}\frac{\mathscr{A}_n\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)(\gamma-\theta_n^-)},$$

where

$$\mathscr{S}_{1,\Delta,l} = \frac{\mathrm{i}(\kappa_0^- - \kappa_l^-)}{J_+(\kappa_l^-)}\mathscr{T}_{1,\Delta,l},$$

$$\mathscr{A}_{1,\Delta,n} = \frac{\mathrm{i}(\theta_n^- - \kappa_0^-)}{J'_+(\theta_n^-)}\left(\sum_{m=-\infty}^{\infty}\frac{\mathscr{R}_{1,\Delta,m}}{\theta_n^- - \lambda_m^-} + \sum_{l=-\infty}^{\infty}\frac{\mathscr{T}_{1,\Delta,l}}{\theta_n^- - \kappa_l^-)}\right).$$

Combining these expressions with (S:2.46) obtains the identity

$$D^{*(2)}_{1,\Delta,+}(\gamma) = -\sum_{l=-\infty}^{\infty}\frac{\mathscr{S}_{1,\Delta,l}\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)(\gamma-\kappa_l^-)} - \sum_{n=0}^{\infty}\frac{\mathscr{A}_{1,\Delta,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_l^-)(\gamma-\theta_n^-)} - \frac{P^{*(2)}_{1,\Delta}}{\gamma-\kappa_0^-}. \tag{S:2.47}$$

Substitution of (S:2.47) into (S:2.45) and application of the additive factorisation (S:2.5), and an analogous version of (S:2.42), yields

$$\frac{F^{*(2)}_{1,\Delta,+}(\gamma)}{4\pi J_+(\gamma)} - \sum_{l=-\infty}^{\infty}\frac{\mathscr{S}_{1,\Delta,l}\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)}\left[\tilde{J}_-(\gamma,\kappa_l^-)\right]_+ - \sum_{n=0}^{\infty}\frac{\mathscr{A}_{1,\Delta,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)}\left[\tilde{J}_-(\gamma,\theta_n^-)\right]_+$$

$$-\frac{k(\gamma)G^{*(2)}_{\Delta}(\gamma)}{J_+(\gamma)} + \sum_{m=-\infty}^{\infty}\frac{k^+_{r,m}G^{*(2)}_{\Delta}(\lambda_m^+)}{(\gamma-\lambda_m^+)J_+(\lambda_m^+)} - P^{*(2)}_{1,\Delta}\left[\tilde{J}_-(\gamma,\kappa_0^-)\right]_+$$

$$= - J_-(\gamma) D^{*(2)}_{1,\Delta,-}(\gamma) + \sum_{l=-\infty}^{\infty} \frac{\mathscr{S}_{1,\Delta,l} \mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)} \left[\tilde{J}_-(\gamma, \kappa_l^-) \right]_- \tag{S:2.48}$$

$$+ \sum_{n=0}^{\infty} \frac{\mathscr{A}_{1,\Delta,n} \mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)} \left[\tilde{J}_-(\gamma, \theta_n^-) \right]_- + \sum_{m=-\infty}^{\infty} \frac{k^+_{r,m} G^{*(2)}_\Delta(\lambda^+_m)}{(\gamma - \lambda^+_m) J_+(\lambda^+_m)} + P^{*(2)}_{1,\Delta} \left[\tilde{J}_-(\gamma, \kappa_0^-) \right]_- . \tag{S:2.49}$$

By applying the unsteady Kutta condition via a similar argument to section S:2.2.2, we conclude that each side of the above equation must be equal to zero. Multiplying by γ and letting $|\gamma| \to \infty$ in $\mathscr{M}^+$, the left-hand side of (S:2.49) gives

$$P^{*(2)}_{1,\Delta} = - \sum_{l=-\infty}^{\infty} \frac{\mathscr{S}_{1,\Delta,l} \mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)} \cdot \frac{J_-(\kappa_l^-)}{J_-(\kappa_0^-)} - \sum_{n=0}^{\infty} \frac{\mathscr{A}_{1,\Delta,n} \mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)} \cdot \frac{J_-(\theta_n^-)}{J_-(\kappa_0^-)}$$
$$+ \sum_{m=-\infty}^{\infty} \frac{\mathscr{U}_{1,\Delta,m}}{\mathrm{i}(\lambda^+_m - \kappa_0^-)} \cdot \frac{1}{J_-(\kappa_0^-)},$$

where

$$\mathscr{U}_{1,\Delta,m} = \mathrm{i}(\lambda^+_m - \kappa_0^-) \cdot \frac{k^+_{r,m} G^{*(2)}_\Delta(\lambda^+_m)}{J_+(\lambda^+_m)}.$$

Substituting in the downstream boundary data (S:2.47), rearranging the right-hand side of (S:2.49), and applying the pressure representation (S:2.50) gives the final expression

$$D^{(2)}_{1,\Delta}(\gamma) = - \sum_{l=-\infty}^{\infty} \frac{\mathscr{S}_{1,\Delta,l} \mathrm{e}^{\mathrm{i}(\gamma - \kappa_l^-) c_\phi}}{\mathrm{i}(\gamma - \kappa_0^-)(\gamma - \kappa_l^-)} \cdot \frac{J_-(\kappa_l^-)}{J_-(\gamma)} - \sum_{n=0}^{\infty} \frac{\mathscr{A}_{1,\Delta,n} \mathrm{e}^{\mathrm{i}(\gamma - \theta_n^-) c_\phi}}{\mathrm{i}(\gamma - \kappa_0^-)(\gamma - \theta_n^-)} \cdot \frac{J_-(\theta_n^-)}{J_-(\gamma)}$$
$$- \sum_{m=-\infty}^{\infty} \frac{\mathscr{U}_{1,\Delta,m} \mathrm{e}^{\mathrm{i}\gamma c_\phi}}{\mathrm{i}(\gamma - \kappa_0^-)(\gamma - \lambda^+_m)} \cdot \frac{1}{J_-(\gamma)}. \tag{S:2.50}$$

S:2.3.3. *Solution to third and fourth Wiener–Hopf problems* – $D^{(3)}_{1,\Delta}$, $D^{(4)}_{1,\Delta}$

The method of solution for $D^{(3)}_{1,\Delta}$ and $D^{(4)}_{1,\Delta}$ is identical to section S:2.2.3 except that the residues of $D^{(2)}_{1,\Delta}$ at $\gamma = \theta_n^+$ are given by

$$D^{(2)}_{r,\Delta,k} = \frac{-\mathrm{e}^{\mathrm{i}\theta_k^+ c_\phi}}{\mathrm{i}(\theta_k^+ - \kappa_0^-) J'_-(\theta_k^+)} \left\{ \sum_{l=-\infty}^{\infty} \frac{\mathscr{S}_{1,\Delta,l} \mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{(\theta_k^+ - \kappa_l^-)} \cdot J_-(\kappa_l^-) \right.$$
$$\left. + \sum_{n=0}^{\infty} \frac{\mathscr{A}_{1,\Delta,n} \mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{(\theta_k^+ - \theta_n^-)} \cdot J_-(\theta_n^-) + \sum_{m=-\infty}^{\infty} \frac{\mathscr{U}_{1,\Delta,m}}{\theta_k^+ - \lambda^+_m} \right\}.$$

Consequently, analogous relations to (S:2.30) and (S:2.34) may be derived, and the calculations of $\mathscr{B}_{1,\Delta,n}$ and $\mathscr{C}_{1,\Delta,n}$ are achieved via (S:2.32) and (S:2.35).

S:2.4. *Solution for $D_{1,\Gamma}$*

The integral equation and boundary values for $\Delta h_{1,\Gamma}$ and $D_{1,\Gamma}$ are summarised in (3.35a–e). Similarly to section S:2.2, we split this problem into four coupled problems, such as in (S:2.6) and (S:2.7), resulting in the semi-infinite integral equations

$$f^{(n)}_{1,\Gamma}(\phi) = 4\pi \int_{-\infty}^{\infty} \left\{ -D^{(n)}_{1,\Gamma}(\gamma) j(\gamma) + \delta_{2,n} G_\Gamma(\gamma) k(\gamma) \right\} \mathrm{e}^{-\mathrm{i}\gamma\phi} \, \mathrm{d}\gamma, \tag{S:2.51}$$

for $n = 1, 2, 3, 4$. The corresponding boundary values are given by

$$f^{(1)}_{1,\Gamma}(\phi) = 0, \qquad \phi > 0; \quad \text{(S:2.52.a)}$$

$$\Delta_0\left[h^{(1)}_{1,\Gamma}\right](\phi) = 0, \qquad \phi < 0; \quad \text{(S:2.52.b)}$$

$$f^{(2)}_{1,\Gamma}(\phi) = 0, \qquad \phi < c_\phi; \quad \text{(S:2.53.a)}$$

$$\Delta_0\left[h^{(1)}_{1,\Gamma}\right](\phi) + \Delta_0\left[h^{(2)}_{1,\Gamma}\right](\phi) =$$

$$2\pi\mathrm{i}P^{(2)}_{1,\Gamma}\mathrm{e}^{-\mathrm{i}\kappa_0^-\phi} + \Gamma\left(\sum_{\pm}\mathscr{H}^{\pm\prime}_{0,\kappa_0^-}\mathrm{e}^{-\mathrm{i}\kappa_0^-\phi} + \sum_{m=-\infty}^{\infty}\mathscr{H}'_{0,m}\mathrm{e}^{-\mathrm{i}\lambda_m^-\phi}\right), \quad \phi > c_\phi; \quad \text{(S:2.53.b)}$$

$$f^{(3)}_{1,\Gamma}(\phi) = 0, \qquad \phi > 0; \quad \text{(S:2.54.a)}$$

$$\Delta_0\left[h^{(2)}_{1,\Gamma}\right](\phi) + \Delta_0\left[h^{(3)}_{1,\Sigma}\right] + \Delta_0\left[h^{(4)}_{1,\Gamma}\right](\phi) = 0, \qquad \phi < 0; \quad \text{(S:2.54.b)}$$

$$f^{(4)}_{1,\Gamma}(\phi) = 0, \qquad \phi < c_\phi; \quad \text{(S:2.55.a)}$$

$$\Delta_0\left[h^{(3)}_{1,\Gamma}\right](\phi) + \Delta_0\left[h^{(4)}_{1,\Sigma}\right](\phi) = 2\pi\mathrm{i}P^{(4)}_{1,\Gamma}\mathrm{e}^{-\mathrm{i}\kappa_0^-\phi}, \qquad \phi > c_\phi; \quad \text{(S:2.55.b)}$$

where, as before, $P^{(2)}_{1,\Gamma}$ and $P^{(4)}_{1,\Gamma}$ are two constants of integration that will be specified to enforce the Kutta condition.

S:2.4.1. *Solution to first Wiener–Hopf problem – $D^{(1)}_{1,\Gamma}$*

By applying a similar argument to section S:2.2.1, we conclude that

$$D^{(1)}_{1,\Gamma}(\gamma) = 0. \qquad \text{(S:2.56)}$$

S:2.4.2. *Solution to second Wiener–Hopf problem – $D^{(2)}_{1,\Gamma}$*

Taking the Fourier transform of (S:2.51) for $n = 2$, using the kernel splitting property (S:2.1), and applying the upstream boundary values (S:2.53.a) yields

$$\frac{F^{*(2)}_{1,\Gamma,+}(\gamma)}{4\pi J_+(\gamma)} = -\left(D^{*(2)}_{1,\Gamma,-}(\gamma) + D^{*(2)}_{1,\Gamma,+}(\gamma)\right)J_-(\gamma) + G^*_\Gamma(\gamma)\frac{k(\gamma)}{J_+(\gamma)}, \qquad \text{(S:2.57)}$$

where $F^{*(2)}_{1,\Gamma,+}$, $D^{*(2)}_{1,\Gamma,-}$ and $D^{*(2)}_{1,\Gamma,+}$ are defined analogously to (S:2.21.a), (S:2.21.b) and (S:2.21.c) respectively. The solution to the first integral equation (S:2.56) means that $\Delta_0\left[h^{(1)}_{1,\Gamma}\right](\phi) \equiv 0$. Therefore, we may use the downstream boundary values (S:2.53.b) to write

$$D^{*(2)}_{1,\Gamma,+}(\gamma) = -\frac{P^{*(2)}_{1,\Gamma}}{\gamma - \kappa_0^-} - \frac{\Gamma}{2\pi\mathrm{i}}\left(\sum_{\pm}\frac{\mathscr{H}^{\pm\prime}_{0,\kappa_0^-}\mathrm{e}^{-\mathrm{i}\kappa_0^- c_\phi}}{\gamma - \kappa_0^-} + \sum_{m=-\infty}^{\infty}\frac{\mathscr{H}'_{0,m}\mathrm{e}^{-\mathrm{i}\lambda_m^- c_\phi}}{\gamma - \lambda_m^-}\right).$$

Consequently, we use pole removal to obtain the additive splitting

$$J_-(\gamma)D^{*(2)}_{1,\Gamma,+}(\gamma) = \left[J_-(\gamma)D^{*(2)}_{1,\Gamma,+}(\gamma)\right]_+ + \left[J_-(\gamma)D^{*(2)}_{1,\Gamma,+}(\gamma)\right]_-, \qquad \text{(S:2.58)}$$

where

$$\left[J_-(\gamma)D^{*(2)}_{1,\Gamma,+}(\gamma)\right]_\pm = -P^{*(2)}_{1,\Gamma}\left[\tilde{J}_-(\gamma,\kappa_0^-)\right]_\pm$$
$$-\frac{\Gamma}{2\pi\mathrm{i}}\left(\sum_\pm \mathscr{H}^{\pm\prime}_{0,\kappa_0^-}\mathrm{e}^{-\mathrm{i}\kappa_0^- c_\phi}\left[\tilde{J}_-(\gamma,\kappa_0^-)\right]_\pm + \sum_{m=-\infty}^{\infty}\mathscr{H}'_{0,m}\mathrm{e}^{-\mathrm{i}\lambda_m^- c_\phi}\left[\tilde{J}_-(\gamma,\lambda_m^-)\right]_\pm\right).$$

We may also use pole removal to obtain the additive splitting

$$G^*_\Gamma(\gamma)\frac{k(\gamma)}{J_+(\gamma)} = \underbrace{G^*_\Gamma(\gamma)\frac{k(\gamma)}{J_+(\gamma)} - \sum_{m=-\infty}^{\infty}\frac{\mathscr{U}_{1,\Gamma,m}}{\mathrm{i}(\lambda_m^+-\kappa_0^-)(\gamma-\lambda_m^+)}}_{+}$$
$$+\underbrace{\sum_{m=-\infty}^{\infty}\frac{\mathscr{U}_{1,\Gamma,m}}{\mathrm{i}(\lambda_m^+-\kappa_0^-)(\gamma-\lambda_m^+)}}_{-}, \tag{S:2.59}$$

where

$$\mathscr{U}_{1,\Gamma,m} = \frac{\Gamma\zeta_m^+(\lambda_m^+-\kappa_0^-)}{8\pi^2\Delta_\phi\sqrt{k^2w^2-f_m}}\cdot\frac{1}{J_+(\lambda_m^+)}$$
$$\times\left(\sum_\pm\frac{\pm\zeta_{\kappa_0^-}\mathscr{H}^{\pm\prime}_{0,\kappa_0^-}\mathrm{e}^{-\mathrm{i}\kappa_0^- c_\phi}}{\lambda_m^+-\kappa_0^-} + \sum_{k=-\infty}^{\infty}\frac{\zeta_k^-\mathscr{H}'_{0,k}\mathrm{e}^{-\mathrm{i}\lambda_k^- c_\phi}}{\lambda_m^+-\lambda_k^-}\right).$$

Substituting the additive splitting (S:2.58) and wake coefficient (S:2.59) into the Wiener–Hopf equation (S:2.57) yields

$$\frac{F^{*(2)}_{1,\Gamma,+}(\gamma)}{4\pi J_+(\gamma)} - G^*_\Gamma(\gamma)\frac{k(\gamma)}{J_+(\gamma)} + \sum_{m=-\infty}^{\infty}\frac{\mathscr{U}_{1,\Gamma,m}}{\mathrm{i}(\lambda_m^+-\kappa_0^-)(\gamma-\lambda_m^+)} + \left[J_-(\gamma)D^{*(2)}_{1,\Gamma,+}(\gamma)\right]_+$$
$$= -D^{*(2)}_{1,\Gamma,-}(\gamma)J_-(\gamma) - \left[J_-(\gamma)D^{*(2)}_{1,\Gamma,+}(\gamma)\right]_- + \sum_{m=-\infty}^{\infty}\frac{\mathscr{U}_{1,\Gamma,m}}{\mathrm{i}(\lambda_m^+-\kappa_0^-)(\gamma-\lambda_m^+)}.$$

Following section S:2.2.2, we apply the standard Wiener–Hopf method and obtain

$$P^{*(2)}_{1,\Gamma} = \sum_{m=-\infty}^{\infty}\frac{\mathscr{U}_{1,\Delta,m}}{\mathrm{i}(\lambda_m^+-\kappa_0^-)}\cdot\frac{1}{J_-(\kappa_0^-)}$$
$$-\frac{\Gamma}{2\pi\mathrm{i}}\left(\sum_\pm\mathscr{H}^{\pm\prime}_{0,\kappa_0^-}\mathrm{e}^{-\mathrm{i}\kappa_0^- c_\phi} + \sum_{m=-\infty}^{\infty}\mathscr{H}'_{0,m}\mathrm{e}^{-\mathrm{i}\lambda_m^- c_\phi}\cdot\frac{J_-(\lambda_m^-)}{J_-(\kappa_0^-)}\right), \tag{S:2.60}$$

and

$$D^{*(2)}_{1,\Gamma,-}(\gamma) = \sum_{m=-\infty}^{\infty}\frac{\mathscr{U}_{1,\Gamma,m}}{\mathrm{i}(\lambda_m^+-\kappa_0^-)(\gamma-\lambda_m^+)}\cdot\frac{1}{J_-(\gamma)} + P^{*(2)}_{1,\Gamma}\frac{\left[\tilde{J}_-(\gamma,\kappa_0^-)\right]_-}{J_-(\gamma)}$$
$$+\frac{\Gamma}{2\pi\mathrm{i}}\left(\sum_\pm\mathscr{H}^{\pm\prime}_{0,\kappa_0^-}\mathrm{e}^{-\mathrm{i}\kappa_0^- c_\phi}\frac{\left[\tilde{J}_-(\gamma,\kappa_0^-)\right]_-}{J_-(\gamma)} + \sum_{m=-\infty}^{\infty}\mathscr{H}'_{0,m}\mathrm{e}^{-\mathrm{i}\lambda_m^- c_\phi}\frac{\left[\tilde{J}_-(\gamma,\lambda_m^-)\right]_-}{J_-(\gamma)}\right).$$

Consequently, substituting in the downstream representation (S:2.58) and (S:2.60) results in

$$D^{(2)}_{1,\Gamma}(\gamma) = -\frac{\Gamma}{2\pi \mathrm{i}} \sum_{m=-\infty}^{\infty} \frac{J_-(\lambda_m^-)}{J_-(\gamma)} \cdot \frac{(\lambda_m^- - \kappa_0^-)\mathscr{H}'_{0,m}\mathrm{e}^{\mathrm{i}(\gamma-\lambda_m^-)c_\phi}}{(\gamma-\kappa_0^-)(\gamma-\lambda_m^-)} - \sum_{m=-\infty}^{\infty} \frac{\mathscr{U}_{1,\Gamma,m}\mathrm{e}^{\mathrm{i}\gamma c_\phi}}{\mathrm{i}(\gamma-\kappa_0^-)(\gamma-\lambda_m^+)} \cdot \frac{1}{J_-(\gamma)}.$$

S:2.4.3. *Solution to third and fourth Wiener–Hopf problems – $D^{(3)}_{1,\Gamma}$, $D^{(4)}_{1,\Gamma}$*

The solution to the coupled 3rd and 4th Wiener–Hopf equations is identical to that in previous sections except

$$D^{(2)}_{r,\Delta,k} = \frac{-\mathrm{e}^{\mathrm{i}\theta_k^+ c_\phi}}{\mathrm{i}(\theta_k^+ - \kappa_0^-)J'_-(\theta_k^+)} \left\{ \sum_{m=-\infty}^{\infty} \frac{\mathscr{U}_{1,\Gamma,m}}{\theta_k^+ - \lambda_m^+} + \frac{\Gamma}{2\pi} \sum_{m=-\infty}^{\infty} \frac{\lambda_m^- - \kappa_0^-}{\theta_k^+ - \lambda_m^-} J_-(\lambda_m^-)\mathscr{H}'_{0,m}\mathrm{e}^{-\mathrm{i}\lambda_m^- c_\phi} \right\}.$$

S:2.5. *Solution for $D_{1,S}$*

The integral equation and boundary values for $\Delta_0\left[h_{1,S}\right]$ and $D_{1,S}$ are summarised in (3.37a–d). We perform an analogous splitting to that in (S:2.6) and (S:2.7). The integral equations satisfied by each $\Delta_0\left[h^{(n)}_{1,S}\right]$ and $D^{(n)}_{1,S}$ are

$$f^{(n)}_{1,S}(\phi) = 4\pi \int_{-\infty}^{\infty} \left\{ -D_{1,S}(\gamma)j(\gamma) + \delta_{n,1}S^{(1)}(\gamma) + \delta_{n,2}S^{(2)}(\gamma) \right\} \mathrm{e}^{-\mathrm{i}\gamma\phi}\mathrm{d}\gamma, \qquad \text{(S:2.61)}$$

for $n = 1, 2, 3, 4$, where

$$S^{(1)}(\gamma) = S^u(\gamma) + S^{i,(1)}(\gamma), \qquad S^{(2)}(\gamma) = S^d(\gamma) + S^{i,(2)}(\gamma),$$

which are defined in (S:6.11), (S:6.12) and (S:6.13).

The (homogeneous) boundary values are given by

$$f^{(1)}_{1,S}(\phi) = 0, \qquad \phi > 0; \qquad \text{(S:2.62.a)}$$

$$\Delta_0\left[h^{(1)}_{1,S}\right](\phi) = 0, \qquad \phi < 0; \qquad \text{(S:2.62.b)}$$

$$f^{(2)}_{1,S}(\phi) = 0, \qquad \phi < c_\phi; \qquad \text{(S:2.63.a)}$$

$$\Delta_0\left[h^{(1)}_{1,S}\right](\phi) + \Delta_0\left[h^{(2)}_{1,S}\right](\phi) = 2\pi\mathrm{i}P^{(2)}_{1,S}\mathrm{e}^{-\mathrm{i}\kappa_0^-\phi}, \qquad \phi > c_\phi; \qquad \text{(S:2.63.b)}$$

$$f^{(3)}_{1,S}(\phi) = 0, \qquad \phi > 0; \qquad \text{(S:2.64.a)}$$

$$\Delta_0\left[h^{(2)}_{1,S}\right](\phi) + \Delta_0\left[h^{(3)}_{1,S}\right](\phi) + \Delta_0\left[h^{(4)}_{1,S}\right](\phi) = 0, \qquad \phi < 0; \qquad \text{(S:2.64.b)}$$

$$f^{(4)}_{1,S}(\phi) = 0, \qquad \phi < c_\phi; \qquad \text{(S:2.65.a)}$$

$$\Delta_0\left[h^{(4)}_{1,S}\right](\phi) + \Delta_0\left[h^{(3)}_{1,S}\right](\phi) = 2\pi\mathrm{i}P^{(4)}_{1,S}\mathrm{e}^{-\mathrm{i}\kappa_0^-\phi}, \qquad \phi > c_\phi; \qquad \text{(S:2.65.b)}$$

where $P_{1,S}^{(2)}$ and $P_{1,S}^{(4)}$ are two constants of integration that will be specified to enforce the Kutta condition.

S:2.5.1. *Solution to first Wiener–Hopf problem* – $D_{1,S}^{(1)}$

Taking a Fourier transform of (S:2.61) for $n = 1$, applying the boundary values (S:2.62.a) and (S:2.62.b), and utilising the kernel splitting (S:2.1) results in

$$\frac{F_{1,S,-}^{(1)}(\gamma)}{4\pi J_-(\gamma)} = -D_{1,S,+}^{(1)}(\gamma)J_+(\gamma) + \frac{S^{(1)}(\gamma)}{J_-(\gamma)}, \tag{S:2.66}$$

where $F_{1,S,-}^{(1)}$ and $D_{1,S,+}^{(1)}$ are defined in an analogous way to section S:2.2.1. We now note that the only poles of $S^{(1)}$ in $\mathscr{M}^-$ are where $\gamma = \lambda_m^-$, κ_l^-. Therefore, we may apply pole removal to obtain the additive factorisation

$$\begin{aligned}\frac{S^{(1)}(\gamma)}{J_-(\gamma)} = &\underbrace{\sum_{m=-\infty}^{\infty} \frac{S_m^{u,r,(1)}}{(\gamma-\lambda_m^-)J_-(\lambda_m^-)} + \sum_{l=-\infty}^{\infty} \frac{\sum_{l_2=-\infty}^{\infty} f_{l,l_2}\mathscr{F}_{l_2}(\kappa_l^-)}{(\gamma-\kappa_l^-)J_-(\kappa_l^-)}}_{+} \\ &+ \underbrace{\frac{S^{(1)}(\gamma)}{J_-(\gamma)} - \sum_{m=-\infty}^{\infty} \frac{S_m^{u,r,(1)}}{(\gamma-\lambda_m^-)J_-(\lambda_m^-)} - \sum_{l=-\infty}^{\infty} \frac{\sum_{l_2=-\infty}^{\infty} f_{l,l_2}\mathscr{F}_{l_2}(\kappa_l^-)}{(\gamma-\kappa_l^-)J_-(\kappa_l^-)}}_{-},\end{aligned} \tag{S:2.67}$$

where

$$S_m^{u,r,(1)} = S_m^{u,r} + \sum_{l_1,l_2} f_{l_1,l_2} \frac{\mathscr{F}_{l_2,m}^{r,-}}{(\lambda_m^- - \kappa_{l_1}^-)},$$

and $\mathscr{F}_{l_2,m}^{r,-}$ are the residues of $\mathscr{F}_{l_2}(\gamma)$ at the acoustic modes $\gamma = \lambda_m^-$, defined in (S:6.14).

Therefore, (S:2.66) becomes

$$\begin{aligned}&\frac{F_-^{(1)}(\gamma)}{4\pi J_-(\gamma)} - \frac{S^{(1)}(\gamma)}{J_-(\gamma)} + \sum_{m=-\infty}^{\infty} \frac{S_m^{u,r,(1)}}{(\gamma-\lambda_m^-)J_-(\lambda_m^-)} + \sum_{l=-\infty}^{\infty} \frac{\sum_{l_2=-\infty}^{\infty} f_{l,l_2}\mathscr{F}_{l_2}(\kappa_l^-)}{(\gamma-\kappa_l^-)J_-(\kappa_l^-)} \\ &= -D_{1,S,+}^{(1)}(\gamma)J_+(\gamma) + \sum_{m=-\infty}^{\infty} \frac{S_m^{u,r,(1)}}{(\gamma-\lambda_m^-)J_-(\lambda_m^-)} + \sum_{l=-\infty}^{\infty} \frac{\sum_{l_2=-\infty}^{\infty} f_{l,l_2}\mathscr{F}_{l_2}(\kappa_l^-)}{(\gamma-\kappa_l^-)J_-(\kappa_l^-)}.\end{aligned} \tag{S:2.68}$$

We now apply the typical Wiener–Hopf argument of section S:2.2.1 and conclude that

$$D_{1,S}^{(1)}(\gamma) = \sum_{l=-\infty}^{\infty} \frac{\mathscr{T}_{1,S,l}}{\gamma-\kappa_l^-} \cdot \frac{1}{J_+(\gamma)} + \sum_{m=-\infty}^{\infty} \frac{\mathscr{R}_{1,S,m}}{\gamma-\lambda_m^-} \cdot \frac{1}{J_+(\gamma)},$$

where

$$\mathscr{R}_{1,S,m} = \frac{S_m^{u,r,(1)}}{J_-(\lambda_m^-)}, \qquad \mathscr{T}_{1,S,l} = \frac{\sum_{l_2=-\infty}^{\infty} f_{l,l_2}\mathscr{F}_{l_2}(\kappa_l^-)}{J_-(\kappa_l^-)}.$$

S:2.5.2. *Solution to second Wiener–Hopf problem* – $D_{1,S}^{(2)}$

Taking a Fourier transform of (S:2.61) for $n = 2$, applying the boundary value (S:2.63.a), and utilising the kernel splitting (S:2.1) results in

$$\frac{F_{1,S,+}^{*(2)}(\gamma)}{4\pi J_+(\gamma)} = -\left(D_{1,S,-}^{*(2)}(\gamma) + D_{1,S,+}^{*(2)}(\gamma)\right) J_-(\gamma) + \frac{S^{*(2)}(\gamma)}{J_+(\gamma)}, \tag{S:2.69}$$

where $F^{*(2)}_{1,S,+}$, $D^{*(2)}_{1,S,\pm}$ and $S^{*(2)}$ are defined in an analogous way to section S:2.2.2.

Similarly to section S:2.3.2, we may use the downstream boundary values (S:2.63.b) to write

$$D^{*(2)}_{1,S,+}(\gamma) = -\sum_{l=-\infty}^{\infty} \frac{\mathscr{S}_{1,S,l}\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)(\gamma - \kappa_l^-)} - \sum_{n=0}^{\infty} \frac{\mathscr{A}_{1,S,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_l^-)(\gamma - \theta_n^-)} - \frac{P^{*(2)}_{1,S}}{\gamma - \kappa_0^-}, \tag{S:2.70}$$

where $\mathscr{A}_{1,S,n}$ and $\mathscr{S}_{1,S,l}$ are defined in an analogous way to section S:2.3. Substitution of (S:2.70) into (S:2.69), application of the additive factorisation (S:2.5), and an analogous version of (S:2.67) yields

$$\begin{aligned}
&\frac{F^{*(2)}_{1,S,+}(\gamma)}{4\pi J_+(\gamma)} - \sum_{l=-\infty}^{\infty} \frac{\mathscr{S}_{1,S,l}\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)} \left[\tilde{J}_-(\gamma,\kappa_l^-)\right]_+ - \sum_{n=0}^{\infty} \frac{\mathscr{A}_{1,S,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)} \left[\tilde{J}_-(\gamma,\theta_n^-)\right]_+ \\
&- \frac{S^{*(2)}(\gamma)}{J_+(\gamma)} + \sum_{m=-\infty}^{\infty} \frac{S^{*d,r,(2)}_m}{(\gamma - \lambda_m^+)J_+(\lambda_m^+)} - P^{*(2)}_{1,S}\left[\tilde{J}_-(\gamma,\kappa_0^-)\right] \\
&= \sum_{l=-\infty}^{\infty} \frac{\mathscr{S}_{1,S,l}\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)(\gamma - \kappa_l^-)} \left[\tilde{J}_-(\gamma,\kappa_l^-)\right]_- + \sum_{n=0}^{\infty} \frac{\mathscr{A}_{1,S,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)} \left[\tilde{J}_-(\gamma,\theta_n^-)\right]_- \\
&- J_-(\gamma)D^{*(2)}_{1,S,-}(\gamma) + \sum_{m=-\infty}^{\infty} \frac{S^{*d,r,(2)}_m}{(\gamma - \lambda_m^+)J_+(\lambda_m^+)} + P^{*(2)}_{1,S}\left[\tilde{J}_-(\gamma,\kappa_0^-)\right]_- .
\end{aligned} \tag{S:2.71}$$

By applying a similar argument to section S:2.2.2, we conclude that each side of the above equation must be equal to zero. Multiplying both sides by γ and letting $|\gamma| \to \infty$ in $\mathscr{M}^+$, the left-hand side of (S:2.71) gives the value of the wake coefficient

$$\begin{aligned}
P^{*(2)}_{1,S} =& -\sum_{l=-\infty}^{\infty} \frac{\mathscr{S}_{1,S,l}\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\mathrm{i}(\kappa_0^- - \kappa_l^-)} \cdot \frac{J_-(\kappa_l^-)}{J_-(\kappa_0^-)} - \sum_{n=0}^{\infty} \frac{\mathscr{A}_{1,S,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\mathrm{i}(\theta_n^- - \kappa_0^-)} \cdot \frac{J_-(\theta_n^-)}{J_-(\kappa_0^-)} \\
&+ \sum_{m=-\infty}^{\infty} \frac{\mathscr{U}_{1,S,m}}{\mathrm{i}(\lambda_m^+ - \kappa_0^-)} \cdot \frac{1}{J_-(\kappa_0^-)},
\end{aligned} \tag{S:2.72}$$

where

$$\mathscr{U}_{1,S,m} = \mathrm{i}(\lambda_m^+ - \kappa_0^-) \cdot \frac{S^{*d,r,(2)}_m}{J_+(\lambda_m^+)}.$$

Substituting the downstream boundary data (S:2.70) and the wake coefficient (S:2.72) into the Wiener–Hopf equation (S:2.71) gives the final expression

$$\begin{aligned}
D^{(2)}_{1,S}(\gamma) =& -\sum_{l=-\infty}^{\infty} \frac{\mathscr{S}_{1,S,l}\mathrm{e}^{\mathrm{i}(\gamma-\kappa_l^-)c_\phi}}{\mathrm{i}(\gamma - \kappa_0^-)(\gamma - \kappa_l^-)} \cdot \frac{J_-(\kappa_l^-)}{J_-(\gamma)} - \sum_{n=0}^{\infty} \frac{\mathscr{A}_{1,S,n}\mathrm{e}^{\mathrm{i}(\gamma-\theta_n^-)c_\phi}}{\mathrm{i}(\gamma - \kappa_0^-)(\gamma - \theta_n^-)} \cdot \frac{J_-(\theta_n^-)}{J_-(\gamma)} \\
&- \sum_{m=-\infty}^{\infty} \frac{\mathscr{U}_{1,S,m}\mathrm{e}^{\mathrm{i}\gamma c_\phi}}{\mathrm{i}(\gamma - \kappa_0^-)(\gamma - \lambda_m^+)} \cdot \frac{1}{J_-(\gamma)}.
\end{aligned}$$

Similarly to previous sections, the only poles of $D^{*(2)}_{1,S}(\gamma)$ in the upper half plane are at the zeros of $J_-(\gamma)$.

S:2.5.3. *Solution to third and fourth Wiener–Hopf problems* – $D^{(3)}_{1,S}$, $D^{(4)}_{1,S}$

The solution to the coupled 3rd and 4th Wiener–Hopf equations is identical to that in previous sections, except

$$D^{(2)}_{r,S,k} = \frac{-\mathrm{e}^{\mathrm{i}\theta_k^+ c_\phi}}{\mathrm{i}(\theta_k^+ - \kappa_0^-)J'_-(\theta_k^+)}\left\{\sum_{l=-\infty}^{\infty}\frac{\mathcal{S}_{1,S,l}\mathrm{e}^{-\mathrm{i}\kappa_l^- c_\phi}}{\theta_k^+ - \kappa_l^-}\cdot J_-(\kappa_l^-)\right.$$
$$\left. + \sum_{n=0}^{\infty}\frac{\mathscr{A}_{1,S,n}\mathrm{e}^{-\mathrm{i}\theta_n^- c_\phi}}{\theta_k^+ - \theta_n^-}\cdot J_-(\theta_n^-) + \sum_{m=-\infty}^{\infty}\frac{\mathscr{U}_{1,S,m}}{\theta_k^+ - \lambda_m^+}\right\}.$$

S:3. Background flow analysis

In this section we detail the solution of the steady background flow. We shift the coordinate system $z_\beta \mapsto z_\beta - 1$ so that the leading-edge of the zeroth aerofoil is at $z_\beta = -1$.

We consider an ansatz for the complex velocity of the form

$$\Phi(z_\beta) = \frac{1}{2(d+\mathrm{i}s)}\int_{-1}^{1}\gamma(\tau)\left(\coth\left(\frac{\pi\mathrm{i}\,(\tau - z_\beta)}{d+\mathrm{i}s}\right) - 1\right)\mathrm{d}\tau, \tag{S:3.1}$$

where $\gamma(\tau)$ represents a distribution of sources, sinks and vorticity arranged on the chord line so that

$$q(z_\beta) - \mathrm{i}\mu(z_\beta) = -\mathrm{i}\alpha + \Phi(z_\beta).$$

The problem is now to find γ, which we decompose into real and imaginary parts as

$$\gamma(\tau) = \gamma_r(\tau) + \mathrm{i}\gamma_\mathrm{i}(\tau).$$

In a similar way to the unstaggered case considered in Baddoo & Ayton (2018), we may consider the limits of $\Phi(z_\beta)$ as $z_\beta \to t^\pm$, (where $t \in [-1, 1]$) via the Modified Plemelj Formula (Baddoo & Ayton 2018, Appendix A). Taking the difference between the limits on the upper and lower sides, and equating imaginary parts yields

$$\gamma_\mathrm{i}(t) = -2y'_{th}(t).$$

Conversely, summing $\Phi(z_\beta)$ either side of the chord lines the singular integral equation

$$\Sigma u^\pm(t) - 2\mathrm{i}(y'_c(t) - \alpha) = \frac{1}{2(d+\mathrm{i}s)}\fint_{-1}^{1}\gamma(\tau)\left(\coth\left(\frac{\pi\mathrm{i}\,(\tau - t)}{d+\mathrm{i}s}\right) - 1\right)\mathrm{d}\tau. \tag{S:3.2}$$

We now consider the imaginary part of (S:3.2). It is precisely at this point that we diverge from the small-stagger solution in Baddoo & Ayton (2018). In particular, the imaginary part of the right-hand side of (S:3.2) does not result in a simple expression amenable to typical Riemann–Hilbert methods. Instead, we have the singular integral equation

$$4\pi(y'_c(t) - \alpha) = \fint_{-1}^{1}\left(-2y'_{th}(\tau)K_r(\tau, t) + \gamma_r(\tau)K_\mathrm{i}(\tau, t)\right)\mathrm{d}\tau, \tag{S:3.3}$$

where

$$K_r(\tau, t) = \mathbb{R}\mathrm{e}\left[\frac{-\pi}{d+\mathrm{i}s}\left(\coth\left(\frac{\pi\mathrm{i}\,(\tau - t)}{d+\mathrm{i}s}\right) - 1\right)\right],$$
$$K_\mathrm{i}(\tau, t) = \mathbb{I}\mathrm{m}\left[\frac{-\pi}{d+\mathrm{i}s}\left(\coth\left(\frac{\pi\mathrm{i}\,(\tau - t)}{d+\mathrm{i}s}\right) - 1\right)\right].$$

We note that K_i is a singular operator: as $\tau \to t$ we have

$$K_{\mathrm{i}}(\tau, t) \sim \frac{1}{\tau - t}.$$

Consequently, we separate the singular and regular parts of K_{i} and write

$$K_{\mathrm{i}}(\tau, t) = \frac{1}{\tau - t} + \tilde{K}_{\mathrm{i}}(\tau, t), \tag{S:3.4}$$

where

$$\tilde{K}_{\mathrm{i}}(\tau, t) = \begin{cases} \mathbb{I}\mathrm{m}\left[\dfrac{\pi}{d + \mathrm{i}s}\right], & \text{for } t = \tau, \\[2ex] \mathbb{I}\mathrm{m}\left[\dfrac{-\pi}{d + \mathrm{i}s}\left(\coth\left(\dfrac{\pi \mathrm{i}\,(\tau - t)}{d + \mathrm{i}s}\right) - 1\right) - \dfrac{1}{\tau - t}\right], & \text{for } t \neq \tau. \end{cases}$$

Consequently, we may now express (S:3.3) as the singular integral equation

$$⨍_{-1}^{1} \frac{\gamma_r(\tau)}{\tau - t}\mathrm{d}\tau + \int_{-1}^{1} \gamma_r(\tau)\tilde{K}_{\mathrm{i}}(\tau, t)\mathrm{d}\tau = f(t), \tag{S:3.5}$$

where the forcing term f is given by

$$f(t) = 4\pi(y_c'(t) - \alpha) - 2⨍_{-1}^{1} y_{th}'(\tau)K_r(\tau, t)\mathrm{d}\tau.$$

Note that the second integral in (S:3.5) is not considered in the principal value sense because $\tilde{K}_{\mathrm{i}}$ is regular for all $(t, \tau) \in [-1, 1]^2$.

This form of singular integral is amenable to numerical methods by expanding γ_r into a suitable series of basis functions. This method was first proposed by Erdogan & Gupta (1972), who proposed an expansion of the unknown function γ_r as a series of weighted Chebyshev polynomials. This expansion accurately captures the endpoint behaviour of γ_r, which necessarily possesses a square root singularity and zero at the leading- and trailing-edge respectively. However, the weighted Chebyshev expansion is entirely appropriate for the present problem, and is widely used in isolated aerofoil analyses (Glauert 1926; Rienstra 1992).

Accordingly, we expand γ_r as a series of weighted Chebyshev polynomials in the form

$$\gamma_r(t) = \gamma_{r,0}\sqrt{\frac{1 - t}{1 + t}} + \sqrt{1 - t^2}\sum_{n=1}^{\infty} \gamma_{r,n}\, U_{n-1}(t), \tag{S:3.6}$$

where U_n represent the Chebyshev polynomials of the second kind and $\gamma_{r,n}$ represent unknown coefficients. The Chebyshev polynomials U_n may be expressed as

$$U_n(t) = \frac{\sin\left[(n + 1)\theta\right]}{\sin(\theta)}, \qquad t = \cos(\theta).$$

Moreover, they satisfy the orthogonality relation

$$\int_{-1}^{1} \sqrt{1 - \tau^2}\, U_n(\tau)U_m(\tau)\mathrm{d}\tau = \delta_{n,m}\frac{\pi}{2}.$$

In order to apply the expansion (S:3.6) to the singular integral equation (S:3.5), we use the finite Hilbert transforms

$$⨍_{-1}^{1} \sqrt{1 - \tau^2}\,\frac{U_{n-1}(\tau)}{\tau - t}\mathrm{d}\tau = -\pi\, T_n(t), \tag{S:3.7}$$

$$\fint_{-1}^{1} \sqrt{\frac{1-\tau}{1+\tau}}\,\frac{\mathrm{d}\tau}{\tau - t} = -\pi, \tag{S:3.8}$$

where T_n are the Chebyshev polynomials of the first kind.

We now substitute (S:3.6) into (S:3.5) and apply the identities (S:3.7) and (S:3.8) to obtain

$$-\pi\gamma_{r,0} - \pi\sum_{n=1}^{\infty}\gamma_{r,n}T_n(t) + \int_{-1}^{1}\gamma_r(\tau)\tilde{K}_{\mathrm{i}}(\tau,t)\mathrm{d}\tau = f(t). \tag{S:3.9}$$

The remaining integral in the above expression is amenable to Chebyshev-Gauss quadrature. We note the quadrature rules (Olver *et al.* 2010, §3.5)

$$\int_{-1}^{1} g(\tau)\sqrt{\frac{1-\tau}{1+\tau}}\,\mathrm{d}\tau \sim \sum_{m=1}^{M} w_m^{(1)} g\left(\tau_m^{(1)}\right) \qquad \text{as } M\to\infty,$$

for

$$\tau_k^{(1)} = \cos\left(\frac{2m}{2M+1}\pi\right),$$
$$w_k^{(1)} = \frac{4\pi}{2M+1}\sin^2\left(\frac{m}{2M+1}\pi\right),$$

and

$$\int_{-1}^{1} g(\tau)\sqrt{1-\tau^2}\,\mathrm{d}\tau \sim \sum_{m=1}^{M} w_m^{(2)} g\left(\tau_m^{(2)}\right) \qquad \text{as } M\to\infty,$$

for

$$\tau_k^{(2)} = \cos\left(\frac{m}{M+1}\pi\right),$$
$$w_k^{(2)} = \frac{\pi}{M+1}\sin^2\left(\frac{m}{M+1}\pi\right).$$

Consequently, we may express (S:3.9) as

$$-\pi\gamma_{r,0} - \pi\sum_{n=1}^{\infty}\gamma_{r,n}T_n(t)$$
$$+\sum_{m=1}^{\infty}\left[\gamma_{r,0}w_m^{(1)}\tilde{K}_{\mathrm{i}}\left(\tau_m^{(1)},t\right) + w_m^{(2)}\tilde{K}_{\mathrm{i}}\left(\tau_m^{(2)},t\right)\sum_{n=1}^{\infty}\gamma_{r,n}U_{n-1}\left(\tau_m^{(2)}\right)\right] = f(t). \tag{S:3.10}$$

We now truncate the infinite sums in (S:3.10) at N and collocate at the zeros of U_{N+1} to obtain an $(N+1)\times(N+1)$ linear system for the $N+1$ unknown coefficients $\gamma_{r,n}$.

S:3.1. *Upstream and downstream flow decompositions*

We now use the solution for γ (S:3.6) to express the background flow (S:3.1) in a form that is amenable to the Wiener–Hopf method. Essentially, we wish to express the mean flow perturbation (S:3.1) in a Fourier-type basis, as this will yield a simple form in the spectral plane under a Fourier transform.

It is straightforward to show that

$$\coth\left(\frac{\pi\mathrm{i}(\tau - z_\beta)}{d+\mathrm{i}s}\right) - 1 = -\frac{2}{1-\exp\left(\frac{2\pi\mathrm{i}(\tau-z_\beta)}{d+\mathrm{i}s}\right)}.$$

The above expression can be expanded as a geometric series depending on whether z_β is upstream or downstream of the cascade:

$$\coth\left(\frac{\pi\mathrm{i}(\tau - z_\beta)}{d+\mathrm{i}s}\right) - 1 = \begin{cases} 2\displaystyle\sum_{n=1}^{\infty} \exp\left(\frac{-2\pi\mathrm{i}n(\tau - z_\beta)}{d+\mathrm{i}s}\right), & (x+1)s < \beta_\infty yd, \\ -2\displaystyle\sum_{n=0}^{\infty} \exp\left(\frac{2\pi\mathrm{i}n(\tau - z_\beta)}{d+\mathrm{i}s}\right), & (x-1)s > \beta_\infty yd. \end{cases}$$

Substituting this into the expression for the mean flow (S:3.1) yields the modal representation

$$\varPhi(z_\beta) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} v_n^+ \exp\left(\frac{2\pi\mathrm{i}nz_\beta}{d+\mathrm{i}s}\right), & (x+1)s < \beta_\infty yd, \\ -\displaystyle\sum_{n=0}^{\infty} v_n^- \exp\left(\frac{-2\pi\mathrm{i}nz_\beta}{d+\mathrm{i}s}\right), & (x-1)s > \beta_\infty yd, \end{cases} \tag{S:3.11}$$

where the coefficients are given by

$$v_n^\pm = \frac{1}{d+\mathrm{i}s}\int_{-1}^{1} \gamma(\tau)\exp\left(\frac{\mp 2\pi\mathrm{i}n\tau}{d+\mathrm{i}s}\right)\mathrm{d}\tau.$$

Analytical expressions for the above coefficients are available by using results for the finite Fourier transforms of weighted Chebyshev polynomials (Olver *et al.* 2010, §18.17.16), but it is also straightforward to integrate the above numerically using Gauss–Chebyshev quadrature. Note that v_0^- is expressible in terms of the circulation $\varGamma$ as

$$v_0^- = -\frac{\varGamma}{d+\mathrm{i}s}.$$

Integrating (S:3.11) yields analytic forms for the full velocity potential and streamfunction. We select the constant of integration so that the leading-edge $z_\beta = -1$ corresponds to $\phi(-1) + \mathrm{i}\psi(-1) = 0$. In the upstream region, $(x+1)s < \beta_\infty yd$, we have

$$\phi(z_\beta) + \mathrm{i}\psi(z_\beta) = (1 - \mathrm{i}\epsilon\alpha)(z_\beta + 1) \tag{S:3.12}$$

$$+\,\epsilon\frac{d+\mathrm{i}s}{2\pi\mathrm{i}}\sum_{n=1}^{\infty}\frac{v_n^+}{n}\left[\exp\left(\frac{2\pi\mathrm{i}nz_\beta}{d+\mathrm{i}s}\right) - \exp\left(\frac{-2\pi\mathrm{i}n}{d+\mathrm{i}s}\right)\right], \tag{S:3.13}$$

and in downstream region, $(x-1)s > yd$, we have

$$\begin{aligned}\phi(z_\beta) + \mathrm{i}\psi(z_\beta) = {} & \left(1 - \mathrm{i}\epsilon\alpha - \epsilon\frac{\varGamma}{d+\mathrm{i}s}\right)(z_\beta + 1) \\ & - \frac{\epsilon}{2\pi\mathrm{i}}\int_{-1}^{1}\gamma(t)\left(\log\left(\frac{\sinh\left(\frac{\pi\mathrm{i}(t-1)}{d+\mathrm{i}s}\right)}{\sinh\left(\frac{\pi\mathrm{i}(t+1)}{d+\mathrm{i}s}\right)}\right) + \frac{2\pi\mathrm{i}}{d+\mathrm{i}s}\right)\mathrm{d}t \\ & + \epsilon\frac{d+\mathrm{i}s}{2\pi\mathrm{i}}\sum_{n=1}^{\infty}\frac{v_n^-}{n}\left[\exp\left(\frac{-2\pi\mathrm{i}nz_\beta}{d+\mathrm{i}s}\right) - \exp\left(\frac{-2\pi\mathrm{i}n}{d+\mathrm{i}s}\right)\right].\end{aligned} \tag{S:3.14}$$

S:4. Asymptotic behaviour of half-range Fourier integrals

Before we may apply the Wiener–Hopf argument, we must determine the asymptotic behaviour of the half range Fourier transforms terms using the physical leading-edge condition. Here we consider the symmetric solution $h_{1,\Sigma}$, although the approach can be generalised to each of the other components. Firstly, the singularity of the velocity field at the leading-edge ($\phi = 0$) can be, at worst, integrable. Therefore, we can write

$$\Sigma_0 \left[\frac{\partial h_{1,\Sigma}}{\partial \psi} \right] (\phi) \sim A\phi^{\eta}, \qquad \text{as } \phi \to 0^-,$$

for some constant A and $-1 < \eta < 0$. By adapting result (1.74) from (Noble 1958, p. 36), we determine that

$$F^{(1)}_{1,\Sigma,-}(\gamma) \sim -\frac{A}{2\pi}\Gamma(\eta+1)\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}(\eta+1)}\gamma^{-(\eta+1)}, \qquad \text{as } |\gamma| \to \infty \text{ in } \mathscr{M}^-, \tag{S:4.1.a}$$

where Γ here refers to the Gamma function.

We also assume that $\Delta_0[h_{1,\Sigma}]$ does not admit singular behaviour at the leading-edge: this assumption is physically supported since, if it did not hold, the pressure at the leading-edge would be non-integrable and result in an infinite force there. Therefore, $\Delta_0[h_{1,\Sigma}](0)$ is finite. Consequently, we may apply integration by parts to obtain

$$D^{(1)}_{1,\Sigma,+}(\gamma) \sim -\frac{\Delta_0\left[h^{(1)}_{1,\Sigma}\right](0)}{2\pi\mathrm{i}\gamma} - \frac{1}{2\pi\mathrm{i}\gamma}\int_0^{\infty} \Delta_0\left[h^{(1)}_{1,\Sigma}\right](\phi)\mathrm{e}^{\mathrm{i}\gamma\phi}\mathrm{d}\phi, \qquad \text{as } |\gamma| \to \infty \text{ in } \mathscr{M}^+, \tag{S:4.1.b}$$

and Riemann–Lebesgue lemma tells us that the second term in the above expression is $o(\gamma^{-1})$.

S:5. List of identities

In this section we present some identities that are used throughout the paper.

S:5.1. *Radical equation*

Consider the equation

$$s_\phi \zeta = \pm\left(\sigma'_\phi + d_\phi\gamma - 2m\pi\right). \tag{S:5.1}$$

We may express $(\sigma'_\phi - 2m\pi)/\Delta_\phi = f_m$ to write

$$s_\phi\zeta = \pm\left(d_\phi\gamma + \Delta_\phi f_m\right),$$

which has solutions

$$\begin{aligned}\gamma = \lambda^{\pm}_m &= -\frac{d_\phi}{\Delta_\phi}f_m \pm \frac{\sqrt{4d_\phi^2\Delta_\phi^2 f_m^2 - 4(\Delta_\phi^2 f_m^2 - s_\phi^2 k^2 w^2)(d_\phi^2 + s_\phi^2)}}{2(d_\phi^2 + s_\phi^2)} \\ &= -\frac{d_\phi}{\Delta_\phi}f_m \pm \frac{s_\phi}{\Delta_\phi}\sqrt{k^2w^2 - f_m^2}.\end{aligned}$$

We can see that λ^+_m must correspond to the "+" equation, because the branch cut on the right-hand side gives positive imaginary part. Therefore, the roots of (S:5.1) are

$$s_\phi\zeta^{\pm}_m = \pm\left(\sigma'_\phi + d_\phi\lambda^{\pm}_m - 2m\pi\right), \tag{S:5.2}$$

where

$$\zeta_m^{\pm} = \sqrt{k^2w^2 - \lambda_m^{\pm 2}} = \pm\frac{s_\phi}{\Delta_\phi} f_m + \frac{d_\phi}{\Delta_\phi}\sqrt{k^2w^2 - f_m^2}.$$

This solution gives us several expressions that are useful for simplification at various points in the analysis:

$$d_\phi \zeta_m^{\pm} \pm s_\phi \lambda_m^{\pm} = \Delta_\phi \sqrt{k^2w^2 - f_m^2}, \tag{S:5.3}$$

$$\Delta_\phi^2 \left(\gamma - \lambda_m^-\right)\left(\gamma - \lambda_m^+\right) = \left(s_\phi\zeta + d_\phi\gamma + \sigma_\phi' - 2m\pi\right)\left(-s_\phi\zeta + d_\phi\gamma + \sigma_\phi' - 2m\pi\right), \tag{S:5.4}$$

$$\lambda_m^+ - \lambda_m^- = 2\frac{s_\phi}{\Delta_\phi}\sqrt{k^2w^2 - f_m^2}. \tag{S:5.5}$$

Accordingly, we have the following expressions for some relevant residues:

$$\operatorname{Res}\left[\frac{1}{1 - \mathrm{e}^{\mathrm{i}(\mp s_\phi\zeta + d_\phi\gamma + \sigma_\phi')}}, \lambda_m^{\pm}\right] = \frac{\mathrm{i}\zeta_m^{\pm}}{\Delta_\phi\sqrt{k^2w^2 - f_m^2}}. \tag{S:5.6}$$

S:5.2. *Integrals*

We note the integrals

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{i}\mu}{k^2w^2 - \mu^2 - \gamma^2}\mathrm{e}^{\mathrm{i}\mu(ns_\phi - \psi)}\mathrm{d}\mu = \frac{1}{2}\operatorname{sgn}\left(ns_\phi - \psi\right)\mathrm{e}^{\mathrm{i}\zeta|ns_\phi - \psi|}, \tag{S:5.7a}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{k^2w^2 - \mu^2 - \gamma^2}\mathrm{e}^{\mathrm{i}\mu(ns_\phi - \psi)}\mathrm{d}\mu = \frac{1}{2\mathrm{i}\zeta}\mathrm{e}^{\mathrm{i}\zeta|ns_\phi - \psi|}, \tag{S:5.7b}$$

where $\zeta = \sqrt{k^2w^2 - \gamma^2}$. These may be calculated via contour integration and specifying the branch of ζ to have positive imaginary part.

S:5.3. *Doubly-infinite sums*

For $0 < \psi_1 < s_\phi$, we have the following identities

$$\begin{aligned} j(\gamma) &= \frac{\mathrm{i}\zeta}{4\pi}\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n(\sigma_\phi' + d_\phi\gamma) + \mathrm{i}\zeta|n|s_\phi} \\ &= \frac{\mathrm{i}\zeta}{4\pi}\left(\sum_{n=0}^{\infty}\mathrm{e}^{\mathrm{i}n(\zeta s_\phi - d_\phi\gamma - \sigma_\phi')} + \sum_{n=1}^{\infty}\mathrm{e}^{\mathrm{i}n(\zeta s_\phi + d_\phi\gamma + \sigma_\phi')}\right) \\ &= \frac{1}{4\pi}\left(\frac{1}{1 - \mathrm{e}^{\mathrm{i}(\zeta s_\phi - d_\phi\gamma - \sigma_\phi')}} - \frac{1}{1 - \mathrm{e}^{-\mathrm{i}(s_\phi\zeta + d_\phi\gamma + \sigma_\phi')}}\right) \\ &= \frac{\zeta}{4\pi}\cdot\frac{\sin\left(\zeta s_\phi\right)}{\cos\left(\zeta s_\phi\right) - \cos\left(\gamma d_\phi + \sigma_\phi'\right)}, \end{aligned} \tag{S:5.8}$$

$$
\begin{aligned}
k(\gamma) &= \frac{1}{4\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \operatorname{sgn}(ns_\phi)\,\mathrm{e}^{\mathrm{i}n(\sigma'_\phi+d_\phi\gamma)+\mathrm{i}\zeta|ns_\phi|}\\
&= \frac{1}{4\pi}\left(-\sum_{n=1}^{\infty}\mathrm{e}^{\mathrm{i}n(s_\phi\zeta-d_\phi\gamma-\sigma'_\phi)}+\sum_{n=1}^{\infty}\mathrm{e}^{\mathrm{i}n(s_\phi\zeta+d_\phi\gamma+\sigma'_\phi)}\right)\\
&= \frac{\mathrm{i}}{4\pi}\left(\frac{1}{1-\mathrm{e}^{-\mathrm{i}(s_\phi\zeta-d_\phi\gamma-\sigma'_\phi)}}-\frac{1}{1-\mathrm{e}^{-\mathrm{i}(s_\phi\zeta+d_\phi\gamma+\sigma'_\phi)}}\right)\\
&= \frac{1}{4\pi\mathrm{i}}\cdot\frac{\sin\left(\gamma d_\phi+\sigma'_\phi\right)}{\cos(\zeta s_\phi)-\cos\left(\gamma d_\phi+\sigma'_\phi\right)},
\end{aligned}
\tag{S:5.9}
$$

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}\operatorname{sgn}(\psi_1+ns_\phi)\mathrm{e}^{\mathrm{i}n(\sigma'_\phi+d_\phi\gamma)}\mathrm{e}^{\mathrm{i}\zeta|\psi_1+ns_\phi|}\\
&\quad= -\mathrm{e}^{-\mathrm{i}\zeta\psi_1}\sum_{n=1}^{\infty}\mathrm{e}^{\mathrm{i}n(s_\phi\zeta-d_\phi\gamma-\sigma'_\phi)}+\mathrm{e}^{\mathrm{i}\zeta\psi_1}\sum_{n=0}^{\infty}\mathrm{e}^{\mathrm{i}n(s_\phi\zeta+d_\phi\gamma+\sigma'_\phi)}\\
&\quad= \frac{\mathrm{e}^{-\mathrm{i}\zeta\psi_1}}{1-\mathrm{e}^{\mathrm{i}(d_\phi\gamma+\sigma'_\phi-s_\phi\zeta)}}+\frac{\mathrm{e}^{\mathrm{i}\zeta\psi_1}}{1-\mathrm{e}^{\mathrm{i}(d_\phi\gamma+\sigma'_\phi+s_\phi\zeta)}},
\end{aligned}
\tag{S:5.10}
$$

$$
\begin{aligned}
I_D(\gamma,\psi) &= -\frac{1}{2}\sum_{n=-\infty}^{\infty}\operatorname{sgn}(ns_\phi-\psi)\,\mathrm{e}^{\mathrm{i}n(\sigma'_\phi+\gamma d_\phi)+\mathrm{i}\zeta|ns_\phi-\psi|}\\
&= \frac{1}{2}\cdot\left(\frac{\mathrm{e}^{\mathrm{i}\zeta\psi}}{1-\mathrm{e}^{\mathrm{i}(s_\phi\zeta-d_\phi\gamma-\sigma'_\phi)}}+\frac{\mathrm{e}^{-\mathrm{i}\zeta\psi}}{1-\mathrm{e}^{-\mathrm{i}(s_\phi\zeta+d_\phi\gamma+\sigma'_\phi)}}\right)\\
&= \frac{\cos(\zeta(\psi-s))-\mathrm{e}^{\mathrm{i}d\gamma+\mathrm{i}\sigma}\cos(\zeta\psi)}{2(\cos(s\zeta)-\cos(d\gamma+\sigma)},
\end{aligned}
\tag{S:5.11}
$$

$$
\begin{aligned}
I_G(\gamma,\psi) &= \frac{1}{2\mathrm{i}\zeta}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}n(\sigma'_\phi+\gamma d_\phi)+\mathrm{i}\zeta|ns_\phi-\psi|}\\
&= \frac{\mathrm{i}}{2\zeta}\cdot\left(\frac{\mathrm{e}^{\mathrm{i}\zeta\psi}}{1-\mathrm{e}^{\mathrm{i}(s_\phi\zeta-d_\phi\gamma-\sigma'_\phi)}}-\frac{\mathrm{e}^{-\mathrm{i}\zeta\psi}}{1-\mathrm{e}^{-\mathrm{i}(s_\phi\zeta+d_\phi\gamma+\sigma'_\phi)}}\right)\\
&= \frac{\sin(\zeta(\psi-s))-\mathrm{e}^{\mathrm{i}d\gamma+\mathrm{i}\sigma}\sin(\zeta\psi)}{2\zeta\left(\cos(s\zeta)-\cos(d\gamma+\sigma)\right)},
\end{aligned}
\tag{S:5.12}
$$

$$
\begin{aligned}
I_S(\gamma,\psi,\psi_1) &= \frac{1}{4\pi\mathrm{i}\zeta}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}n(\sigma'_\phi+\gamma d_\phi)+\mathrm{i}\zeta|\psi_1+ns_\phi-\psi|}\\
&= \frac{1}{4\pi\mathrm{i}\zeta}\cdot\left(\mathrm{e}^{\mathrm{i}\zeta|\psi_1-\psi|}-\frac{\mathrm{e}^{\mathrm{i}\zeta(\psi_1-\psi)}}{1-\mathrm{e}^{-\mathrm{i}(\zeta s_\phi+d_\phi\gamma+\sigma'_\phi)}}-\frac{\mathrm{e}^{-\mathrm{i}\zeta(\psi_1-\psi)}}{1-\mathrm{e}^{-\mathrm{i}(\zeta s_\phi-d_\phi\gamma-\sigma'_\phi)}}\right)\\
&= \frac{\sin(\zeta(|\psi-\psi_1|-s_\phi))-\mathrm{e}^{\mathrm{i}(d_\phi\gamma+\sigma'_\phi)\operatorname{sgn}(\psi-\psi_1)}\sin(\zeta|\psi-\psi_1|)}{4\pi\zeta\left(\cos(s_\phi\zeta)-\cos\left(d_\phi\gamma+\sigma'_\phi\right)\right)}.
\end{aligned}
\tag{S:5.13}
$$

It should be noted that, although these sums are strictly only valid in the Wiener–Hopf strip where $\mathbb{I}\mathrm{m}[\zeta]>0$, they also represent the analytic continuation of the sums from the strip to the entire complex plane.

S:6. Analysis of source terms

In this section, we present our novel, quasi-analytic method of decomposing the source terms arising in the rapid distortion theory framework. The inclusion of the source terms in the modified Helmholtz equation is an essential feature of the analysis and is generalisable to similar problems. Since the solution terms satisfies the quasi-periodicity relation (3.14), we may restrict our analysis to a single infinite channel of height s_ϕ, and extrapolate to the entire domain. For example, in (3.28) we used the quasi-periodic relation to reduce the transformed source terms from an integral in the entire space to an integral along a channel. We now split the region of interest into the upstream ($\phi s_\phi < \psi d_\phi$), downstream ($\phi s_\phi > \psi d_\phi + c_\phi s_\phi$) and inter-blade ($\psi d_\phi < \phi s_\phi < \psi d_\phi + c_\phi s_\phi$) regions.

We will first outline our decomposition of the relevant source terms into exponential functions. Then we will analyse the Fourier transformation of the decompositions in each of the three regions. Finally, we provide some details of how we invert the Fourier transform and obtain the final acoustic field.

S:6.1. *Exponential decomposition of source terms*

In this section we perform the exponential decomposition of the source terms. This step is necessary in order to know the exact structure of the transformed source terms in the complex plane, which is a requirement for the application of the Wiener–Hopf method.

In each region (upstream, downstream and inter-blade), we decompose the source terms into Fourier series (S:6.2, S:6.4, S:6.7). We denote the total source contribution to (3.28) by

$$S(\gamma) = S^u(\gamma) + S^d(\gamma) + S^i(\gamma),$$

where u, d, i correspond to the upstream, downstream and inter-blade regions as defined in section S:6.2.

S:6.1.1. *Upstream region decomposition*

In the upstream region, the source terms satisfy the quasi-periodic relation (3.14). Accordingly, we may express the upstream source terms as a series of exponential functions in the form

$$\mathbb{S}^u(\phi,\psi) = \sum_{M_m^+} \mathbb{S}^+_{P,M_m^+} \mathrm{e}^{-\frac{2m\pi\mathrm{i}}{\Delta_\phi^2}(d_\phi\phi + s_\phi\psi)} \mathrm{e}^{\mathrm{i}M_m^+(\psi d_\phi - \phi s_\phi)} \mathrm{e}^{\mathrm{i}kk_n\psi} \mathrm{e}^{-\mathrm{i}\kappa_0^-\phi}, \tag{S:6.1}$$

which we write as

$$\mathbb{S}^u(\phi,\psi) = \sum_{M_m^+} \mathbb{S}^+_{P,M_m^+} \exp\left[-\mathrm{i}\tilde{M}_m^+\phi\right] \exp\left[\mathrm{i}\left(\Delta_\phi f_m + d_\phi \tilde{M}_m^+\right)\frac{\psi}{s_\phi}\right], \tag{S:6.2}$$

where

$$\tilde{M}_m^+ = s_\phi M_m^+ + \kappa_0^- + \frac{2\pi d_\phi m}{\Delta_\phi^2}.$$

S:6.1.2. *Downstream region decomposition*

The source terms in the downstream region take a more complicated form than those in the upstream region. This increased complexity can be attributed to two factors. Firstly, the perturbation to the background velocity in the horizontal direction, q, decays in the

		Resonance with $\mathcal{O}(1)$ terms?	
		yes	no
Quasi-periodic?	yes	$\mathbb{S}^-_{P,M^-_m}$	$\mathbb{S}^-_{P,\lambda^-_m}$
	no	$\mathbb{S}^-_{N,\kappa^-_0,\pm}$	$\mathbb{S}^-_{N,N^-_{\phi,n}}$

Table 3.2: A summary of the meaning of the coefficients of the source terms (S:6.3) in the downstream region.

upstream region, whereas in the downstream region q tends to a constant value as a result of the cascade deflecting the flow. Consequently, some source terms that vanish in the upstream region do not vanish in the downstream region. These terms cause resonance with the $\mathcal{O}(1)$ solution, and this issue is resolved in section S:7. Secondly, the source terms in the downstream region possess contributions that do not automatically satisfy the quasi-periodic relation (3.14). For example, the κ^-_0 mode in (S:1.3) does not satisfy (3.14) and, therefore, neither does qh_0 in (3.4). Consequently, we express the downstream source terms as

$$\begin{aligned}\mathbb{S}^d(\phi,\psi) = &\sum_{M^-_m} \mathbb{S}^-_{P,M^-_m} \mathrm{e}^{-\frac{2m\pi\mathrm{i}}{\Delta_\phi^2}(d_\phi(\phi-c_\phi)+s_\phi\psi)} \mathrm{e}^{\mathrm{i}M^-_m(\psi d_\phi-(\phi-c_\phi)s_\phi)} \mathrm{e}^{\mathrm{i}kk_n\psi} \mathrm{e}^{-\mathrm{i}\kappa^-_0(\phi-c_\phi)} \\ &+ \sum_{m=-\infty}^{\infty} \mathbb{S}^-_{P,\lambda^-_m} \mathrm{e}^{-\mathrm{i}\left(\lambda^-_m(\phi-c_\phi)+\zeta^-_m\psi\right)} \\ &+ \sum_{N^-_{\phi,n}} \mathbb{S}^-_{N,N^-_{\phi,n}} \mathrm{e}^{-\mathrm{i}\left(N^-_{\phi,n}(\phi-c_\phi)+N^-_{\psi,n}\psi\right)} + \sum_{\pm} \mathbb{S}^-_{N,\kappa^-_0,\pm} \mathrm{e}^{-\mathrm{i}\left(\kappa^-_0(\phi-c_\phi)\pm\zeta_{\kappa^-_0}\psi\right)},\end{aligned} \tag{S:6.3}$$

The role of the coefficients is summarised in table 3.2. We rewrite (S:6.3) as

$$\begin{aligned}\mathbb{S}^d(\phi,\psi) = &\sum_{M^-_m} \mathbb{S}^-_{P,M^-_m} \exp\left[-\mathrm{i}\tilde{M}^-_m(\phi-c_\phi)\right] \exp\left[\mathrm{i}\left(\Delta_\phi f_m + d_\phi\tilde{M}^-_m\right)\frac{\psi}{s_\phi}\right] \\ &+ \sum_{m=-\infty}^{\infty} \mathbb{S}^-_{P,\lambda^-_m} \exp\left[-\mathrm{i}\lambda^-_m(\phi-c_\phi)\right] \exp\left[-\mathrm{i}\zeta^-_m\psi\right] \\ &+ \sum_{N^-_{\phi,n}} \mathbb{S}^-_{N,N^-_{\phi,n}} \exp\left[-\mathrm{i}N^-_{\phi,n}(\phi-c_\phi)\right] \exp\left[-\mathrm{i}N^-_{\psi,n}\psi\right] \\ &+ \sum_{\pm} \mathbb{S}^-_{N,\kappa^-_0,\pm} \exp\left[-\mathrm{i}\kappa^-_0(\phi-c_\phi)\right] \exp\left[\mp\mathrm{i}\zeta_{\kappa^-_0}\psi\right],\end{aligned} \tag{S:6.4}$$

where

$$\tilde{M}^-_m = s_\phi M^-_m + \kappa^-_0 + \frac{2\pi d_\phi m}{\Delta_\phi^2}.$$

S:6.1.3. *Inter-blade region decomposition*

We seek to express the source terms in the inter-blade region in a similar way to previous sections. Unfortunately, the decaying Fourier series representation used for the

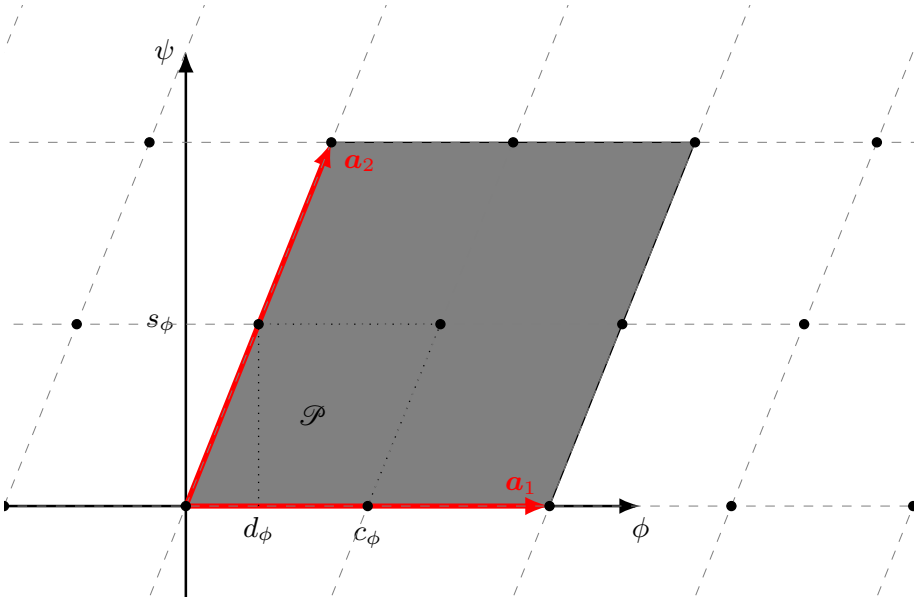

Figure S:6.1: The two-dimensional oblique Bravais lattice for source terms in the inter-blade region.

background flow in previous sections is not applicable since the kernel of the background cascade flow cannot be expanded as a sum of exponentials. Therefore, we seek a Fourier series representation of the entire source terms in the inter-blade region. Strictly speaking, this representation is only necessary for the mean flow terms since we have the exponential form of the acoustic field in this region. However, the structure of the $\mathscr{O}(1)$ acoustic field in the inter-blade region is highly complex, so we choose to decompose the entire source terms using this method.

A straightforward 2-D Fourier series where the period is the inter-blade region will break down. This is because the source terms are discontinuous either side of the blade in (ϕ, ψ)-space and, more importantly, posses (integrable) singularities at the leading-edge of every blade. If we were to take a naive Fourier series where the period window is simply $\mathscr{P}$ of figure S:6.1, we would not be able to accurately capture the behaviour at the leading- and trailing-edges.

We parametrise every point in the plane $\boldsymbol{z} = (\phi, \psi)$ as

$$\boldsymbol{z} = x_1 \frac{\boldsymbol{a}_1}{|\boldsymbol{a}_1|} + x_2 \frac{\boldsymbol{a}_2}{|\boldsymbol{a}_2|}, \tag{S:6.5}$$

where $\boldsymbol{a}_1 = (2c_\phi, 0)$ and $\boldsymbol{a}_2 = (2d_\phi, 2s_\phi)$ are the periods of the extended source terms.

Rearranging (S:6.5) results in

$$x_1 = \phi_1 - \frac{d_\phi}{s_\phi}\psi, \qquad x_2 = \frac{\Delta_\phi}{s_\phi}\psi_1,$$

and consequently, the source terms in the inter-blade region may be written in a Fourier series as

$$\mathbb{S}^i(\phi, \psi) = \sum_{l_1, l_2} \mathbb{S}^i_{l_1, l_2} \exp\left[-\pi \mathrm{i} l_1 \frac{\phi}{c_\phi}\right] \exp\left[-\pi \mathrm{i} \frac{\psi}{s_\phi}\left(l_2 - \frac{l_1 d_\phi}{c_\phi}\right)\right] \exp\left[\mathrm{i}(k k_n \psi - \kappa_0^- \phi)\right], \tag{S:6.6}$$

which reduces to

$$\mathbb{S}^i(\phi,\psi) = \sum_{l_1,l_2} \mathbb{S}^i_{l_1,l_2} \exp\left[-\mathrm{i}\kappa^-_{l_1}\phi\right] \exp\left[\mathrm{i}\left(\Delta_\phi f_{l_2/2} + d_\phi \kappa^-_{l_1}\right)\frac{\psi}{s_\phi}\right], \tag{S:6.7}$$

where the coefficients $\mathbb{S}^i_{l_1,l_2}$ are defined by

$$\mathbb{S}^i_{l_1,l_2} = \frac{1}{|\boldsymbol{a_1}|}\frac{1}{|\boldsymbol{a_2}|}\int_0^{|\boldsymbol{a_2}|}\int_0^{|\boldsymbol{a_1}|} \mathbb{S}^i(\phi(x_1,x_2),\psi(x_1,x_2))\mathrm{e}^{2\pi\mathrm{i}\left(l_1\frac{x_1}{|\boldsymbol{a_1}|}+l_2\frac{x_2}{|\boldsymbol{a_2}|}\right)}\mathrm{d}x_1\mathrm{d}x_2. \tag{S:6.8}$$

S:6.2. *Transformation of source terms*

We segregate the integral (3.28) into upstream, downstream and inter-blade contributions, as in the previous section, and write

$$S(\gamma) = S^u(\gamma) + S^i(\gamma) + S^d(\gamma),$$

where

$$S^x(\gamma) = -\frac{1}{8\pi^2}\int_0^{s_\phi} S^x_\phi(\gamma,\psi_1)\sum_{\pm}\frac{\mathrm{e}^{\pm\mathrm{i}\zeta\psi_1}}{1-\mathrm{e}^{\mathrm{i}(\pm\zeta s_\phi+\sigma'_\phi+\gamma d_\phi)}}\,\mathrm{d}\psi_1. \tag{S:6.9}$$

for $x = u$, i, d, and S^x_ϕ are defined as

$$S^u_\phi(\gamma) = \int_{-\infty}^{\psi_1 d_\phi/s_\phi} \mathbb{S}(\phi_1,\psi_1)\mathrm{e}^{\mathrm{i}\gamma\phi_1}\mathrm{d}\phi_1,$$
$$S^i_\phi(\gamma) = \int_{\psi_1 d_\phi/s_\phi}^{\psi_1 d_\phi/s_\phi+c_\phi} \mathbb{S}(\phi_1,\psi_1)\mathrm{e}^{\mathrm{i}\gamma\phi_1}\mathrm{d}\phi_1,$$
$$S^d_\phi(\gamma) = \int_{\psi_1 d_\phi/s_\phi+c_\phi}^{\infty} \mathbb{S}(\phi_1,\psi_1)\mathrm{e}^{\mathrm{i}\gamma\phi_1}\mathrm{d}\phi_1.$$

S:6.2.1. *Upstream region transformation*

We may used the upstream decomposition of the source terms (S:6.2) to evaluate the upstream Fourier transform of the source terms as

$$S^u_\phi(\gamma,\phi_1) = \sum_{m=-\infty}^{\infty} \mathbb{S}^+_{P,M^+_m}\frac{\mathrm{e}^{\mathrm{i}(d_\phi\gamma-2m\pi+\sigma'_\phi)\psi/s_\phi}}{\mathrm{i}(\gamma-\tilde{M}^+_m)}, \tag{S:6.10}$$

Therefore we may calculate the constituent parts of the transformed source terms in (S:6.9) as

$$S^u(\gamma) = -\frac{s_\phi}{4\pi^2\Delta_\phi^2}\sum_{m=-\infty}^{\infty}\frac{\mathbb{S}^+_{P,M^+_m}}{\gamma-\tilde{M}^+_m}\cdot\frac{d_\phi\gamma-2m\pi+\sigma'_\phi}{(\gamma-\lambda^-_m)(\gamma-\lambda^+_m)}. \tag{S:6.11}$$

The residues of $S^u(\gamma)$ at $\gamma = \lambda^-_m$ are given by

$$S^{u,r}_m = -\frac{s_\phi\zeta^-_m}{8\pi^2\Delta_\phi\sqrt{k^2w^2-f_m^2}}\cdot\frac{\mathbb{S}^+_{P,M^+_m}}{\lambda^-_m-\tilde{M}^+_m},$$

where we have used the identities (S:5.5) and (S:5.2).

S:6.2.2. *Downstream region transformation*

Similarly, we may used the downstream decomposition of the source terms (S:6.4) to evaluate the downstream Fourier transform of the source terms as

$$\begin{aligned}
S^d_\phi(\gamma,\phi_1) = &-\sum_{m=-\infty}^{\infty} \mathbb{S}^-_{P,M^-_m} \frac{\mathrm{e}^{\mathrm{i}(d_\phi\gamma-2m\pi+\sigma'_\phi)\psi/s_\phi}}{\mathrm{i}(\gamma-\tilde{M}^-_m)}\mathrm{e}^{\mathrm{i}\gamma c_\phi}\\
&-\sum_{m=-\infty}^{\infty} \mathbb{S}^-_{P,\lambda^-_m} \frac{\mathrm{e}^{\mathrm{i}(d_\phi\gamma-2m\pi+\sigma'_\phi)\psi/s_\phi}}{\mathrm{i}(\gamma-\lambda^-_m)}\mathrm{e}^{\mathrm{i}\gamma c_\phi}\\
&-\sum_{n} \mathbb{S}^-_{N,N^-_{\phi,n}} \frac{\mathrm{e}^{\mathrm{i}(d_\phi(\gamma-N^-_{\phi,n})s_\phi-N^-_{\psi,n})\psi/s_\phi}}{\mathrm{i}(\gamma-N^-_{\phi,n})}\mathrm{e}^{\mathrm{i}\gamma c_\phi}\\
&-\sum_{\pm} \mathbb{S}^-_{N,\kappa^-_0,\pm} \frac{\mathrm{e}^{\mathrm{i}(d_\phi(\gamma-\kappa^-_0)\pm s_\phi\zeta_{\kappa^-_0})\psi/s_\phi}}{\mathrm{i}(\gamma-\kappa^-_0)}\mathrm{e}^{\mathrm{i}\gamma c_\phi}.
\end{aligned}$$

Therefore, we may calculate (S:6.9) as

$$\begin{aligned}
S^d(\gamma) = &\frac{s_\phi}{4\pi^2\Delta_\phi^2}\sum_{m=-\infty}^{\infty}\frac{\mathbb{S}^-_{P,M^-_m}}{\gamma-\tilde{M}^-_m}\cdot\frac{d_\phi\gamma-2m\pi+\sigma'_\phi}{(\gamma-\lambda^-_m)(\gamma-\lambda^+_m)}\mathrm{e}^{\mathrm{i}\gamma c_\phi}\\
&+\frac{s_\phi}{4\pi^2\Delta_\phi^2}\sum_{m=-\infty}^{\infty}\mathbb{S}^-_{P,\lambda^-_m}\frac{d_\phi\gamma-2m\pi+\sigma'_\phi}{(\gamma-\lambda^-_m)^2(\gamma-\lambda^+_m)}\mathrm{e}^{\mathrm{i}\gamma c_\phi}\\
&+\frac{s_\phi}{8\pi^2}\sum_{\pm}\frac{\mathbb{S}^-_{N,\kappa^-_0,\pm}}{\gamma-\kappa^-_0}\sum_{\pm'}\frac{\mathrm{e}^{\mathrm{i}(d_\phi(\gamma-\kappa^-_0)-s_\phi(\pm\zeta_{\kappa^-_0}\mp'\zeta))}-1}{d_\phi(\gamma-\kappa^-_0)-s_\phi(\pm\zeta_{\kappa^-_0}\mp'\zeta)}\cdot\frac{1}{1-\mathrm{e}^{\mathrm{i}(\pm'\zeta s_\phi+\gamma d_\phi+\sigma'_\phi)}}\mathrm{e}^{\mathrm{i}\gamma c_\phi}\\
&+\frac{s_\phi}{8\pi^2}\sum_{N^-_{\phi,n}}\frac{\mathbb{S}^-_{N,N^-_{\phi,n}}}{\gamma-N^-_{\phi,n}}\sum_{\pm}\frac{\mathrm{e}^{\mathrm{i}(d_\phi(\gamma-N^-_{\phi,n})-s_\phi(N^-_{\psi,n}\mp\zeta))}-1}{d_\phi(\gamma-N^-_{\phi,n})-s_\phi(N^-_{\psi,n}\mp\zeta)}\cdot\frac{1}{1-\mathrm{e}^{\mathrm{i}(\pm\zeta s_\phi+\gamma d_\phi+\sigma'_\phi)}}\mathrm{e}^{\mathrm{i}\gamma c_\phi}.
\end{aligned}\tag{S:6.12}$$

Using (S:5.6), we calculate the residues of $S^d(\gamma)$ at $\gamma=\lambda^+_m$ as

$$\begin{aligned}
S^{d,r}_m = &\frac{s_\phi\zeta^+_m}{8\pi^2\Delta_\phi\sqrt{k^2w^2-f_m^2}}\cdot\frac{\mathbb{S}^-_{P,M^-_m}}{\lambda^+_m-\tilde{M}^-_m}\mathrm{e}^{\mathrm{i}c_\phi\lambda^+_m}\\
&+\frac{s_\phi\zeta^+_m}{8\pi^2\Delta_\phi\sqrt{k^2w^2-f_m^2}}\cdot\frac{\mathbb{S}^-_{P,\lambda^-_m}}{\lambda^+_m-\lambda^-_m}\mathrm{e}^{\mathrm{i}c_\phi\lambda^+_m}\\
&+\frac{s_\phi}{8\pi^2}\sum_{\pm}\frac{\mathbb{S}^-_{N,\kappa^-_0,\pm}}{\lambda^+_m-\kappa^-_0}\cdot\frac{\mathrm{e}^{\mathrm{i}(d_\phi(\lambda^+_m-\kappa^-_0)-s_\phi(\pm\zeta_{\kappa^-_0}+\zeta^+_m))}-1}{d_\phi(\lambda^+_m-\kappa^-_0)-s_\phi(\pm\zeta_{\kappa^-_0}+\zeta^+_m)}\cdot\frac{\zeta^+_m}{\mathrm{i}\Delta_\phi\sqrt{k^2w^2-f_m^2}}\mathrm{e}^{\mathrm{i}c_\phi\lambda^+_m}\\
&+\frac{s_\phi}{8\pi^2}\sum_{N^-_{\phi,n}}\frac{\mathbb{S}^-_{N,N^-_{\phi,n}}}{\lambda^+_m-N^-_{\phi,n}}\cdot\frac{\mathrm{e}^{\mathrm{i}(d_\phi(\lambda^+_m-N^-_{\phi,n})-s_\phi(N^-_{\psi,n}+\zeta^+_m))}-1}{d_\phi(\lambda^+_m-N^-_{\phi,n})-s_\phi(N^-_{\psi,n}+\zeta^+_m)}\cdot\frac{\zeta^+_m}{\mathrm{i}\Delta_\phi\sqrt{k^2w^2-f_m^2}}\mathrm{e}^{\mathrm{i}c_\phi\lambda^+_m}.
\end{aligned}$$

S:6.2.3. *Inter-blade region transformation*

We may calculate the inter-blade Fourier transform by substituting in our exponential expression (S:6.7) to obtain

$$S^i_\phi(\gamma,\psi_1) = S^{i,(1)}_\phi(\gamma,\psi_1) + S^{i,(2)}_\phi(\gamma,\psi_1),$$

where

$$S_\phi^{i,(1)}(\gamma,\psi_1)=\sum_{l_1,l_2}\mathbb{S}^i_{l_1,l_2}\frac{-1}{\mathrm{i}\left(\gamma-\kappa_{l_1}^-\right)}\exp\left[\mathrm{i}(d_\phi\gamma+\sigma'_\phi-\pi l_2)\frac{\psi_1}{s_\phi}\right],$$

$$S_\phi^{i,(2)}(\gamma,\psi_1)=\sum_{l_1,l_2}\mathbb{S}^i_{l_1,l_2}\frac{\mathrm{e}^{\mathrm{i}(\gamma-\kappa_{l_1}^-)c_\phi}}{\mathrm{i}\left(\gamma-\kappa_{l_1}^-\right)}\exp\left[\mathrm{i}(d_\phi\gamma+\sigma'_\phi-\pi l_2)\frac{\psi_1}{s_\phi}\right].$$

Using these representations, the ψ_1 integral in (S:6.9) may now be performed to obtain

$$S^i(\gamma)=S^{i,(1)}(\gamma)+S^{i,(2)}(\gamma),$$

where

$$S^{i,(1)}(\gamma)=\sum_{l_1,l_2}\mathbb{S}^i_{l_1,l_2}\frac{\mathscr{F}_{l_2}(\gamma)}{\left(\gamma-\kappa_{l_1}^-\right)},\quad S^{i,(2)}(\gamma)=-\sum_{l_1,l_2}\mathbb{S}^i_{l_1,l_2}\mathrm{e}^{(\gamma-\kappa_{l_1}^-)c_\phi}\frac{\mathscr{F}_{l_2}(\gamma)}{\left(\gamma-\kappa_{l_1}^-\right)},\tag{S:6.13}$$

and

$$\mathscr{F}_{l_2}(\gamma)=\frac{s_\phi}{8\pi^2}\sum_{\pm}\frac{(-1)^{l_2}\mathrm{e}^{\mathrm{i}(d_\phi\gamma+\sigma'_\phi\pm s_\phi\zeta)}-1}{\pm s_\phi\zeta+d_\phi\gamma+\sigma'_\phi-\pi l_2}\cdot\frac{1}{1-\mathrm{e}^{\mathrm{i}(\pm\zeta s_\phi+\gamma d_\phi+\sigma'_\phi)}}.$$

When $l_2=2m$ is even, this reduces to

$$\mathscr{F}_{l_2}(\gamma)=\frac{-s_\phi}{4\pi^2\Delta_\phi^2}\cdot\frac{d_\phi\gamma+\sigma'_\phi-2m\pi}{(\gamma-\lambda_m^+)(\gamma-\lambda_m^-)}.$$

Consequently, the residues of $\mathscr{F}_{l_2}$ at $\lambda_m^\pm$ are

$$\mathscr{F}^{r,\pm}_{l_2,m}=\frac{s_\phi}{8\pi^2}\cdot\frac{\zeta_m^\pm}{\Delta_\phi\sqrt{k^2w^2-f_m^2}}\times\begin{cases}\dfrac{2}{\pi\mathrm{i}(2m-l_2)}, & l_2\text{ odd},\\ -\delta_{l_2,2m}, & l_2\text{ even}.\end{cases}\tag{S:6.14}$$

S:6.3. *Inversion of source terms*

When we invert the Fourier transform in section 4, we encounter integrals of the form

$$\begin{aligned}I(\gamma,\psi)=\int_0^{s_\phi}\mathrm{e}^{\mathrm{i}a(\gamma)\psi_1/s_\phi}&\left\{\mathrm{e}^{\mathrm{i}\zeta|\psi_1-\psi|}-\frac{\mathrm{e}^{\mathrm{i}\zeta(\psi_1-\psi)}}{1-\mathrm{e}^{-\mathrm{i}(\zeta s_\phi+\gamma d_\phi+\sigma'_\phi)}}-\frac{\mathrm{e}^{-\mathrm{i}\zeta(\psi_1-\psi)}}{1-\mathrm{e}^{\mathrm{i}(-\zeta s_\phi+\gamma d_\phi+\sigma'_\phi)}}\right\}\mathrm{d}\psi_1\\
=&\frac{\mathrm{i}s_\phi}{a(\gamma)-s_\phi\zeta}\left[\mathrm{e}^{\mathrm{i}\zeta\psi}-\mathrm{e}^{\mathrm{i}\psi a(\gamma)/s_\phi}+\frac{\mathrm{e}^{\mathrm{i}(a(\gamma)-(s_\phi-\psi)\zeta)}-\mathrm{e}^{\mathrm{i}\psi\zeta}}{1-\mathrm{e}^{\mathrm{i}(-s_\phi\zeta+d_\phi\gamma+\sigma'_\phi)}}\right]\\
&+\frac{\mathrm{i}s_\phi}{a(\gamma)+s_\phi\zeta}\left[\mathrm{e}^{\mathrm{i}\psi a(\gamma)/s_\phi}-\mathrm{e}^{\mathrm{i}(a(\gamma)+(s_\phi-\psi)\zeta)}+\frac{\mathrm{e}^{\mathrm{i}(a(\gamma)+\zeta(s_\phi-\psi))}-\mathrm{e}^{-\mathrm{i}\psi\zeta}}{1-\mathrm{e}^{-\mathrm{i}(\zeta s_\phi+\gamma d_\phi+\sigma'_\phi)}}\right]\end{aligned}$$

for some linear functions $a(\gamma)$. Note that the residues of I at $\gamma=\lambda_m^\pm$ are

$$\operatorname{Res}\left[I,\lambda_m^\pm\right]=\frac{\mathrm{i}s_\phi\zeta_m^\pm}{a(\lambda_m^\pm)\mp s_\phi\zeta_m^\pm}\cdot\frac{\mathrm{e}^{\mathrm{i}(a(\lambda_m^\pm)\mp s_\phi\zeta_m^\pm)}-1}{\Delta_\phi\sqrt{k^2w^2-f_m^2}}\mathrm{e}^{\pm\mathrm{i}\zeta_m^\pm\psi}.$$

In the special case $a(\gamma) = \mathrm{i}(\sigma'_\phi - \pi l_2 + d_\phi \gamma)$, we have

$$
\begin{aligned}
&I(\gamma,\psi)\\
&= \int_0^{s_\phi} \mathrm{e}^{\mathrm{i}(\sigma'_\phi - \pi l_2 + d_\phi\gamma)\psi_1/s_\phi} \left\{ \mathrm{e}^{\mathrm{i}\zeta|\psi_1-\psi|} - \frac{\mathrm{e}^{\mathrm{i}\zeta(\psi_1-\psi)}}{1-\mathrm{e}^{-\mathrm{i}(\zeta s_\phi + \gamma d_\phi + \sigma'_\phi)}} - \frac{\mathrm{e}^{-\mathrm{i}\zeta(\psi_1-\psi)}}{1-\mathrm{e}^{\mathrm{i}(-\zeta s_\phi + \gamma d_\phi + \sigma'_\phi)}} \right\} \mathrm{d}\psi_1\\
&= -2\pi\zeta \left\{ \frac{s_\phi^2}{\Delta_\phi^2} \cdot \frac{\mathrm{e}^{\mathrm{i}(\sigma'_\phi + d_\phi\gamma - \pi l_2)\psi/s_\phi}}{(\gamma - \lambda^+_{l_2/2})(\gamma - \lambda^-_{l_2/2})} \right.\\
&\left. + \sum_{\pm} \frac{\pm s_\phi}{2\zeta} \cdot \frac{\mathrm{e}^{\pm\mathrm{i}\zeta\psi}}{1-\mathrm{e}^{\mathrm{i}(\pm s_\phi\zeta - d_\phi\gamma - \sigma'_\phi)}} \cdot \frac{\left((-1)^{l_2}-1\right)}{\pm s_\phi\zeta - \sigma'_\phi - d_\phi\gamma + \pi l_2} \right\}.
\end{aligned}
$$

During the inversion of the Fourier transform, the ensuing γ-integral is then closed in the upper or lower half plane, depending on which region of the physical domain is under consideration. In particular, the residues of the function in the curly brackets at $\gamma = \lambda_m^\pm$ are given by

$$
\frac{s_\phi \mathrm{e}^{\mathrm{i}\zeta_m^\pm \psi}}{2\Delta_\phi\sqrt{k^2w^2 - f_m^2}} \times \begin{cases} \dfrac{2}{\pi\mathrm{i}(2m - l_2)}, & l_2 \text{ odd},\\ -\delta_{l_2,2m}, & l_2 \text{ even}. \end{cases}
$$

S:7. Resolving resonance

In this section we observe that the regular perturbation expansion (3.1) breaks down when $\phi = \mathcal{O}(\epsilon^{-1})$ in the downstream region. In particular, the $\mathcal{O}(\epsilon)$ solution grows to $\mathcal{O}(1)$ and therefore has the same magnitude as the leading-order solution. Consequently, the series (3.1) is no longer an asymptotic expansion of the solution. The growth of the $\mathcal{O}(\epsilon)$ solution is caused by resonance between the $\mathcal{O}(\epsilon)$ solution and terms in the $\mathcal{O}(1)$ solution. We now show how to regularise our solution by appealing to a multiple-scales argument so that the solution is valid in a larger region up to $\phi = \mathcal{O}(\epsilon^{-2})$.

We first consider the resonance arising in the source term solution $h_{1,S}$, and then apply a similar argument to regularise the resonance arising in the $h_{1,\Gamma}$ solution.

S:7.1. *Resonance in the $h_{1,S}$ solution*

In the downstream region we have

$$
\begin{aligned}
h_{1,S}(\phi,\psi) = &\sum_{m=-\infty}^{\infty} \mathscr{H}^-_{S,m} \exp\left[-\mathrm{i}\left(\lambda_m^-\phi + \zeta_m^-\psi\right)\right]\\
&+ \sum_{M_m^-} \mathbb{S}^-_{P,M_m^-}\mathbb{I}^-_{P,M_m^-}(\phi,\psi) + \sum_{m=-\infty}^{\infty} \mathbb{S}^-_{P,\lambda_m^-}\mathbb{I}^-_{P,\lambda_m^-}(\phi,\psi)\\
&+ \sum_{N^-_{\phi,n}} \mathbb{S}^-_{N,N^-_{\phi,n}}\mathbb{I}^-_{N,N^-_{\phi,n}}(\phi,\psi) + \sum_{\pm} \mathbb{S}^-_{N,\kappa_0^-,\pm}\mathbb{I}^-_{N,\kappa_0^-,\pm}(\phi,\psi)\\
&+ \sum_{\pm} \mathscr{H}^-_{1,S,\kappa_0^-,\pm} \mathrm{e}^{-\mathrm{i}(\kappa_0^-\phi \pm \zeta_{\kappa_0^-}\psi)},
\end{aligned}
\tag{S:7.1}
$$

where the coefficients are defined as

$$
\begin{aligned}
\mathscr{H}^{-}_{S,m} = &-\frac{\pi}{\Delta_\phi}\sum_{m=-\infty}^{\infty}\frac{\zeta^-_m D^{(2,4)}_{1,S}(\lambda^-_m)}{\sqrt{k^2w^2-f_m^2}}+\\
&-\sum_{M^-_m}\mathbb{S}^-_{P,M^-_m}\frac{s_\phi}{\Delta_\phi}\cdot\frac{\exp\left[\mathrm{i}\lambda^-_m c_\phi\right]}{2\sqrt{k^2w^2-f_m^2}(\tilde{M}^-_m-\lambda^-_m)}\\
&+\mathbb{S}^-_{P,\lambda^-_m}\frac{1}{4(k^2w^2-f_m^2)}\exp\left[\mathrm{i}\lambda^-_m c_\phi\right]\\
&-\sum_{N^-_{\phi,n}}\mathbb{S}^-_{N,N^-_{\phi,n}}\frac{\mathrm{i}s_\phi(\mathrm{e}^{\mathrm{i}(d_\phi(\lambda^-_m-N^-_{\phi,n})+s_\phi(\zeta^-_m-N^-_{\psi,n}))}-1)}{d_\phi(\lambda^-_m-N^-_{\phi,n})+s_\phi(\zeta^-_m-N^-_{\psi,n})}\frac{\exp\left[\mathrm{i}\lambda^-_m c_\phi\right]}{2(\lambda^-_m-N^-_{\phi,n})\Delta_\phi\sqrt{k^2w^2-f_m^2}}\\
&-\sum_{\pm}\mathbb{S}^-_{N,\kappa^-_0,\pm}\frac{\mathrm{i}s_\phi(\mathrm{e}^{\mathrm{i}(d_\phi(\lambda^-_m-\kappa^-_0)+s_\phi(\zeta^-_m\pm\zeta_{\kappa^-_0}))}-1)}{d_\phi(\lambda^-_m-\kappa^-_0)+s_\phi(\zeta^-_m\pm\zeta_{\kappa^-_0})}\frac{\exp\left[\mathrm{i}\lambda^-_m c_\phi\right]}{2(\lambda^-_m-\kappa^-_0)\Delta_\phi\sqrt{k^2w^2-f_m^2}}\\
&+\frac{4\pi^2}{\zeta^-_m}\sum_{l_1,l_2}\frac{\mathbb{S}^i_{l_1,l_2}\mathscr{F}^{r,-}_{l_2,m}}{\lambda^-_m-\kappa^-_{l_1}}\mathrm{e}^{\mathrm{i}(\lambda^-_m-\kappa^-_{l_1})c_\phi},\\
\mathscr{H}^{-}_{1,S,\kappa^-_0,\pm} = &\frac{\pi\mathrm{i}P_{1,S}}{2}\cdot\frac{\mathrm{e}^{\mathrm{i}(d_\phi\kappa^-_0+\sigma'_\phi)}-\mathrm{e}^{\pm\mathrm{i}s_\phi\zeta_{\kappa^-_0}}}{\cos(d_\phi\kappa^-_0+\sigma'_\phi)-\cos(s_\phi\zeta_{\kappa^-_0})},
\end{aligned}
\tag{S:7.2}
$$

and the inhomogeneous functions are defined as

$$
\begin{aligned}
\mathbb{I}^-_{P,M^-_m}(\phi,\psi) &= -\frac{s_\phi^2}{\Delta_\phi^2}\cdot\frac{\exp\left[-\mathrm{i}\left(\tilde{M}^-_m(\phi-c_\phi)-(\Delta_\phi f_m+d_\phi\tilde{M}^-_m)\frac{\psi}{s_\phi}\right)\right]}{(\tilde{M}^-_m-\lambda^+_m)(\tilde{M}^-_m-\lambda^-_m)},\\
\mathbb{I}^-_{N,N^-_{\phi,n}}(\phi,\psi) &= \sum_{\pm}\frac{1}{\zeta(N^-_{\phi,n})\pm N^-_{\psi,n}}\cdot\Bigg(\mathrm{e}^{-\mathrm{i}N^-_{\psi,n}\psi}-\mathrm{e}^{\mathrm{i}(\pm\zeta(N^-_{\phi,n})\psi+H(\mp1)(\zeta(N^-_{\phi,n})-N^-_{\psi,n})s_\phi)}\\
&\qquad\mp\frac{\mathrm{e}^{-\mathrm{i}(s_\phi N^-_{\psi,n}\pm(s_\phi-\psi)\zeta(N^-_{\phi,n}))}-\mathrm{e}^{\pm\mathrm{i}\zeta(N^-_{\phi,n})\psi}}{1-\mathrm{e}^{-\mathrm{i}(\zeta(N^-_{\phi,n})\mp d_\phi N^-_{\phi,n}\mp\sigma'_\phi)}}\Bigg)\frac{\exp\left[-\mathrm{i}N^-_{\phi,n}(\phi-c_\phi)\right]}{2\zeta(N^-_{\phi,n})},\\
\mathbb{I}^-_{P,\lambda^-_m}(\phi,\psi) &= (s_\phi(\phi-c_\phi)-d_\phi\psi)\frac{\exp\left[-\mathrm{i}\left(\lambda^-_m(\phi-c_\phi)+\zeta^-_m\psi\right)\right]}{2\mathrm{i}\Delta_\phi\sqrt{k^2w^2-f_m^2}},\\
\mathbb{I}^-_{N,\kappa^-_0,\pm}(\phi,\psi) &= \Bigg\{\pm\mathrm{e}^{\pm\mathrm{i}\zeta_{\kappa^-_0}\psi}\left(\psi-\frac{s_\phi}{1-\mathrm{e}^{\mathrm{i}(\mp s_\phi\zeta_{\kappa^-_0}+d_\phi\kappa^-_0+\sigma'_\phi)}}\right)\\
&-\frac{\mathrm{e}^{-\mathrm{i}H(\mp1)\zeta_{\kappa^-_0}s_\phi}}{2\mathrm{i}\zeta_{\kappa^-_0}}\left[\mathrm{e}^{\pm\mathrm{i}\zeta_{\kappa^-_0}(\psi-H(\mp1)s_\phi)}-\mathrm{e}^{\mp\mathrm{i}\zeta_{\kappa^-_0}(\psi-H(\mp1)s_\phi)}\frac{1-\mathrm{e}^{-\mathrm{i}(-s_\phi\zeta_{\kappa^-_0}+d_\phi\kappa^-_0+\sigma'_\phi)}}{1-\mathrm{e}^{-\mathrm{i}(s_\phi\zeta_{\kappa^-_0}+d_\phi\kappa^-_0+\sigma'_\phi)}}\right]\Bigg\}\\
&\qquad\times\frac{\exp\left[-\mathrm{i}\left(\kappa^-_0(\phi-c_\phi)\right)\right]}{2\zeta_{\kappa^-_0}}.
\end{aligned}
$$

It may be verified that applying the Helmholtz operator (3.3) to the inhomogeneous functions yields

$$\begin{aligned}
\mathscr{L}(\mathbb{I}^-_{P,M^-_m}) &= \exp\left[-\mathrm{i}\left(\tilde{M}^-_m(\phi-c_\phi)-(\Delta_\phi f_m+d_\phi\tilde{M}^-_m)\frac{\psi}{s_\phi}\right)\right],\\
\mathscr{L}(\mathbb{I}^-_{P,\lambda^-_m}) &= \exp\left[-\mathrm{i}\left(\lambda^-_m(\phi-c_\phi)+\zeta^-_m\psi\right)\right],\\
\mathscr{L}(\mathbb{I}^-_{N,N^-_{\phi,n}}) &= \exp\left[-\mathrm{i}N^-_{\phi,n}(\phi-c_\phi)+N^-_{\psi,n}\psi\right],\\
\mathscr{L}(\mathbb{I}^-_{N,\kappa^-_0,\pm}) &= \exp\left[-\mathrm{i}\left(\kappa^-_0(\phi-c_\phi)\pm\zeta_{\kappa^-_0}\psi\right)\right],
\end{aligned}$$

so that (S:7.1) is in fact the solution to the inhomogeneous Helmholtz equation.

We now combine the $h_{1,S}$ solution (S:7.1) with the $\mathscr{O}(1)$ solution h_0 (S:1.3) to write

$$\begin{aligned}
h_0(\phi,\psi)+\epsilon h_{1,S}(\phi,\psi) = &\sum_{m=-\infty}^{\infty}\left(\mathscr{H}^-_{0,m}+\epsilon\mathscr{H}^-_{\Gamma,m}\right)\exp\left[-\mathrm{i}\left(\lambda^-_m\phi+\zeta^-_m\psi\right)\right] \qquad \text{(S:7.3)}\\
&+\sum_{M^-_m}\epsilon\mathbb{S}^-_{P,M^-_m}\mathbb{I}^-_{P,M^-_m}(\phi,\psi)+\sum_{m=-\infty}^{\infty}\epsilon\mathbb{S}^-_{P,\lambda^-_m}\mathbb{I}^-_{P,\lambda^-_m}(\phi,\psi)\\
&+\sum_{N^-_{\phi,n}}\epsilon\mathbb{S}^-_{N,N^-_{\phi,n}}\mathbb{I}^-_{N,N^-_{\phi,n}}(\phi,\psi)+\sum_{\pm}\epsilon\mathbb{S}^-_{N,\kappa^-_0,\pm}\mathbb{I}^-_{N,\kappa^-_0,\pm}(\phi,\psi)\\
&+\sum_{\pm}\left(\mathscr{H}^-_{0,\kappa^-_0,\pm}+\epsilon\mathscr{H}^-_{1,S,\kappa^-_0,\pm}\right)\mathrm{e}^{-\mathrm{i}(\kappa^-_0\phi\pm\zeta_{\kappa^-_0}\psi)}.
\end{aligned}$$

We note that the $\mathbb{I}^-_{P,\lambda^-_m}$ terms are proportional to $\epsilon\phi$. These are *secular terms* (van Dyke 1964), and present a problem for our asymptotic series: the expansion (3.1) breaks down when $\phi=\mathscr{O}(\epsilon^{-1})$. Moreover, the solution does not satisfy the Sommerfeld radiation condition as the acoustic field is not bounded in the far-field when even a single mode is cut-on. Consequently, we need a way of regularising these terms.

We now combine the resonant $\mathbb{I}^-_{P,\lambda^-_m}$ terms with the terms on line (S:7.3) and write

$$\begin{aligned}
\tilde{h}_{0,S}(\phi,\psi) = &\sum_{m=-\infty}^{\infty}\left(\mathscr{H}^-_{0,m}+\epsilon\mathscr{H}^-_{S,m}\right)\left\{1-\mathrm{i}\epsilon\left(s_\phi(\phi-c_\phi)-d_\phi\psi\right)O_{S,m}\right\}\\
&\times\exp\left[-\mathrm{i}\left(\lambda^-_m\phi+\zeta^-_m\psi\right)\right], \qquad \text{(S:7.4)}
\end{aligned}$$

where

$$O_{S,m}=\frac{\mathbb{S}^-_{P,\lambda^-_m}\mathrm{e}^{\mathrm{i}\lambda^-_m c_\phi}}{2\Delta_\phi\sqrt{k^2w^2-f_m^2}\mathscr{H}^-_{0,m}},$$

and we have ignored $\mathscr{O}(\epsilon^2)$ terms. We now note that, when $s_\phi(\phi-c_\phi)-d_\phi\psi=\mathscr{O}(1)$, we may express the term in the curly brackets in (S:7.4) as

$$1-\mathrm{i}\epsilon\left(s_\phi(\phi-c_\phi)-d_\phi\psi\right)O_{S,m}=\exp\left[-\mathrm{i}\epsilon\left(s_\phi(\phi-c_\phi)-d_\phi\psi\right)O_{S,m}\right]+\mathscr{O}((\epsilon O_{S,m})^2),$$

In particular, the above relation is exact along the matching line $s_\phi(\phi-c_\phi)-d_\phi\psi$ where the downstream region meets the lower triangular region. Consequently, we may express

(S:7.4) as

$$\tilde{h}_{0,S}(\phi,\psi) = \sum_{m=-\infty}^{\infty} \left(\mathscr{H}^-_{0,m} + \epsilon\mathscr{H}^-_{S,m}\right) \exp\left[-\mathrm{i}\left(\tilde{\lambda}^-_{m,S}\phi + \tilde{\zeta}^-_{m,S}\psi\right)\right] \exp[\mathrm{i}\epsilon s_\phi c_\phi O_{S,m}], \tag{S:7.5}$$

where we have introduced the modified modes

$$\tilde{\lambda}^-_{S,m} = \lambda^-_m + \epsilon s_\phi O_{S,m},$$
$$\tilde{\zeta}^-_{S,m} = \zeta^-_m - \epsilon d_\phi O_{S,m}.$$

Although we have apparently performed an illegal maneuver in regularising (S:7.4), the resulting function satisfies the required inhomogeneous Helmholtz equation up to $\mathscr{O}(\phi\epsilon^2)$. Applying the Helmholtz operator to $\tilde{h}_{0,S}$ as defined in (S:7.5) yields

$$\begin{aligned}\mathscr{L}\left(\tilde{h}_{0,S}\right) &= \sum_{m=-\infty}^{\infty} \left(\mathscr{H}^-_{0,m} + \epsilon\mathscr{H}^-_{S,m}\right) \left\{2\epsilon O_{S,m}\left(d_\phi\zeta^-_m - s_\phi\lambda^-_m\right) + \mathscr{O}(\epsilon^2)\right\} \\ &\qquad \times \exp\left[-\mathrm{i}\left(\tilde{\lambda}^-_{S,m}\phi + \tilde{\zeta}^-_{S,m}\psi\right)\right] \exp[\mathrm{i}\epsilon s_\phi c_\phi O_{S,m}] \\ &= \sum_{m=-\infty}^{\infty} \epsilon\mathbb{S}^-_{P,\lambda^-_m} \exp\left[-\mathrm{i}\left(\lambda^-_m(\phi - c_\phi) + \zeta^-_m\psi\right)\right] + \mathscr{O}(\phi\epsilon^2),\end{aligned}$$

which is the form of the desired forcing terms.

We now note that (S:7.5) is exactly the solution that would have been obtained if a multiple-scales perturbation expansion had been applied to the downstream region as opposed to the regular perturbation series (3.1). In that case, we would have written

$$\tilde{h}_{0,S}(\phi,\psi) = H_0(\phi,\Phi,\psi) + \epsilon H_1(\phi,\Phi,\psi) + \cdots$$

where $\Phi = \epsilon\phi$, and consequently obtained the solution (S:7.5). Accordingly, we have obtained the multiple-scales solution albeit via a circuitous route. The solution could be further expanded into the region $\phi = \mathscr{O}(\epsilon^{-2})$ via a second multiple-scales type argument, but it is not necessary in the present work.

S:7.2. *Resonance in the $h_{1,\Gamma}$ solution*

We now apply a similar procedure to regularise terms in the $h_{1,\Gamma}$ problem so that the solution is valid in the region $\phi = \mathscr{O}(\epsilon^{-1})$. In the downstream region, we have

$$\begin{aligned}h_{1,\Gamma}(\phi,\psi) = &\sum_{m=-\infty}^{\infty} \mathscr{H}^-_{\Gamma,m} \exp\left[-\mathrm{i}\left(\lambda^-_m\phi + \zeta^-_m\psi\right)\right] \\ &- \sum_{m=-\infty}^{\infty} \Gamma\mathscr{H}'_{0,m}\left(s_\phi(\phi - c_\phi) - d_\phi\psi\right) \frac{\zeta^-_m \exp\left[-\mathrm{i}\left(\lambda^-_m\phi + \zeta^-_m\psi\right)\right]}{s_\phi\Delta_\phi\sqrt{k^2w^2 - f_m^2}} \\ &- \sum_{m=-\infty}^{\infty} \Gamma\mathscr{H}'_{0,m}\frac{\psi}{s_\phi} \exp\left[-\mathrm{i}\left(\lambda^-_m\phi + \zeta^-_m\psi\right)\right] \\ &+ \sum_{\pm} \mathscr{H}^-_{1,\Gamma,\kappa^-_0,\pm} \mathrm{e}^{-\mathrm{i}(\kappa^-_0\phi \pm \zeta_{\kappa^-_0}\psi)},\end{aligned}$$

where

$$
\begin{aligned}
\mathscr{H}^{-}_{\Gamma,m} = &-\frac{\pi}{\Delta_\phi}\sum_{m=-\infty}^{\infty}\frac{\zeta^-_m D^{(2,4)}_{1,\Gamma}(\lambda^-_m)}{\sqrt{k^2w^2-f_m^2}} \\
&-\sum_{\pm}\Gamma\mathscr{H}^{\pm\prime}_{0,\kappa^-_0}\frac{\pm\mathrm{i}\zeta_{\kappa^-_0}\mathrm{e}^{\mathrm{i}(\lambda^-_m-\kappa^-_0)c_\phi}}{2(\lambda^-_m-\kappa^-_0)}\frac{1}{\Delta_\phi\sqrt{k^2w^2-f_m^2}} \\
&-\sum_{\substack{n=-\infty\\ n\neq m}}^{\infty}\left[\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}(\lambda^-_m-\lambda^-_n)c_\phi}}{2(\lambda^-_m-\lambda^-_n)}\Gamma\mathscr{H}'_{0,n}\left\{\zeta^-_n+\zeta^-_m\frac{\lambda^-_n-\kappa^-_0}{\lambda^-_m-\kappa^-_0}\cdot\frac{J_-(\lambda^-_n)}{J_-(\lambda^-_m)}\right\}\right]\frac{1}{\Delta_\phi\sqrt{k^2w^2-f_m^2}} \\
&-\sum_{m=-\infty}^{\infty}\Gamma\mathscr{H}'_{0,m}\frac{\left(((s_\phi\lambda^-_m)^2-(d_\phi\zeta^-_m)^2)\cot(s_\phi\zeta^-_m)-s_\phi/\zeta^-_m((\zeta^-_m)^2+(\lambda^-_m)^2)\right)}{2\mathrm{i}\Delta_\phi^2(k^2w^2-f_m^2)} \\
&-\Gamma\mathscr{H}'_{0,m}\left\{\frac{\zeta^-_m}{\lambda^-_m-\kappa^-_0}-\frac{\lambda^-_m}{\zeta^-_m}+\zeta^-_m\frac{J'_-(\lambda^-_m)}{J_-(\lambda^-_m)}\right\}\frac{\mathrm{i}}{2\Delta_\phi\sqrt{k^2w^2-f_m^2}} \\
&+\Gamma\mathscr{H}'_{0,m}\frac{s_\phi\lambda^-_m\mathrm{e}^{\mathrm{i}s_\phi\lambda^-_m}+d_\phi\zeta^-_m\mathrm{e}^{-\mathrm{i}s_\phi\zeta^-_m}}{2\mathrm{i}\sin(s_\phi\zeta^-_m)}\frac{1}{\Delta_\phi\sqrt{k^2w^2-f_m^2}},
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{H}^{-}_{1,\Gamma,\kappa^-_0,\pm} = &\frac{\pi\mathrm{i}P_{1,\Gamma}}{2}\cdot\frac{\mathrm{e}^{\mathrm{i}(d_\phi\kappa^-_0+\sigma'_\phi)}-\mathrm{e}^{\pm\mathrm{i}s_\phi\zeta_{\kappa^-_0}}}{\cos(d_\phi\kappa^-_0+\sigma'_\phi)-\cos(s_\phi\zeta_{\kappa^-_0})} \\
&+\frac{\Gamma}{2}\mathscr{H}^{\pm\prime}_{0,\kappa^-_0}\frac{\mathrm{e}^{\mathrm{i}(d_\phi\kappa^-_0+\sigma'_\phi)}-\mathrm{e}^{\pm\mathrm{i}s_\phi\zeta_{\kappa^-_0}}}{\cos(d_\phi\kappa^-_0+\sigma'_\phi)-\cos(s_\phi\zeta_{\kappa^-_0})}.
\end{aligned}
\tag{S:7.6}
$$

In a similar way to the previous section, we combine the $h_{1,\Gamma}$ solution with the h_0 solution to write

$$
h_0(\phi,\psi)+\epsilon h_{1,\Gamma}(\phi,\psi) = \sum_{m=-\infty}^{\infty}\left(\mathscr{H}^-_{0,m}+\epsilon\mathscr{H}^-_{\Gamma,m}\right)\exp\left[-\mathrm{i}\left(\lambda^-_m\phi+\zeta^-_m\psi\right)\right] \tag{S:7.7}
$$

$$
-\sum_{m=-\infty}^{\infty}\epsilon\Gamma\mathscr{H}'_{0,m}\left(s_\phi(\phi-c_\phi)-d_\phi\psi\right)\frac{\zeta^-_m\exp\left[-\mathrm{i}\left(\lambda^-_m\phi+\zeta^-_m\psi\right)\right]}{s_\phi\Delta_\phi\sqrt{k^2w^2-f_m^2}} \tag{S:7.8}
$$

$$
-\sum_{m=-\infty}^{\infty}\epsilon\Gamma\mathscr{H}'_{0,m}\frac{\psi}{s_\phi}\exp\left[-\mathrm{i}\left(\lambda^-_m\phi+\zeta^-_m\psi\right)\right] \tag{S:7.9}
$$

$$
+\sum_{\pm}\left(\mathscr{H}^-_{0,\kappa^-_0,\pm}+\epsilon\mathscr{H}^-_{1,\Gamma,\kappa^-_0,\pm}\right)\mathrm{e}^{-\mathrm{i}(\kappa^-_0\phi\pm\zeta_{\kappa^-_0}\psi)}.
$$

Again, we note that the terms on line (S:7.8) are proportional to $\epsilon\phi$ and, accordingly, the asymptotic expansion (3.1) breaks down when $\phi=\mathscr{O}(\epsilon^{-1})$. We combine these resonant terms with the terms on line (S:7.7) and write

$$
\begin{aligned}
\tilde{h}_{0,\Gamma}(\phi,\psi) = &\sum_{m=-\infty}^{\infty}\left(\mathscr{H}^-_{0,m}+\epsilon\mathscr{H}^-_{\Gamma,m}\right)\left\{1-\mathrm{i}\epsilon\left(s_\phi(\phi-c_\phi)-d_\phi\psi\right)O_{\Gamma,m}\right\} \\
&\times\exp\left[-\mathrm{i}\left(\lambda^-_m\phi+\zeta^-_m\psi\right)\right],
\end{aligned}
\tag{S:7.10}
$$

where

$$O_{\Gamma,m} = -\frac{\Gamma\zeta_m^-(\lambda_m^- + kM_\infty^2\delta)}{s_\phi\Delta_\phi\sqrt{k^2w^2 - f_m^2}},$$

and we have ignored $\mathcal{O}(\epsilon^2)$ terms and used the definition (S:1.4.b). Similarly to the previous section, we may express (S:7.10) as

$$\tilde{h}_{0,\Gamma}(\phi,\psi) = \sum_{m=-\infty}^{\infty}\left(\mathscr{H}_{0,m}^- + \epsilon\mathscr{H}_{\Gamma,m}^-\right)\exp\left[-\mathrm{i}\left(\tilde{\lambda}_{\Gamma,m}^-\phi + \tilde{\zeta}_{\Gamma,m}^-\psi\right)\right]\exp[\mathrm{i}\epsilon s_\phi c_\phi O_{\Gamma,m}], \tag{S:7.11}$$

where we have introduced the modified modes

$$\tilde{\lambda}_{\Gamma,m}^- = \lambda_m^- + \epsilon s_\phi O_{\Gamma,m},$$
$$\tilde{\zeta}_{\Gamma,m}^- = \zeta_m^- - \epsilon d_\phi O_{\Gamma,m}.$$

When combined with the term on line (S:7.9), the ensuing function satisfies the homogeneous Helmholtz equation up to $\mathcal{O}(\phi\epsilon^2)$. Moreover, we have

$$\Delta_0\left[\tilde{h}_{0,\Gamma}\right](\phi) = 0,$$

and the jump in acoustic potential (3.35e) is obtained exclusively by the term on line (S:7.9).

S:7.3. *Summary*

In conclusion, we can resolve the resonance terms by modifying the terms proportional to $\mathrm{e}^{-\mathrm{i}(\lambda_m^-\phi+\zeta_m^-\psi)}$. In this context, we change the acoustic modes λ_m^- and ζ_m^- to

$$\tilde{\lambda}_m^- = \lambda_m^- + \epsilon s_\phi(O_{S,m} + O_{\Gamma,m}),$$
$$\tilde{\zeta}_m^- = \zeta_m^- - \epsilon d_\phi(O_{S,m} + O_{\Gamma,m}),$$

and rescale that part of the solution by a factor of $\exp\left[\mathrm{i}\epsilon s_\phi c_\phi(O_{S,m} + O_{\Gamma,m})\right]$.

S:8. Coefficients of acoustic potential function

In this section we define the coefficients of the acoustic potential function found in section 4. In this section we use the notation

$$D_{n,x}^{(i,j)} = D_{n,x}^{(i)} + D_{n,x}^{(j)},$$
$$D^{(i,j)} = D_0^{(i,j)} + \epsilon\left(D_{1,\Sigma}^{(i,j)} + D_{1,\Delta}^{(i,j)} + D_{1,\Gamma}^{(i,j)} + D_{1,S}^{(i,j)}\right).$$

In the upstream region, we define

$$\mathscr{H}_m^+ = \mathscr{H}_{0,m}^+ + \epsilon\left(\mathscr{H}_{\Sigma,m}^+ + \mathscr{H}_{\Delta,m}^+ + \mathscr{H}_{\Gamma,m}^+ + \mathscr{H}_{S,m}^+\right),$$

where

$$\mathscr{H}^{+}_{n,x,m} = \frac{\pi\zeta^{+}_{m}}{\Delta_\phi\sqrt{k^2w^2-f_m^2}}D^{(1,3)}_{n,x}(\lambda^{+}_{m}), \qquad \text{for } n=0 \text{ and } n=1,\ x=\Sigma,\Gamma,$$

$$\mathscr{H}^{+}_{\Delta,m} = \frac{\pi\zeta^{+}_{m}}{\Delta_\phi\sqrt{k^2w^2-f_m^2}}D^{(1,3)}_{1,\Delta}(\lambda^{+}_{m}) + \frac{1}{\Delta_\phi\sqrt{k^2w^2-f_m^2}}\sum_{l=-\infty}^{\infty}\frac{\mathscr{c}_{\Delta,l}}{2(\lambda^{+}_{m}-\kappa^{-}_{l})},$$

$$\mathscr{H}^{+}_{S,m} = \frac{\pi\zeta^{+}_{m}}{\Delta_\phi\sqrt{k^2w^2-f_m^2}}D^{(1,3)}_{1,S}(\lambda^{+}_{m}) - \frac{4\pi^2}{\zeta^{+}_{m}}\sum_{l_1,l_2}\frac{\mathbb{S}^{i}_{l_1,l_2}\mathscr{F}^{r,+}_{l_2,m}}{\lambda^{+}_{m}-\kappa^{-}_{l_1}}$$
$$+\sum_{\tilde{M}^{+}_{m}}\frac{s_\phi}{2\Delta_\phi\sqrt{k^2w^2-f_m^2}(\tilde{M}^{+}_{m}-\lambda^{+}_{m})}.$$

In the downstream region, we define

$$\mathscr{H}^{-}_{m} = \left\{\mathscr{H}^{-}_{0,m} + \epsilon\left(\mathscr{H}^{-}_{\Sigma,m}+\mathscr{H}^{-}_{\Delta,m}+\mathscr{H}^{-}_{\Gamma,m}+\mathscr{H}^{-}_{S,m}\right)\right\}\exp\left[\mathrm{i}\epsilon s_\phi c_\phi(O_{S,m}+O_{\Gamma,m})\right],$$

where

$$\mathscr{H}^{-}_{n,x,m} = -\frac{\pi\zeta^{-}_{m}}{\Delta_\phi\sqrt{k^2w^2-f_m^2}}D^{(2,4)}_{n,x}(\lambda^{-}_{m}), \qquad \text{for } n=0 \text{ and } n=1,\ x=\Sigma,$$

$$\mathscr{H}^{-}_{\Delta,m} = -\frac{\pi\zeta^{-}_{m}}{\Delta_\phi\sqrt{k^2w^2-f_m^2}}D^{(2,4)}_{1,\Delta}(\lambda^{-}_{m}) - \frac{1}{\Delta_\phi\sqrt{k^2w^2-f_m^2}}\sum_{l=-\infty}^{\infty}\frac{\mathscr{c}_{\Delta,l}\mathrm{e}^{\mathrm{i}(\lambda^{-}_{m}-\kappa^{-}_{l})c_\phi}}{2(\lambda^{+}_{m}-\kappa^{-}_{l})},$$

and $\mathscr{H}^{-}_{S,m}$ and $\mathscr{H}^{-}_{\Gamma,m}$ are defined in section S:7.

The coefficients responsible for enforcing the boundary conditions on the blades are given by

$$\mathscr{H}_{\theta^+,b,m} = \pi\mathrm{i}\mathscr{B}_n\cdot\frac{\mathrm{e}^{\mathrm{i}(d_\phi\theta^{+}_{n}+\sigma'_\phi)}}{\cos(d_\phi\theta^{+}_{n}+\sigma'_\phi)-(-1)^n},$$

$$\mathscr{H}_{\theta^-,a,m} = -\pi\frac{\mathscr{A}_n+\mathscr{C}_n}{\theta^{-}_{n}-\kappa^{-}_{0}}\cdot\frac{(-1)^{n+1}}{\cos(d_\phi\theta^{-}_{n}+\sigma'_\phi)-(-1)^n},$$

$$\mathscr{H}_{\theta^-,b,m} = -\pi\frac{\mathscr{A}_n+\mathscr{C}_n}{\theta^{-}_{n}-\kappa^{-}_{0}}\cdot\frac{\mathrm{e}^{\mathrm{i}(d_\phi\theta^{-}_{n}+\sigma'_\phi)}}{\cos(d_\phi\theta^{-}_{n}+\sigma'_\phi)-(-1)^n},$$

$$\mathscr{H}_{\Sigma,a,l} = -\pi\mathrm{i}\frac{\mathscr{T}_l}{J_+(\kappa^{-}_{l})}\cdot\frac{-1}{\cos(d_\phi\kappa^{-}_{l}+\sigma'_\phi)-\cos\left(s_\phi\zeta_{\kappa^{-}_{l}}\right)},$$

$$\mathscr{H}_{\Sigma,b,l} = -\pi\mathrm{i}\frac{\mathscr{T}_l}{J_+(\kappa^{-}_{l})}\cdot\frac{\mathrm{e}^{\mathrm{i}(d_\phi\kappa^{-}_{l}+\sigma'_\phi)}}{\cos(d_\phi\kappa^{-}_{l}+\sigma'_\phi)-\cos\left(s_\phi\zeta_{\kappa^{-}_{l}}\right)},$$

$$\mathscr{H}_{\Delta,a,l} = \frac{-\mathscr{c}_{\Delta,l}}{2\zeta_{\kappa^{-}_{l}}\left(\cos(d_\phi\kappa^{-}_{l}+\sigma'_\phi)-\cos\left(s_\phi\zeta_{\kappa^{-}_{l}}\right)\right)},$$

$$\mathscr{H}_{\Delta,b,l} = \frac{\mathscr{c}_{\Delta,l}\mathrm{e}^{\mathrm{i}(d_\phi\kappa^{-}_{l}+\sigma'_\phi)}}{2\zeta_{\kappa^{-}_{l}}\left(\cos(d_\phi\kappa^{-}_{l}+\sigma'_\phi)-\cos\left(s_\phi\zeta_{\kappa^{-}_{l}}\right)\right)},$$

where

$$\mathscr{T}_l = \mathscr{T}_{0,l} + \epsilon\left(\mathscr{T}_{1,\Sigma,l}+\mathscr{T}_{1,\Delta,l}+\mathscr{T}_{1,\Gamma,l}+\mathscr{T}_{1,S,l}\right),$$
$$\mathscr{A}_n = \mathscr{A}_{0,n} + \epsilon\left(\mathscr{A}_{1,\Sigma,n}+\mathscr{A}_{1,\Delta,n}+\mathscr{A}_{1,\Gamma,n}+\mathscr{A}_{1,S,n}\right),$$

$$\mathscr{B}_n = \mathscr{B}_{0,n} + \epsilon\left(\mathscr{B}_{1,\Sigma,n} + \mathscr{B}_{1,\Delta,n} + \mathscr{B}_{1,\Gamma,n} + \mathscr{B}_{1,S,n}\right),$$
$$\mathscr{C}_n = \mathscr{C}_{0,n} + \epsilon\left(\mathscr{C}_{1,\Sigma,n} + \mathscr{C}_{1,\Delta,n} + \mathscr{C}_{1,\Gamma,n} + \mathscr{C}_{1,S,n}\right).$$

The coefficients responsible for enforcing the Kutta condition in the inter-blade downstream region are

$$\mathscr{H}^-_{\kappa_0^-,b} = \mathscr{H}^-_{0,\kappa_0^-,b} + \epsilon\left(\mathscr{H}^-_{1,\Sigma,\kappa_0^-,b} + \mathscr{H}^-_{1,\Delta,\kappa_0^-,b} + \mathscr{H}^-_{\Gamma,\kappa_0^-,\pm,b} + \mathscr{H}^-_{1,S,\kappa_0^-,b}\right),$$

where

$$\mathscr{H}^-_{n,x,\kappa_0^-,b} = -\pi \mathrm{i} P_{n,x} \cdot \frac{\cos(s_\phi \zeta_{\kappa_0^-})}{\cos(d_\phi \kappa_0^- + \sigma'_\phi) - \cos(s_\phi \zeta_{\kappa_0^-})}.$$

The coefficients responsible for enforcing the Kutta condition in the downstream region are

$$\mathscr{H}^-_{\kappa_0^-,\pm} = \mathscr{H}^-_{0,\kappa_0^-,\pm} + \epsilon\left(\mathscr{H}^-_{1,\Sigma,\kappa_0^-,\pm} + \mathscr{H}^-_{1,\Delta,\kappa_0^-,\pm} + \mathscr{H}^-_{1,\Gamma,\kappa_0^-,\pm} + \mathscr{H}^-_{1,S,\kappa_0^-,\pm}\right),$$

where

$$\mathscr{H}^-_{n,x,\kappa_0^-,\pm} = \frac{\pi \mathrm{i} P_{n,x}}{2} \cdot \frac{\mathrm{e}^{\mathrm{i}(d_\phi \kappa_0^- + \sigma'_\phi)} - \mathrm{e}^{\pm \mathrm{i} s_\phi \zeta_{\kappa_0^-}}}{\cos(d_\phi \kappa_0^- + \sigma'_\phi) - \cos(s_\phi \zeta_{\kappa_0^-})}, \quad \text{for } n = 0 \text{ and } n = 1, x = \Sigma,\ \Delta,$$

and $\mathscr{H}^-_{1,\Gamma,\kappa_0^-,\pm}$ and $\mathscr{H}^-_{1,S,\kappa_0^-,\pm}$ are defined in (S:7.6) and (S:7.2).

Further coefficients are defined as

$$\mathscr{H}_{S,a,l} = -\sum_{l_2=-\infty}^{\infty} \frac{s_\phi^2 \epsilon \mathbb{S}^i_{l,l_2}}{\Delta_\phi^2\left(\lambda^+_{l_2/2} - \kappa_l^-\right)\left(\lambda^-_{l_2/2} - \kappa_l^-\right)} \times \frac{1-(-1)^{l_2}}{2\left(\cos(s_\phi \zeta_{\kappa_l^-}) - \cos(d_\phi \kappa_l^- + \sigma'_\phi)\right)},$$

$$\mathscr{H}_{S,b,l} = \sum_{l_2=-\infty}^{\infty} \frac{s_\phi^2 \epsilon \mathbb{S}^i_{l,l_2} \mathrm{e}^{\mathrm{i}(d_\phi \kappa_l^- + \sigma'_\phi)}}{\Delta_\phi^2\left(\lambda^+_{l_2/2} - \kappa_l^-\right)\left(\lambda^-_{l_2/2} - \kappa_l^-\right)} \times \frac{1-(-1)^{l_2}}{2\left(\cos(s_\phi \zeta_{\kappa_l^-}) - \cos(d_\phi \kappa_l^- + \sigma'_\phi)\right)},$$

$$\mathscr{H}_{S,c,l} = \sum_{l_2=-\infty}^{\infty} \frac{s_\phi^2 \epsilon \mathbb{S}^i_{l,l_2}}{\Delta_\phi^2\left(\lambda^+_{l_2/2} - \kappa_l^-\right)\left(\lambda^-_{l_2/2} - \kappa_l^-\right)} \times \frac{1-(-1)^{l_2}}{2\left(\cos(s_\phi \zeta_{\kappa_l^-}) - \cos(d_\phi \kappa_l^- + \sigma'_\phi)\right)} \cdot \frac{\mathrm{i}(\pi l_2 - d_\phi \kappa_l^- - \sigma'_\phi)}{s_\phi \zeta_{\kappa_l^-}},$$

$$\mathscr{H}_{S,d,l} = -\sum_{l_2=-\infty}^{\infty} \frac{s_\phi^2 \epsilon \mathbb{S}^i_{l,l_2} \mathrm{e}^{\mathrm{i}(d_\phi \kappa_l^- + \sigma'_\phi)}}{\Delta_\phi^2\left(\lambda^+_{l_2/2} - \kappa_l^-\right)\left(\lambda^-_{l_2/2} - \kappa_l^-\right)} \times \frac{1-(-1)^{l_2}}{2\left(\cos(s_\phi \zeta_{\kappa_l^-}) - \cos(d_\phi \kappa_l^- + \sigma'_\phi)\right)} \cdot \frac{\mathrm{i}(\pi l_2 - d_\phi \kappa_l^- - \sigma'_\phi)}{s_\phi \zeta_{\kappa_l^-}},$$

$$\mathscr{H}^+_{\mathscr{U},b,m} = \pi^2 \frac{4\epsilon \mathscr{U}_{1,S,m} J_+(\lambda_m^+)}{(\lambda_m^+ - \kappa_0^-)\zeta_m^+ \sin(s_\phi \zeta_m^+)} \cdot \mathrm{e}^{\mathrm{i}\lambda_m^+ c_\phi},$$

$$\mathscr{H}^-_{\mathscr{R},a,m} = -\pi^2 \mathrm{i} \frac{4\epsilon \mathscr{R}_{1,S,m} J_-(\lambda^-_m)}{\zeta^-_m \sin(s_\phi \zeta^-_m)} \cdot \mathrm{e}^{-\mathrm{i} s_\phi \zeta^-_m},$$

$$\mathscr{H}^-_{G,a,m} = \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i} s_\phi \zeta^-_m}}{\Delta_\phi \sqrt{k^2 w^2 - f_m^2}} \cdot \sum_{l=-\infty}^{\infty} \frac{\epsilon \mathscr{C}_{\Delta,l}}{2(\lambda^-_m - \kappa^-_l)},$$

$$\mathscr{H}^+_{G,a,m} = \frac{\mathrm{e}^{\mathrm{i} s_\phi \zeta^+_m}}{\Delta_\phi \sqrt{k^2 w^2 - f_m^2}} \cdot \frac{1}{\sin(s_\phi \zeta^+_m)} \cdot \sum_{l=-\infty}^{\infty} \frac{\mathscr{C}_{\Delta,l}}{2(\lambda^+_m - \kappa^-_l)},$$

$$\begin{aligned}
\mathscr{H}^-_{G,b,m} = {} & \frac{1}{\sin(s_\phi \zeta^-_m) \Delta_\phi \sqrt{k^2 w^2 - f_m^2}} \cdot \sum_{l=-\infty}^{\infty} \frac{\epsilon \mathscr{C}_{\Delta,l} \mathrm{e}^{\mathrm{i}(\lambda^-_m - \kappa^-_l) c_\phi}}{2(\lambda^-_m - \kappa^-_l)} \\
& + \frac{\mathrm{i}\epsilon \Gamma}{\Delta_\phi \sqrt{k^2 w^2 - f_m^2} \sin(\zeta^-_m s_\phi)} \\
& \times \left(\sum_{n \neq m} \zeta^-_n \mathscr{H}'_{0,n} \frac{\mathrm{e}^{\mathrm{i}(\lambda^-_m - \lambda^-_n) c_\phi}}{2(\lambda^-_m - \lambda^-_n)} + \sum_{\pm} \pm \zeta_{\kappa^-_0} \mathscr{H}^{\pm\prime}_{0,\kappa^-_0} \frac{\mathrm{e}^{\mathrm{i}(\lambda^-_m - \kappa^-_0) c_\phi}}{2(\lambda^-_m - \kappa^-_0)} \right) \\
& - \frac{\epsilon \Gamma \mathscr{H}'_{0,m}}{2\Delta_\phi \sqrt{k^2 w^2 - f_m^2} \sin(\zeta^-_m s_\phi)} \left(c_\phi \zeta^-_m - \mathrm{i} \frac{\lambda^-_m}{\zeta^-_m} \right),
\end{aligned}$$

$$\begin{aligned}
\mathscr{H}^+_{G,b,m} = {} & \frac{-\mathrm{i}\mathrm{e}^{\mathrm{i} s_\phi \zeta^+_m}}{\sin(s_\phi \zeta^+_m) \Delta_\phi \sqrt{k^2 w^2 - f_m^2}} \cdot \sum_{l=-\infty}^{\infty} \frac{\epsilon \mathscr{C}_{\Delta,l} \mathrm{e}^{\mathrm{i}(\lambda^+_m - \kappa^-_l) c_\phi}}{2(\lambda^+_m - \kappa^-_l)} \\
& + \frac{\epsilon \Gamma \mathrm{e}^{\mathrm{i} s_\phi \zeta^+_m}}{\Delta_\phi \sqrt{k^2 w^2 - f_m^2} \sin(s_\phi \zeta^+_m)} \\
& \times \left(\sum_{n=-\infty}^{\infty} \zeta^-_n \mathscr{H}'_{0,n} \frac{\mathrm{e}^{\mathrm{i}(\lambda^+_m - \lambda^-_n) c_\phi}}{2(\lambda^+_m - \lambda^-_n)} + \sum_{\pm} \mp \zeta_{\kappa^-_0} \mathscr{H}^{\pm\prime}_{0,\kappa^-_0} \frac{\mathrm{e}^{\mathrm{i}(\lambda^+_m - \kappa^-_0) c_\phi}}{2(\lambda^+_m - \kappa^-_0)} \right),
\end{aligned}$$

$$\mathscr{H}^+_{a,m} = \frac{-\pi \mathrm{i}}{\Delta_\phi \sqrt{k^2 w^2 - f_m^2}} \cdot \frac{\zeta^+_m \mathrm{e}^{\mathrm{i} s_\phi \zeta^+_m}}{\sin(s_\phi \zeta^+_m)} D^{(1,3)}(\lambda^+_m),$$

$$\begin{aligned}
\mathscr{H}^-_{b,m} = {} & \pi \mathrm{i} \frac{\zeta^-_m}{\Delta_\phi \sqrt{k^2 w^2 - f_m^2}} \cdot \frac{D^{(2,4)}(\lambda^-_m)}{\sin(s_\phi \zeta^-_m)} \\
& + \frac{-\epsilon \Gamma \zeta^-_m}{\Delta_\phi \sqrt{k^2 w^2 - f_m^2} \sin(s_\phi \zeta^-_m)} \sum_{n \neq m} \mathscr{H}'_{0,n} \frac{\mathrm{e}^{\mathrm{i}(\lambda^-_m - \lambda^-_n) c_\phi}}{2(\lambda^-_m - \lambda^-_n)} \cdot \frac{\lambda^-_n - \kappa^-_0}{\lambda^-_m - \kappa^-_0} \cdot \frac{J_-(\lambda^-_n)}{J_-(\lambda^-_m)} \\
& + \frac{\epsilon \Gamma \zeta^-_m}{\Delta_\phi \sqrt{k^2 w^2 - f_m^2} \sin(s_\phi \zeta^-_m)} \frac{\mathscr{H}'_{0,m}}{2} \left(\frac{1}{\lambda^-_m - \kappa^-_0} + \frac{J'_-(\lambda^-_m)}{J_-(\lambda^-_m)} - \mathrm{i} c_\phi \right),
\end{aligned}$$

$$\begin{aligned}
\mathscr{H}^-_{m,b} = {} & \epsilon \Gamma \mathscr{H}'_{0,m} \mathrm{e}^{\mathrm{i} \zeta^-_m s_\phi} \\
& \times \frac{\left((s_\phi \lambda^-_m)^2 - (d_\phi \zeta^-_m)^2 \right) \cos(s_\phi \zeta^-_m) - \frac{s_\phi}{\zeta^-_m} \left((\zeta^-_m)^2 + (\lambda^-_m)^2 \right) \sin(s_\phi \zeta^-_m)}{4 \sin(s_\phi \zeta^-_m)^2 \Delta_\phi^2 (k^2 w^2 - f_m^2)},
\end{aligned}$$

$$\mathscr{H}^-_{\Gamma,\kappa^-_0,\pm,b} = \frac{-\epsilon \Gamma \mathscr{H}^{\pm\prime}_{0,\kappa^-_0} \mathrm{e}^{\pm \mathrm{i} \zeta_{\kappa^-_0} s_\phi}}{2 \left(\cos(d_\phi \kappa^-_0 + \sigma'_\phi) - \cos\left(s_\phi \zeta_{\kappa^-_0} \right) \right)},$$

$$\mathscr{H}^-_{\Gamma,\lambda^-_m} = -\mathscr{H}'_{0,m} \frac{\epsilon \Gamma}{s_\phi} \exp\left[\mathrm{i} \epsilon s_\phi c_\phi (O_{S,m} + O_{\Gamma,m}) \right],$$

$$\mathscr{H}^{-}_{\Gamma,m,b} = \frac{-\epsilon\Gamma\mathscr{H}'_{0,m}\mathrm{i}}{2\sin(s_\phi\zeta^-_m)\Delta_\phi\sqrt{k^2w^2-f_m^2}}\mathrm{e}^{\mathrm{i}\zeta^-_m s_\phi},$$

$$\mathscr{H}^{-}_{\Gamma,\lambda^-_m} = -\mathscr{H}'_{0,m}\frac{\epsilon\Gamma}{s_\phi}\exp\left[\mathrm{i}\epsilon s_\phi c_\phi(O_{S,m}+O_{\Gamma,m})\right].$$

The inhomogeneous functions in the upstream and inter-blade regions are

$$\mathbb{I}^{+}_{P,M^+_m}(\phi,\psi) = -\frac{s_\phi^2}{\Delta_\phi^2}\cdot\frac{\exp\left[-\mathrm{i}\left(\phi\tilde{M}^+_m - \frac{\psi}{s_\phi}(\Delta_\phi f_m + d_\phi\tilde{M}^+_m)\right)\right]}{(\tilde{M}^+_m-\lambda^+_m)(\tilde{M}^+_m-\lambda^-_m)},$$

$$\mathbb{I}^{i}_{l_1,l_2}(\phi,\psi) = -\frac{s_\phi^2}{\Delta_\phi^2}\cdot\frac{\exp\left[-\mathrm{i}\left(\kappa^-_{l_1}\phi - \frac{\psi}{s_\phi}\left(\Delta_\phi f_{l_2/2} + d_\phi\kappa^-_{l_1}\right)\right)\right]}{(\kappa^-_{l_1}-\lambda^+_{l_2/2})(\kappa^-_{l_1}-\lambda^-_{l_2/2})}.$$

Finally, the inhomogeneous functions in the downstream region are defined in section S:7.

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