Filament mechanics in a half-space via regularised Stokeslet segments - Supplementary Material

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Following the notation of the main text, we present the explicit matricial form of the velocity contribution at $\tilde{\boldsymbol{x}}$ from the j^{th} linear segment of the filament in terms of the force densities $\boldsymbol{f}_j$ and $\boldsymbol{f}_{j+1}$ at the segment endpoints $\boldsymbol{x}_j$ and $\boldsymbol{x}_{j+1}$ respectively. Defining this velocity contribution as $\boldsymbol{u}^{[j]}(\tilde{\boldsymbol{x}})$, explicitly we have the linear relation

$$\boldsymbol{u}^{[j]}(\tilde{\boldsymbol{x}}) = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \end{bmatrix} \begin{bmatrix} f_{j,1} \\ f_{j,2} \\ f_{j+1,1} \\ f_{j+1,2} \end{bmatrix}, \quad j \in \{1 \ldots, N\}, \tag{1}$$

where we have written $\boldsymbol{f}_j = f_{j,1}\boldsymbol{e}_x + f_{j,2}\boldsymbol{e}_y$. We restrict ourselves to a filament moving in a plane that is perpendicular to an infinite planar boundary at $y = 0$, with filaments moving parallel to a plane being dealt with similarly. Additionally, we present only the contributions from regularised image singularities, with the matricial form of those pertaining to the original regularised Stokeslet being given by Cortez (2018). We define the image points $\hat{\boldsymbol{x}}_j$ and $\hat{\boldsymbol{x}}_{j+1}$ as the reflections of $\boldsymbol{x}_j$ and $\boldsymbol{x}_{j+1}$ in the plane $y = 0$, additionally defining the vectors $\boldsymbol{Y} = \tilde{\boldsymbol{x}} - \hat{\boldsymbol{x}}_j$ and $\boldsymbol{v} = \hat{\boldsymbol{x}}_j - \hat{\boldsymbol{x}}_{j+1}$ where the segment label, j, is suppressed for $\boldsymbol{Y}$ and $\boldsymbol{v}$.

As noted in the main text, each entry $M_{\alpha\beta}$ is a linear combination of integrals $T_{m,p}$ computed along the segment. For convenience we form the matrix $\boldsymbol{T}$ with entries

$$\boldsymbol{T} = \begin{bmatrix} T_{4,5} & 0 & 0 \\ T_{3,5} & T_{3,3} & 0 \\ T_{2,5} & T_{2,3} & 0 \\ T_{1,5} & T_{1,3} & T_{1,1} \\ T_{0,5} & T_{0,3} & T_{0,1} \end{bmatrix}, \tag{2}$$

and we write

$$M_{\alpha\beta} = \langle \boldsymbol{N}_{\alpha\beta}, \boldsymbol{T} \rangle_F, \tag{3}$$

where $\boldsymbol{N}_{\alpha\beta}$ is a family of 5×3 auxiliary matrices indexed by $\alpha \in \{1,2\}$ and $\beta \in \{1,2,3,4\}$. Here $\langle \cdot, \cdot \rangle_F$ denotes the Frobenius inner product, equivalent in this case to element-wise multiplication and summation over all elements. Below we provide the auxiliary matrices $\boldsymbol{N}_{\alpha\beta}$ in terms of $\boldsymbol{v} = v_1\boldsymbol{e}_x + v_2\boldsymbol{e}_y$, $\boldsymbol{Y} = Y_1\boldsymbol{e}_x + Y_2\boldsymbol{e}_y$, the regularisation parameter ϵ, and the signed distance h of the image point to the boundary, given by $h = \hat{\boldsymbol{x}}_j \cdot \boldsymbol{e}_y$ and negative for filament motion in $y > 0$.

As determined by symbolic algebra, we have

$$\boldsymbol{N}_{11} =$$

$$\begin{bmatrix} -6v_1^2 v_2(h+Y_2) & 0 & 0 \\ 6v_1(h+Y_2)(v_1v_2 + v_1h - 2v_2Y_1) & v_1^2 & 0 \\ -6(h+Y_2)(\epsilon^2 v_2 + v_1^2 h + v_2 Y_1^2 - 2v_1v_2Y_1 - 2v_1hY_1) & -v_1^2 + 2Y_1v_1 + 2v_2h + 2v_2Y_2 & 0 \\ 6(h+Y_2)(\epsilon^2 v_2 + \epsilon^2 h + v_2Y_1^2 + hY_1^2 - 2v_1hY_1) & \epsilon^2 - 2v_1Y_1 - 2v_2Y_2 - 2hY_2 - 2v_2h - 2h^2 + Y_1^2 & 1 \\ -6h(h+Y_2)(\epsilon^2 + Y_1^2) & -\epsilon^2 + 2h^2 + 2Y_2h - Y_1^2 & -1 \end{bmatrix} \tag{4}$$

$$\boldsymbol{N}_{12} =$$

$$\begin{bmatrix} 6v_1v_2^2(h+Y_2) & 0 & 0 \\ -6v_2(h+Y_2)(v_1v_2 + v_1h - v_1Y_2 - v_2Y_1) & -v_1v_2 & 0 \\ -6(h+Y_2)(v_2^2Y_1 - v_1v_2h + v_1v_2Y_2 + v_1hY_2 + v_2hY_1 - v_2Y_1Y_2) & v_1v_2 + 2v_1h + v_1Y_2 - v_2Y_1 & 0 \\ 6(h+Y_2)(v_1hY_2 + v_2hY_1 - v_2Y_1Y_2 - hY_1Y_2) & v_2Y_1 - v_1Y_2 - 2v_1h + 2hY_1 + Y_1Y_2 & 0 \\ 6hY_1Y_2(h+Y_2) & -Y_1(2h+Y_2) & 0 \end{bmatrix} \tag{5}$$

$$\boldsymbol{N}_{13} = \begin{bmatrix} 6v_1^2 v_2(h+Y_2) & 0 & 0 \\ -6v_1(h+Y_2)(v_1 h - 2v_2 Y_1) & -v_1^2 & 0 \\ 6(h+Y_2)(v_2\epsilon^2 + v_2 Y_1^2 - 2v_1 h Y_1) & -2v_2 h - 2v_1 Y_1 - 2v_2 Y_2 & 0 \\ -6h(h+Y_2)(\epsilon^2 + Y_1^2) & -\epsilon^2 + 2h^2 + 2Y_2 h - Y_1^2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

$$\boldsymbol{N}_{14} = \begin{bmatrix} -6v_1 v_2^2(h+Y_2) & 0 & 0 \\ -6v_2(h+Y_2)(v_1 Y_2 - v_1 h + v_2 Y_1) & v_1 v_2 & 0 \\ 6(h+Y_2)(v_1 h Y_2 + v_2 h Y_1 - v_2 Y_1 Y_2) & v_2 Y_1 - v_1 Y_2 - 2v_1 h & 0 \\ 6hY_1 Y_2(h+Y_2) & -Y_1(2h+Y_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

$$\boldsymbol{N}_{21} = \begin{bmatrix} -6v_1 v_2^2(h+Y_2) & 0 & 0 \\ 6v_2(h+Y_2)(v_1 v_2 + v_1 h - v_1 Y_2 - v_2 Y_1) & -v_1 v_2 & 0 \\ 6(h+Y_2)(v_2^2 Y_1 - v_1 v_2 h + v_1 v_2 Y_2 + v_1 h Y_2 + v_2 h Y_1 - v_2 Y_1 Y_2) & v_1 v_2 + 2v_1 h + v_1 Y_2 - v_2 Y_1 & 0 \\ -6(h+Y_2)(v_1 h Y_2 + v_2 h Y_1 - v_2 Y_1 Y_2 - hY_1 Y_2) & v_2 Y_1 - v_1 Y_2 - 2v_1 h + 2hY_1 + Y_1 Y_2 & 0 \\ -6hY_1 Y_2(h+Y_2) & -Y_1(2h+Y_2) & 0 \end{bmatrix} \tag{8}$$

$$\boldsymbol{N}_{22} = \begin{bmatrix} 6v_2^3(h+Y_2) & 0 & 0 \\ -6v_2^2(h+Y_2)(v_2 + h - 2Y_2) & v_2^2 & 0 \\ 6v_2(h+Y_2)(v_2 h - 2v_2 Y_2 - 2hY_2 + \epsilon^2 + Y_2^2) & -v_2^2 - 2hv_2 & 0 \\ -6(h+Y_2)(\epsilon^2 v_2 + \epsilon^2 h + v_2 Y_2^2 + hY_2^2 - 2v_2 h Y_2) & \epsilon^2 + 2h^2 + 2hY_2 + 2v_2 h + Y_2^2 & 1 \\ 6h(h+Y_2)(\epsilon^2 + Y_2^2) & -\epsilon^2 - 2h^2 - 2hY_2 - Y_2^2 & -1 \end{bmatrix} \tag{9}$$

$$\boldsymbol{N}_{23} = \begin{bmatrix} 6v_1 v_2^2(h+Y_2) & 0 & 0 \\ 6v_2(h+Y_2)(v_1 Y_2 - v_1 h + v_2 Y_1) & v_1 v_2 & 0 \\ -6(h+Y_2)(v_1 h Y_2 + v_2 h Y_1 - v_2 Y_1 Y_2) & v_2 Y_1 - v_1 Y_2 - 2v_1 h & 0 \\ -6hY_1 Y_2(h+Y_2) & -Y_1(2h+Y_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

$$\boldsymbol{N}_{24} = \begin{bmatrix} -6v_2^3(h+Y_2) & 0 & 0 \\ -6v_2^2(-h^2 + hY_2 + 2Y_2^2) & -v_2^2 & 0 \\ -6v_2(h+Y_2)(\epsilon^2 + Y_2^2 - 2hY_2) & 2v_2 h & 0 \\ 6h(h+Y_2)(\epsilon^2 + Y_2^2) & -\epsilon^2 - 2h^2 - 2hY_2 - Y_2^2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

References

Cortez, Ricardo 2018 Regularized Stokeslet segments. *Journal of Computational Physics* **375**, 783–796.