Supplementary Material on

"The effect of uniform electric field on the cross-stream migration of a drop in plane Poiseuille flow"

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A. Method to obtain the unknown solid harmonics present in $O\left(Re_E\right)$ flow field

At $O\left(Re_E\right)$, the velocity and pressure fields are expressed in terms of the following solid harmonics using the Lamb solution: $p_n^{(Re_E)}$, $\Phi_n^{(Re_E)}$, $\chi_n^{(Re_E)}$, $p_{-n-1}^{(Re_E)}$, $\Phi_{-n-1}^{(Re_E)}$ and $\chi_{-n-1}^{(Re_E)}$. These solid harmonics are contain the following unknown coefficients: $A_n^{(Re_E)}$, $B_n^{(Re_E)}$, $C_n^{(Re_E)}$, $A_{-n-1}^{(Re_E)}$, $B_{-n-1}^{(Re_E)}$, $C_{-n-1}^{(Re_E)}$, $\hat{A}_n^{(Re_E)}$, $\hat{B}_n^{(Re_E)}$, $\hat{C}_n^{(Re_E)}$, $\hat{A}_{-n-1}^{(Re_E)}$, $\hat{B}_{-n-1}^{(Re_E)}$ and $\hat{C}_{-n-1}^{(Re_E)}$. These can be obtained by invoking the boundary conditions given in equation (2.14). However, to facilitate the application of orthogonality property of surface harmonics, we represent the boundary conditions at $O\left(Re_E\right)$ in the following form (Haber & Hetsroni 1971; Happel & Brenner 1981; Brenner 1964; Bandopadhyay et al. 2016)

$$\left.\begin{aligned}
\left[\mathbf{u}_i^{(Re_E)}\cdot\mathbf{r}\right]\Big|_{r=1} &= 0,\\
\left[\mathbf{u}_e^{(Re_E)}\cdot\mathbf{r}\right]\Big|_{r=1} &= 0,\\
\left[r\frac{\partial}{\partial r}\left(\mathbf{u}_i^{(Re_E)}\cdot\mathbf{e}_r\right)\right]\Big|_{r=1} &= \left[r\frac{\partial}{\partial r}\left(\mathbf{u}_e^{(Re_E)}\cdot\mathbf{e}_r\right)\right]\Big|_{r=1},\\
\left[\mathbf{r}\cdot\nabla\times\mathbf{u}_i^{(Re_E)}\right]\Big|_{r=1} &= \left[\mathbf{r}\cdot\nabla\times\mathbf{u}_e^{(Re_E)}\right]\Big|_{r=1},\\
\left[\mathbf{r}\cdot\nabla\times\left\{\mathbf{r}\times\left(\mathbf{T}_i^{H(Re_E)}+M\mathbf{T}_i^{E(Re_E)}\right)\right\}\right]\Big|_{r=1} &= \left[\mathbf{r}\cdot\nabla\times\left\{\mathbf{r}\times\left(\mathbf{T}_e^{H(Re_E)}+M\mathbf{T}_e^{E(Re_E)}\right)\right\}\right]\Big|_{r=1},\\
\left[\mathbf{r}\cdot\nabla\times\left(\mathbf{T}_i^{H(Re_E)}+M\mathbf{T}_i^{E(Re_E)}\right)\right]\Big|_{r=1} &= \left[\mathbf{r}\cdot\nabla\times\left(\mathbf{T}_e^{H(Re_E)}+M\mathbf{T}_e^{E(Re_E)}\right)\right]\Big|_{r=1},
\end{aligned}\right\} \tag{A1}$$

where the notation $\left[\xi\right]\big|_{r=1}$ represents the determination of the quantity ξ at $r=1$. The hydrodynamic traction vector at $O\left(Re_E\right)$ can be written as (Leal 2007)

$$\left. \begin{aligned} \mathbf{T}_i^{H(Re_E)} &= \left[-p_i^{(Re_E)}\mathbf{I} + \lambda\left\{ \nabla\mathbf{u}_i^{(Re_E)} + \left(\nabla\mathbf{u}_i^{(Re_E)}\right)^T \right\} \right]\cdot\mathbf{e}_r, \\ \mathbf{T}_e^{H(Re_E)} &= \left[-p_e^{(Re_E)}\mathbf{I} + \nabla\mathbf{u}_e^{(Re_E)} + \left(\nabla\mathbf{u}_e^{(Re_E)}\right)^T \right]\cdot\mathbf{e}_r, \end{aligned} \right\} \tag{A2}$$

where superscript T represents the transpose of respective tensors.

Before expressing the electric traction vector, we represent the electric field in spherical coordinate as $\mathbf{E} = E_r\mathbf{e}_r + E_\theta\mathbf{e}_\theta + E_\phi\mathbf{e}_\phi$. Now, the electrical traction vector inside and outside the drop $\left(\mathbf{T}_{i,e}^{E(Re_E)}\right)$ at $O(Re_E)$ are given as (Im & Kang 2003)

$$\left. \begin{aligned} \mathbf{T}_i^{E(Re_E)} &= S\begin{bmatrix} E_{i,r}^{(0)}E_{i,r}^{(Re_E)} - \dfrac{1}{2}\left(2E_{i,r}^{(0)}E_{i,r}^{(Re_E)} + 2E_{i,\theta}^{(0)}E_{i,\theta}^{(Re_E)} + 2E_{i,\phi}^{(0)}E_{i,\phi}^{(Re_E)}\right) \\ \left(E_{i,r}^{(0)}E_{i,\theta}^{(Re_E)} + E_{i,r}^{(Re_E)}E_{i,\theta}^{(0)}\right) \\ \left(E_{i,r}^{(0)}E_{i,\phi}^{(Re_E)} + E_{i,r}^{(Re_E)}E_{i,\phi}^{(0)}\right) \end{bmatrix}, \\[2em] \mathbf{T}_e^{E(Re_E)} &= \begin{bmatrix} E_{e,r}^{(0)}E_{e,r}^{(Re_E)} - \dfrac{1}{2}\left(2E_{e,r}^{(0)}E_{e,r}^{(Re_E)} + 2E_{e,\theta}^{(0)}E_{e,\theta}^{(Re_E)} + 2E_{e,\phi}^{(0)}E_{e,\phi}^{(Re_E)}\right) \\ \left(E_{e,r}^{(0)}E_{e,\theta}^{(Re_E)} + E_{e,r}^{(Re_E)}E_{e,\theta}^{(0)}\right) \\ \left(E_{e,r}^{(0)}E_{e,\phi}^{(1)} + E_{e,r}^{(1)}E_{e,\phi}^{(0)}\right) \end{bmatrix}. \end{aligned} \right\} \tag{A3}$$

Our first task is to represent different expressions present in equation (A1) in terms of solid harmonics in the following form (Happel & Brenner 1981; Hetsroni & Haber 1970)

$$\left. \begin{aligned} \left[\mathbf{u}_i^{(Re_E)}\cdot\mathbf{r}\right]\Big|_{r=1} &= \sum_{n=1}^{\infty}\left[\frac{n}{2\lambda(2n+3)}p_n^{(Re_E)} + n\Phi_n^{(Re_E)}\right], \\ \left[\mathbf{u}_e^{(Re_E)}\cdot\mathbf{r}\right]\Big|_{r=1} &= \sum_{n=1}^{\infty}\left[\frac{n+1}{2(2n-3)}p_{-n-1}^{(Re_E)} - (n+1)\Phi_{-n-1}^{(Re_E)}\right], \\ \left[r\frac{\partial}{\partial r}\left(\mathbf{u}_i^{(Re_E)}\cdot\mathbf{e}_r\right)\right]\Big|_{r=1} &= -\sum_{n=1}^{\infty}\left[\frac{n(n+1)}{2\lambda(2n+3)}p_n^{(Re_E)} + n(n-1)\Phi_n^{(Re_E)}\right], \\ \left[r\frac{\partial}{\partial r}\left(\mathbf{u}_e^{(Re_E)}\cdot\mathbf{e}_r\right)\right]\Big|_{r=1} &= \sum_{n=1}^{\infty}\left[\frac{n(n+1)}{2(2n-1)}p_{-n-1}^{(Re_E)} - (n+1)(n+2)\Phi_{-n-1}^{(Re_E)}\right], \\ \left[\mathbf{r}\cdot\nabla\times\mathbf{u}_i^{(Re_E)}\right]\Big|_{r=1} &= \sum_{n=1}^{\infty}n(n+1)\chi_n^{(Re_E)}, \\ \left[\mathbf{r}\cdot\nabla\times\mathbf{u}_e^{(Re_E)}\right]\Big|_{r=1} &= \sum_{n=1}^{\infty}n(n+1)\chi_{-n-1}^{(Re_E)}. \end{aligned} \right\} \tag{A4}$$

Next we express $\left[\mathbf{r}\cdot\nabla\times\left\{\mathbf{r}\times\left(\mathbf{T}^{H(Re_E)}+M\mathbf{T}^{E(Re_E)}\right)\right\}\right]\Big|_{r=1}$ and $\left[\mathbf{r}\cdot\nabla\times\left(\mathbf{T}^{H(Re_E)}+M\mathbf{T}^{E(Re_E)}\right)\right]\Big|_{r=1}$ in the following form (Hetsroni & Haber 1970)

$$
\left.\begin{aligned}
\left[\mathbf{r}\cdot\nabla\times\left\{\mathbf{r}\times\mathbf{T}_i^{H(Re_E)}\right\}\right]\bigg|_{r=1} &= -\lambda\sum_{n=1}^{\infty}\left[2(n-1)n(n+1)\Phi_n^{(Re_E)}+\frac{n^2(n+2)}{\lambda(2n+3)}p_n^{(Re_E)}\right], \\
\left[\mathbf{r}\cdot\nabla\times\left\{\mathbf{r}\times\mathbf{T}_e^{H(Re_E)}\right\}\right]\bigg|_{r=1} &= -\sum_{n=1}^{\infty}\left[-2(n+2)n(n+1)\Phi_{-n-1}^{(Re_E)}+\frac{(n+1)^2(n-1)}{(2n-1)}p_{-n-1}^{(Re_E)}\right], \\
\left[\mathbf{r}\cdot\nabla\times\mathbf{T}_i^{H(Re_E)}\right]\bigg|_{r=1} &= \lambda\sum_{n=1}^{\infty}(n-1)n(n+1)\chi_n^{(Re_E)}, \\
\left[\mathbf{r}\cdot\nabla\times\mathbf{T}_e^{H(Re_E)}\right]\bigg|_{r=1} &= -\sum_{n=1}^{\infty}(n+2)n(n+1)\chi_{-n-1}^{(Re_E)}, \\
\left[\mathbf{r}\cdot\nabla\times\left(\mathbf{r}\times\mathbf{T}_i^{E(Re_E)}\right)\right]_{r=1} &= \sum_{n=0}^{\infty}\left[g_{n,m}^{i(Re_E)}\cos(m\phi)+\hat{g}_{n,m}^{i(Re_E)}\sin(m\phi)\right]P_{n,m}, \\
\left[\mathbf{r}\cdot\nabla\times\left(\mathbf{r}\times\mathbf{T}_e^{E(Re_E)}\right)\right]_{r=1} &= \sum_{n=0}^{\infty}\left[g_{n,m}^{e(Re_E)}\cos(m\phi)+\hat{g}_{n,m}^{e(Re_E)}\sin(m\phi)\right]P_{n,m}, \\
\left[\mathbf{r}\cdot\nabla\times\mathbf{T}_i^{E(Re_E)}\right]_{r=1} &= \sum_{n=0}^{\infty}\left[h_{n,m}^{e(Re_E)}\cos(m\phi)+\hat{h}_{n,m}^{e(Re_E)}\sin(m\phi)\right]P_{n,m}, \\
\left[\mathbf{r}\cdot\nabla\times\mathbf{T}_e^{E(Re_E)}\right]_{r=1} &= \sum_{n=0}^{\infty}\left[h_{n,m}^{e(Re_E)}\cos(m\phi)+\hat{h}_{n,m}^{e(Re_E)}\sin(m\phi)\right]P_{n,m},
\end{aligned}\right\} \quad (A5)
$$

where the terms $g_{n,m}^{i(Re_E)}$, $\hat{g}_{n,m}^{i(Re_E)}$, $h_{n,m}^{i(Re_E)}$, $\hat{h}_{n,m}^{i(Re_E)}$, $g_{n,m}^{e(Re_E)}$, $\hat{g}_{n,m}^{e(Re_E)}$, $h_{n,m}^{e(Re_E)}$ and $\hat{h}_{n,m}^{e(Re_E)}$ can be easily evaluated from the known electric potential at $O(Re_E)$. Next, we substitute the expressions from equations (A4) and (A5) in the equation (A1) and obtain the six independent linear algebraic equations. By solving those six equations, we obtain the coefficients $A_n^{(Re_E)}$, $B_n^{(Re_E)}$, $C_n^{(Re_E)}$, $A_{-n-1}^{(Re_E)}$, $B_{-n-1}^{(Re_E)}$ and $C_{-n-1}^{(Re_E)}$ in the following general form

$$A_{n,m}^{(Re_E)} = \frac{(2n+3)}{n(2n+1)(\lambda+1)}\left[(2n+1)(2n-1)\beta_{n,m}^{(Re_E)} + M\left(g_{n,m}^{i(Re_E)} - g_{n,m}^{e(Re_E)}\right)\right],$$

$$B_{n,m}^{(Re_E)} = -\frac{A_{n,m}^{(Re_E)}}{2(2n+3)},$$

$$C_{n,m}^{(Re_E)} = \frac{M\left(h_{n,m}^{e(Re_E)} - h_{n,m}^{i(Re_E)}\right)}{n(n+1)(\lambda(n-1)+(n+2))},$$

$$A_{-n-1,m}^{(Re_E)} = -\frac{\left[(4n^2-1)\{(2n+1)\lambda+2\}\beta_{n,m}^{(Re_E)} + M(1-2n)\left(g_{n,m}^{i(Re_E)} - g_{n,m}^{e(Re_E)}\right)\right]}{(n+1)(2n+1)(\lambda+1)},$$

$$B_{-n-1,m}^{(Re_E)} = -\frac{(4n^2-1)\lambda\beta_{n,m}^{(Re_E)} + M\left(g_{n,m}^{e(Re_E)} - g_{n,m}^{i(Re_E)}\right)}{2(2n+1)(n+1)(\lambda+1)},$$

$$C_{-n-1,m}^{(Re_E)} = C_{n,m}^{(Re_E)}.$$

(A6)

Rest of the coefficients $\hat{A}_{n,m}^{(Re_E)}, \hat{B}_{n,m}^{(Re_E)}, \hat{C}_{n,m}^{(Re_E)}$, $\hat{A}_{-n-1,m}^{(Re_E)}, \hat{B}_{-n-1,m}^{(Re_E)}$ and $\hat{C}_{-n-1,m}^{(Re_E)}$ are obtained by replacing $\beta_{n,m}^{(Re_E)}, g_{n,m}^{i(Re_E)}, h_{n,m}^{i(Re_E)}, g_{n,m}^{e(Re_E)}$ and $h_{n,m}^{e(Re_E)}$ by $\hat{\beta}_{n,m}^{(Re_E)}, \hat{g}_{n,m}^{i(Re_E)}, \hat{h}_{n,m}^{i(Re_E)}, \hat{g}_{n,m}^{e(Re_E)}$ and $\hat{h}_{n,m}^{e(Re_E)}$, respectively in equation (A6). $\beta_{n,m}^{(Re_E)}$ and $\hat{\beta}_{n,m}^{(Re_E)}$ present in equation (A6) can be expressed in terms of drop velocity if the following form

$$\beta_{n,m}^{(Re_E)} = \begin{cases} \beta_{1,0}^{(Re_E)} = -U_{dz}^{(Re_E)}, \beta_{1,1}^{(Re_E)} = -U_{dx}^{(Re_E)} \\ 0 \ \forall \ n \geq 2 \end{cases}$$

$$\hat{\beta}_{n,m}^{(Re_E)} = \begin{cases} \hat{\beta}_{1,1}^{(Re_E)} = -U_{dy}^{(Re_E)} \\ 0 \ \forall \ n \geq 2. \end{cases}$$

(A7)

B. Method to obtain the unknown solid harmonics present in $O(Ca)$ flow field

Determination of $O(Ca)$ velocity and pressure fields are much involved than the $O(Re_E)$ calculations. We first express the boundary conditions in the following form (Haber & Hetsroni 1971; Brenner 1964; Chaffey & Brenner 1967)

$$\left[\left. \mathbf{u}_i \right|_{r=1+Caf^{(Ca)}} \cdot \mathbf{n} \right]^{(Ca)} = 0,$$

$$\left[\left. \mathbf{u}_e \right|_{r=1+Caf^{(Ca)}} \cdot \mathbf{n} \right]^{(Ca)} = 0,$$

$$\left[\left. \nabla \cdot \mathbf{u}_i \right|_{r=1+Caf^{(Ca)}} \right]^{(Ca)} = \left[\left. \nabla \cdot \mathbf{u}_e \right|_{r=1+Caf^{(Ca)}} \right]^{(Ca)},$$

$$\left[\left. \mathbf{n} \cdot \nabla \times \mathbf{u}_i \right|_{r=1+Caf^{(Ca)}} \right]^{(Ca)} = \left[\left. \mathbf{n} \cdot \nabla \times \mathbf{u}_e \right|_{r=1+Caf^{(Ca)}} \right]^{(Ca)},$$

$$\left[\mathbf{n} \cdot \nabla \times \left\{ \mathbf{n} \times \left. \left(\mathbf{T}_i^H + M\mathbf{T}_i^E \right) \right|_{r=1+Caf^{(Ca)}} \right\} \right]^{(Ca)} = \left[\mathbf{n} \cdot \nabla \times \left\{ \mathbf{n} \times \left. \left(\mathbf{T}_e^H + M\mathbf{T}_e^E \right) \right|_{r=1+Caf^{(Ca)}} \right\} \right]^{(Ca)},$$

$$\left[\mathbf{n} \cdot \nabla \times \left. \left(\mathbf{T}_i^H + M\mathbf{T}_i^E \right) \right|_{r=1+Caf^{(Ca)}} \right]^{(Ca)} = \left[\mathbf{n} \cdot \nabla \times \left. \left(\mathbf{T}_e^H + M\mathbf{T}_e^E \right) \right|_{r=1+Caf^{(Ca)}} \right]^{(Ca)}, \qquad \text{(B1)}$$

where $\mathbf{n}$ represents the outward unit normal to the drop surface which can be obtained by solving leading order problem. In equation (B1), the notation $\left[\left. \xi \right|_{r=1+Caf^{(Ca)}} \right]^{(Ca)}$ is used to express the $O(Ca)$ contribution of the term ξ which is evaluated at the deformed drop surface $r=1+Caf^{(Ca)}$. Here $f^{(Ca)}$ represents the correction in drop shape from sphericity which can be obtained by solving leading order problem. To evaluate the $O(Ca)$ contribution of each term we use the Taylor series expansion about $r=1$ in the following form (Haber & Hetsroni 1971; Ajayi 1978; Xu & Homsy 2006)

$$\left. \xi \right|_{r=1+Caf^{(Ca)}} = \left. \xi^{(0)} \right|_{r=1} + Re_E \left. \xi^{(Re_E)} \right|_{r=1} + Ca \left(\left. \xi^{(Ca)} \right|_{r=1} + f^{(Ca)} \left. \frac{\partial \xi^{(0)}}{\partial r} \right|_{r=1} \right) + \cdots,$$

$$\Rightarrow \left[\left. \xi \right|_{r=1+Caf^{(Ca)}} \right]^{(Ca)} = \left(\left. \xi^{(Ca)} \right|_{r=1} + f^{(Ca)} \left. \frac{\partial \xi^{(0)}}{\partial r} \right|_{r=1} \right). \qquad \text{(B2)}$$

Using this method the boundary conditions in equation (B1) can be simplified to the following form (Haber & Hetsroni 1971; Brenner 1964)

$$\left.\left[\mathbf{u}_i^{(Ca)}\cdot\mathbf{r}\right]\right|_{r=1}=T_1,$$

$$\left.\left[\mathbf{u}_e^{(Ca)}\cdot\mathbf{r}\right]\right|_{r=1}=T_2,$$

$$\left.\left[r\frac{\partial}{\partial r}\left(\mathbf{u}_e^{(Ca)}\cdot\mathbf{e}_r\right)-r\frac{\partial}{\partial r}\left(\mathbf{u}_i^{(Ca)}\cdot\mathbf{e}_r\right)\right]\right|_{r=1}=T_3,$$

$$\left.\left[\mathbf{r}\cdot\nabla\times\left(\mathbf{u}_e^{(Ca)}-\mathbf{u}_i^{(Ca)}\right)\right]\right|_{r=1}=T_5,$$

$$\left.\left[\mathbf{r}\cdot\nabla\times\left(\mathbf{r}\times\left\{\left(\mathbf{T}_e^{H(Ca)}+M\mathbf{T}_e^{E(Ca)}\right)-\left(\mathbf{T}_i^{H(Ca)}+M\mathbf{T}_i^{E(Ca)}\right)\right\}\right)\right]\right|_{r=1}=T_4,$$

$$\left.\left[\mathbf{r}\cdot\nabla\times\left\{\left(\mathbf{T}_e^{H(Ca)}+M\mathbf{T}_e^{E(Ca)}\right)-\left(\mathbf{T}_i^{H(Ca)}+M\mathbf{T}_i^{E(Ca)}\right)\right\}\right]\right|_{r=1}=T_6,$$

$$\text{(B3)}$$

where the terms T_1-T_6 can be easily obtained using following relations (Haber & Hetsroni 1971)

$$T_1=\sum_{n=1}^{4}\left(T_{1,n,m}^{(Ca)}\cos m\phi+\hat{T}_{1,n,m}^{(Ca)}\sin m\phi\right)P_{n,m}=\left.\mathbf{u}_i^{(0)}\right|_{r=1}\cdot\tilde{\nabla}f^{(Ca)}-f^{(Ca)}\left.\left(\frac{\partial\mathbf{u}_i^{(0)}}{\partial r}\right)\right|_{r=1}\cdot\mathbf{e}_r,$$

$$T_2=\sum_{n=1}^{4}\left(T_{2,n,m}^{(Ca)}\cos m\phi+\hat{T}_{2,n,m}^{(Ca)}\sin m\phi\right)P_{n,m}=\left.\mathbf{u}_e^{(0)}\right|_{r=1}\cdot\tilde{\nabla}f^{(Ca)}-f^{(Ca)}\left.\left(\frac{\partial\mathbf{u}_e^{(0)}}{\partial r}\right)\right|_{r=1}\cdot\mathbf{e}_r,$$

$$T_3=\sum_{n=1}^{4}\left(T_{3,n,m}^{(Ca)}\cos m\phi+\hat{T}_{3,n,m}^{(Ca)}\sin m\phi\right)P_{n,m}=\nabla\cdot\left(f^{(Ca)}\left.\left(\frac{\partial\mathbf{u}_e^{(0)}}{\partial r}-\frac{\partial\mathbf{u}_i^{(0)}}{\partial r}\right)\right|_{r=1}\right),$$

$$T_4=\sum_{n=1}^{4}\left(T_{4,n,m}^{(Ca)}\cos m\phi+\hat{T}_{4,n,m}^{(Ca)}\sin m\phi\right)P_{n,m}=\begin{bmatrix}\mathbf{e}_r\cdot\nabla\times\left\{\mathbf{e}_r\times\left(f^{(Ca)}\left.\left(\frac{\partial\mathbf{T}_i^{(0)}}{\partial r}-\frac{\partial\mathbf{T}_e^{(0)}}{\partial r}\right)\right|_{r=1}\right)\right\}\\+\mathbf{e}_r\cdot\nabla\times\left\{\mathbf{e}_r\times\left(\boldsymbol{\tau}_e-\boldsymbol{\tau}_i\right)\right|_{r=1}\cdot\tilde{\nabla}f^{(Ca)}\right\}\\+\mathbf{e}_r\cdot\nabla\times\left\{\tilde{\nabla}f^{(Ca)}\times\left.\left(\mathbf{T}_e^{(0)}-\mathbf{T}_i^{(0)}\right)\right|_{r=1}\right\}\\+\tilde{\nabla}f^{(Ca)}\cdot\nabla\times\left\{\mathbf{e}_r\times\left.\left(\mathbf{T}_e^{(0)}-\mathbf{T}_i^{(0)}\right)\right|_{r=1}\right\}\end{bmatrix},$$

$$\text{(B4)}$$

$$T_5 = \sum_{n=1}^{4} \left(T_{5,n,m}^{(Ca)} \cos m\phi + \hat{T}_{5,n,m}^{(Ca)} \sin m\phi \right) P_{n,m} = \begin{bmatrix} \mathbf{e}_r \cdot \nabla \times \left(f^{(Ca)} \left(\dfrac{\partial \mathbf{u}_i^{(0)}}{\partial r} - \dfrac{\partial \mathbf{u}_e^{(0)}}{\partial r} \right) \Big|_{r=1} \right) \\[2mm] + \tilde{\nabla} f^{(Ca)} \cdot \nabla \times \left(\mathbf{u}_e^{(0)} - \mathbf{u}_i^{(0)} \right) \Big|_{r=1} \end{bmatrix},$$

$$T_6 = \sum_{n=1}^{4} \left(T_{6,n,m}^{(Ca)} \cos m\phi + \hat{T}_{6,n,m}^{(Ca)} \sin m\phi \right) P_{n,m} = \begin{bmatrix} \tilde{\nabla} f^{(Ca)} \cdot \nabla \times \left(\mathbf{T}_e^{(0)} - \mathbf{T}_i^{(0)} \right) \Big|_{r=1} \\[2mm] + \mathbf{e}_r \cdot \nabla \times \left(\left(\boldsymbol{\tau}_e - \boldsymbol{\tau}_i \right) \big|_{r=1} \cdot \tilde{\nabla} f^{(Ca)} \right) \\[2mm] - \mathbf{e}_r \cdot \nabla \times \left(f^{(Ca)} \left(\dfrac{\partial \mathbf{T}_e^{(0)}}{\partial r} - \dfrac{\partial \mathbf{T}_i^{(0)}}{\partial r} \right) \Big|_{r=1} \right) \end{bmatrix},$$

$$\tag{B5}$$

where the operator $\tilde{\nabla}$ is given by $\tilde{\nabla} \equiv \mathbf{e}_\theta \dfrac{\partial}{\partial \theta} + \mathbf{e}_\phi \dfrac{1}{\sin\theta} \dfrac{\partial}{\partial \phi}$. In equation (B5), the terms $\boldsymbol{\tau}_{i,e}^{(0)}$ and $\mathbf{T}_{i,e}^{(0)}$ represent the stress tensors and the traction vector, respectively which are of the form

$$\boldsymbol{\tau}_{i,e}^{(0)} = \boldsymbol{\tau}_{i,e}^{H(0)} + M \boldsymbol{\tau}_{i,e}^{E(0)}, \quad \mathbf{T}_{i,e}^{(0)} = \boldsymbol{\tau}_{i,e}^{(0)} \cdot \mathbf{e}_r. \tag{B6}$$

$\boldsymbol{\tau}_{i,e}^{H(0)}$ represents the hydrodynamic stress tensor and $\boldsymbol{\tau}_{i,e}^{E(0)}$ represents the Maxwell stress tensor at the leading order and are given by

$$\begin{aligned} \boldsymbol{\tau}_i^{H(0)} &= \left[-p_i^{(0)} \mathbf{I} + \lambda \left\{ \nabla \mathbf{u}_i^{(0)} + \left(\nabla \mathbf{u}_i^{(0)} \right)^T \right\} \right], \\[2mm] \boldsymbol{\tau}_e^{H(0)} &= \left[-p_e^{(0)} \mathbf{I} + \nabla \mathbf{u}_e^{(0)} + \left(\nabla \mathbf{u}_e^{(0)} \right)^T \right], \\[2mm] \boldsymbol{\tau}_i^{E(0)} &= S \left[\mathbf{E}_i^{(0)} \left(\mathbf{E}_i^{(0)} \right)^T - \frac{1}{2} \left| \mathbf{E}_i^{(0)} \right|^2 \mathbf{I} \right], \\[2mm] \boldsymbol{\tau}_e^{E(0)} &= \left[\mathbf{E}_e^{(0)} \left(\mathbf{E}_e^{(0)} \right)^T - \frac{1}{2} \left| \mathbf{E}_e^{(0)} \right|^2 \mathbf{I} \right]. \end{aligned} \tag{B7}$$

Next we represent the left side terms of equation (B3) in terms of solid spherical harmonics by following similar method as given in equations (A4)-(A5) by substituting the superscript Re_E by Ca. After that one can easily obtain the coefficients $A_n^{(Re_E)}$, $B_n^{(Re_E)}$, $C_n^{(Re_E)}$, $A_{-n-1}^{(Re_E)}$, $B_{-n-1}^{(Re_E)}$ and $C_{-n-1}^{(Re_E)}$ in the following general form

$$A_{n,m}^{(Ca)} = -\frac{(2n+3)\left[\begin{array}{l}-(4n^2-1)\beta_{n,m}^{(Ca)}+\{(n-1)((2n(\lambda+1))+(2\lambda+1))\}T_{1,n,m}\\+(n+2)T_{2,n,m}+(2n+1)T_{3,n,m}-T_{4,n,m}+M\left(g_{n,m}^{e(Ca)}-g_{n,m}^{i(Ca)}\right)\end{array}\right]}{n(2n+1)(\lambda+1)},$$

$$B_{n,m}^{(Ca)} = -\frac{A_{n,m}^{(Ca)}}{2(2n+3)}+\frac{T_{1,n,m}}{n},$$

$$C_{n,m}^{(Ca)} = \frac{M\left(h_{n,m}^{e(Ca)}-h_{n,m}^{i(Ca)}\right)-(n+2)T_{5,n,m}-T_{6,n,m}}{n(n+1)(\lambda(n-1)+(n+2))},$$

$$A_{-n-1,m}^{(Ca)} = -\frac{\left[\begin{array}{l}(4n^2-1)(2+\lambda(2n+1))\beta_{n,m}^{(Ca)}+\lambda(n-1)(2n-1)T_{1,n,m}\\-(n+2)(2n-1)(2n+\lambda(2n+1))T_{2,n,m}-\lambda(4n^2-1)T_{3,n,m}\\-(2n-1)T_{4,n,m}+M(2n-1)\left(g_{n,m}^{e(Ca)}-g_{n,m}^{i(Ca)}\right)\end{array}\right]}{(\lambda+1)(n+1)(2n+1)},$$

$$B_{-n-1,m}^{(Ca)} = -\frac{\left[\begin{array}{l}\lambda(4n^2-1)\beta_{n,m}^{(Ca)}+\lambda(n-1)T_{1,n,m}-\lambda(2n+1)T_{3,n,m}-T_{4,n,m}\\+\{2(1-n^2)-\lambda n(2n+1)\}T_{2,n,m}+M\left(g_{n,m}^{e(Ca)}-g_{n,m}^{i(Ca)}\right)\end{array}\right]}{2(\lambda+1)(2n+1)(n+1)},$$

$$C_{-n-1,m}^{(Ca)} = \frac{M\left(h_{n,m}^{e(Ca)}-h_{n,m}^{i(Ca)}\right)+\lambda(n-1)T_{5,n,m}-T_{6,n,m}}{n\{n^2+3n+2+\lambda(n^2-1)\}}.$$

(B8)

The determination of the coefficients $\hat{A}_{n,m}^{(Ca)}, \hat{B}_{n,m}^{(Ca)}, \hat{C}_{n,m}^{(Ca)}, \hat{A}_{-n-1,m}^{(Ca)}, \hat{B}_{-n-1,m}^{(Ca)}$ and $\hat{C}_{-n-1,m}^{(Ca)}$ requires the substitution of $\beta_{n,m}^{(Ca)}, g_{n,m}^{i(Ca)}, h_{n,m}^{i(Ca)}, g_{n,m}^{e(Ca)}, h_{n,m}^{e(Ca)}$ and $T_{1,n,m}-T_{6,n,m}$ by $\hat{\beta}_{n,m}^{(Ca)}, \hat{g}_{n,m}^{i(Ca)}, \hat{h}_{n,m}^{i(Ca)}, \hat{g}_{n,m}^{e(Ca)}$, $\hat{h}_{n,m}^{e(Ca)}$ and $\hat{T}_{1,n,m}-\hat{T}_{6,n,m}$, respectively in equation (B8). The terms $\beta_{n,m}^{(Ca)}$ and $\hat{\beta}_{n,m}^{(Ca)}$ present in equation (B8)c are given by

$$\left.\begin{array}{l}\beta_{n,m}^{(Ca)}=\begin{cases}\beta_{1,0}^{(Ca)}=-U_{dz}^{(Ca)}, \beta_{1,1}^{(Ca)}=-U_{dx}^{(Ca)}\\0 \ \forall \ n\geq 2\end{cases}\\[4mm]\hat{\beta}_{n,m}^{(Ca)}=\begin{cases}\hat{\beta}_{1,1}^{(Ca)}=-U_{dy}^{(Ca)}\\0 \ \forall \ n\geq 2.\end{cases}\end{array}\right\}$$

(B9)

The terms $g_{n,m}^{e(Ca)}, \hat{g}_{n,m}^{e(Ca)}, h_{n,m}^{e(Ca)}$ and $\hat{h}_{n,m}^{e(Ca)}$ which are present in equation (B8) are obtained from the $O(Ca)$ electric field as

$$\left.\begin{array}{l}\sum_{n=0}^{4}\left[g_{n,m}^{i(Ca)}\cos\left(m\phi\right)+\hat{g}_{n,m}^{i(Ca)}\sin\left(m\phi\right)\right]P_{n,m}=\left.\left(\mathbf{r}\cdot\nabla\times\left\{\mathbf{r}\times\mathbf{T}_{i}^{E(Ca)}\right\}\right)\right|_{r=1},\\[2mm]\sum_{n=0}^{4}\left[g_{n,m}^{e(Ca)}\cos\left(m\phi\right)+\hat{g}_{n,m}^{e(Ca)}\sin\left(m\phi\right)\right]P_{n,m}=\left.\left(\mathbf{r}\cdot\nabla\times\left\{\mathbf{r}\times\mathbf{T}_{e}^{E(Ca)}\right\}\right)\right|_{r=1},\\[2mm]\sum_{n=0}^{4}\left[h_{n,m}^{i(Ca)}\cos\left(m\phi\right)+\hat{h}_{n,m}^{i(Ca)}\sin\left(m\phi\right)\right]P_{n,m}=\left.\left(\mathbf{r}\cdot\nabla\times\mathbf{T}_{i}^{E(Ca)}\right)\right|_{r=1},\\[2mm]\sum_{n=0}^{4}\left[h_{n,m}^{e(Ca)}\cos\left(m\phi\right)+\hat{h}_{n,m}^{e(Ca)}\sin\left(m\phi\right)\right]P_{n,m}=\left.\left(\mathbf{r}\cdot\nabla\times\mathbf{T}_{e}^{E(Ca)}\right)\right|_{r=1},\end{array}\right\} \tag{B10}$$

where $\mathbf{T}_{i,e}^{E(Ca)}$ represents the $O(Ca)$ electric traction vector of the following form

$$\mathbf{T}_{i}^{E(Ca)}=S\begin{bmatrix}\left(E_{i,r}^{(Ca)}\right)^{2}-\dfrac{1}{2}\left|\mathbf{E}_{i}^{(Ca)}\right|^{2}\\[2mm]E_{i,r}^{(Ca)}E_{i,\theta}^{(Ca)}\\[2mm]E_{i,r}^{(Ca)}E_{i,\phi}^{(Ca)}\end{bmatrix},\ \mathbf{T}_{e}^{E(Ca)}=\begin{bmatrix}\left(E_{e,r}^{(Ca)}\right)^{2}-\dfrac{1}{2}\left|\mathbf{E}_{e}^{(Ca)}\right|^{2}\\[2mm]E_{e,r}^{(Ca)}E_{e,\theta}^{(Ca)}\\[2mm]E_{e,r}^{(Ca)}E_{e,\phi}^{(Ca)}\end{bmatrix}. \tag{B11}$$

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