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BOOK REVIEWS

Handbook of Mathematical Fluid Dynamics, Volume 2. Edited by S. FRIEDLANDER & D. SERRE. Elsevier. 2003, 614 pp. ISBN 0-444-51287-X. $135.00.

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This book is the second in a three-volume series of review articles written by experts in mathematical fluid dynamics. With the exception of the last article, the second volume is devoted to problems arising in incompressible flows.

The article by R. Robert 'Statistical Hydrodynamics (Onsager revisited)' is an overview of a very intriguing domain of fluid dynamics and statistical physics. It is well known that there exists a unique solution of the two-dimensional Euler equations with sufficiently regular initial condition (say, with bounded vorticity). However, we know practically nothing on the long-time behaviour of a typical solution. In this situation it is natural to apply the ideas of equilibrium statistical mechanics and to look for the most probable flows which are consistent with the initial conditions (say, have the same set of known integrals). The article is devoted to the implementation of this program. To this end the author approximates the Euler equations by a finite-dimensional system with invariant volume and the same basic integrals (energy and enstrophy). Then he introduces the invariant (microcanonical) measure, and passes to the limit. In this case the large deviation property is checked, and the most probable flows are characterized by the maximum of some sort of entropy. This gives us a closed system of nonlinear integral equations defining the most probable flow from the integrals of the initial one. Their solutions describe 'big vortices' which are observed in two-dimensional flows as a result of their self-organization. This line of reasoning meets the same difficulties as all other attempts to give a dynamical basis for statistical mechanics. For this approach to be valid, we have to prove a very strong ergodicity for the approximating finite-dimensional system including the estimate of the mixing time. However, there is an even more fundamental difficulty: it is unclear whether the concepts of equilibrium statistical mechanics are applicable to ideal incompressible fluid. Possibly its phase space is too big to hold any reasonable invariant measure. But this does not diminish the brilliant work of R. Robert. Beside the above theory the paper contains interesting material on the statistical solutions of the Burgers and Vlasov–Poisson equations, and a very remarkable theorem of J. Duchon and R. Robert on the energy balance for weak solutions of the Euler and Navier–Stokes equations: it turns out that the irregularities of the velocity field can generate and absorb energy, and the exact formula for the rate of energy production/absorption is derived. Generally, this paper is very interesting and stimulating.

The paper by Y. Brenier 'Topics on Hydrodynamics and Volume Preserving Maps' is devoted to the foundations of the dynamics of an ideal incompressible fluid. The starting point is the classical configuration space of incompressible fluid, i.e. the group of volume-preserving diffeomorphisms of the flow domain. This is an infinite-dimensional Riemannian manifold, and the solutions of the Euler–Lagrange equations are geodesics. At the same time geodesics are locally the shortest paths, minimizing the action (i.e. the mean kinetic energy). This double meaning suggests the idea of constructing the fluid flows as the shortest trajectories connecting given pairs of fluid configurations. However, Shnirelman (1985) has proved in that for a wide class of boundary conditions this problem has no classical solution, i.e. the minimum cannot be achieved. Then followed the works of Brenier (1989) and Shelukhin (1988) where the different notions of a generalized solution for this problem were introduced. The Shortest Path Problem is closely related to the classical Monge–Kantorovich mass transport theory, and many basic concepts from the latter are successfully used in the Shortest Path Problem. The generalized solutions for this problem, for a wide class of boundary conditions, are multiphase flows with a continuum of phases (generally, every fluid particle gives rise to a new phase). The phases move independently from one another in the common pressure field, and eventually each phase collapses on the target point. The article contains the notions leading to this remarkable result, and some related material, including the density of smooth diffeomorphisms in the semigroup of measure-preserving maps. The main tools are measures in different functional spaces, different kinds of convergence, and the duality theory (note that this theory can be regarded as a special infinite-dimensional linear programming problem). The relevance of this theory to the fluid dynamics itself is an interesting question. Brenier himself considers (Brenier 1999) some asymptotic problems of three-dimensional flows which are described by the so-called hydrostatic limit of the Euler equations. There definitely should be other applications of this profound theory.

A. Shnirelman's article 'Weak Solutions of Incompressible Euler Equation' is devoted to the construction and analysis of several examples of weak solutions of incompressible Euler equations. Weak Euler equations are integral identities expressing essentially the mass and momentum balance in the fluid. There is a hope that some sort of weak solution of the Euler equations could describe turbulent motions of a fluid with vanishing viscosity. However for many years no non-trivial examples of a weak solution have been known. In 1993 the first non-trivial (and pathological) example of a weak solution in the plane with a compact space–time support was published in Scheffer (1993). Scheffer's original construction is notoriously complicated, which prevents its full appreciation. In the present article the author describes a much simpler example of a weak solution with a compact support and shows that the underlying phenomenon is the inverse energy cascade. The second part of the article is devoted to the construction of a weak solution on a three-dimensional torus whose kinetic energy monotonically decreases, as for real turbulence. The construction is based on a simple observation that there exist mechanical systems with decreasing energy but without explicit friction, namely a system of freely moving particles coalescing upon collision. In this flow the particles collide and stick with positive probability, thus dissipating the energy. These collisions are possible because the velocity field is very irregular (in fact, everywhere discontinuous), so the trajectories of different particles can meet. This construction required an extensive machinery of generalized flows introduced by Brenier (1989). In the last part of the paper the author applies the above-mentioned energy balance equation derived by Duchon & Robert (2000) to the weak solution which he has constructed. The author proves that the term in the energy balance corresponding to the energy creation/dissipation due to the non-smoothness of the velocity field is everywhere negative for his solution. This proves that the energy is dissipated everywhere, as in a true turbulent flow. Hopefully, the weak solution constructed by the author is a step to a better understanding of turbulent flows.

The article 'Near Identity Transformations for The Navier–Stokes Equations' by P. Constantin discusses the mathematical questions of global regularity for the incompressible Navier–Stokes equations and their approximations. As is well-known, the mathematical theory for the Navier–Stokes equations is incomplete. In particular, the most important task is to deduce suitable *a priori* estimates for their smooth solutions. Having these estimates one can show the smoothness and uniqueness of the solution for all time. But when such *a priori* estimates are not available one has to consider global approximate solutions. If we obtain uniform *a priori* estimates for approximations then we can get global estimates for weak solutions of the Navier–Stokes equations. The author considers two essentially different classes of approximations. The first class preserves the energy dissipation, but the vorticity equation is not exact. Corresponding typical examples are the Galerkin approximations and various modified equations. The second class of approximations is the class of vortex methods and their generalizations. For this class the vorticity equation is treated exactly whereas the energy dissipation is approximated. The paper describes all the approximations of the Navier–Stokes equations from the point of view of near identity transformations and provides uniform and non-uniform *a priori* estimates for approximate solutions. The second part of the article reviews the author's recent results concerning diffusive-Lagrangian aspects of the Navier–Stokes equations. The mixed Eulerian–Lagrangian approach discovered by the author can also be presented in terms of near identity transformations. It is worth noting that kinetic bounds for the displacement and the virtual velocity as well as for the dispersion and diffusion of particle paths require less regularity than well-known local existence theorems in Lagrangian and Eulerian variables.

The article 'Planar Navier–Stokes equations: Vorticity approach', by M. Ben-Artzi, reviews the existence, uniqueness, and regularity theory for solutions to the Navier–Stokes equations in two space dimensions. First of all, it should be noted that in two dimensions the L_2 theory of existence and uniqueness of weak and classical solutions is complete, but there are still unsolved problems in the L_1 theory when the problem is formulated in 'vorticity form' and initial vorticities are supposed to be in L_1. This question is addressed in the present article. The author concentrates in his survey on the case of the full plane and uses the vorticity formulation that is reduced in two dimensions to a (nonlinear) convection–diffusion equation for the scalar vorticity. For the case of smooth initial data, the theorem on global L_1 well-posedness was proved in McGrath (1967). The proof of this theorem is outlined. The extension of the solution operator to 'rough' initial data in L_1 was first done by Giga, Miyakawa & Osada (1988). Their method relies on Nash estimates for fundamental solutions of parabolic equations. Their result was later improved by the author of the chapter. Unlike the work of Giga *et al.* (1988), his method does not appeal to the Nash estimate. It relies on a simple property of the heat equation. The main advantage of his method is that the constants appearing in *a priori* estimates are specified and depend continuously on the initial vorticity. The author discusses the further extension of these results to measure-valued initial data.

A. Babin's article ' Attractors of Navier–Stokes Equations' deals with the theory of global attractors of the Navier–Stokes equations and related equations of hydrodynamics. The global attractor includes all the regimes which describe large-time dynamics of the equations and all the instabilities the equations possess. Such regimes include equilibria and time-periodic solutions, but may also include much more complicated, time-chaotic solutions. The dimension of the attractor corresponds to the number of degrees of freedom the system eventually has after a long time has elapsed. The article serves two purposes. First, to present the basic concepts and results from the theory of attractors in infinite-dimensional function spaces to readers who are not experts in the theory of dynamical systems. To this end, the author tries to avoid technicalities and when possible illustrates general ideas by examples. From the theory of global attractors he chooses the parts that are most closely related to the Navier–Stokes equations. Namely, he pays most attention to estimates of the dimension of attractors in terms of physical parameters, such as the Grashof number, and to the methods they are based on. In particular, the principal steps of the derivation of the Constantin–Foias–Temam upper dimension estimate for the two-dimensional Navier–Stokes system are explained. The principal idea of the lower estimate of the dimension by Babin-Vishik is also given. The second purpose of the article is to present a review of recent developments in the theory of global attractors of the Navier–Stokes equations not covered in the books by Temam and Babin and Vishik. The results discussed include: dynamics in channels and pipes, in particular the results on the attractor of a forced Poiseuille flow; the theory of exponential attractors; the theory of attractors of 3D Navier–Stokes equations; flows in thin domains and helical flows; global regularity of the dynamics of rotating fluids with a small Rossby number and the corresponding theory of attractors.

The topic of the article 'Stability and instability in viscous fluids', by M. Renardy & Y. Renardy, is very broad and general and could itself be the subject of a multivolume monograph. Therefore, the authors focus on some classical problems in simple geometries and under simple boundary conditions. The article begins with a discussion of mathematical issues in the theory of stability and bifurcation. All the definitions, ideas, and methods are introduced and discussed for the finite-dimensional case and illustrated using examples of ordinary differential equations. Then, a generalization to spatially infinite systems (partial differential equations) is considered. The connection between linear stability, spectral stability, and nonlinear stability is discussed and the classification of bifurcations is given. Hydrodynamic applications of the mathematical analysis of stability and bifurcation is the subject of the remainder of the article. The authors consider four main groups of hydrodynamical problems: thermal convection, flows between rotating cylinders, parallel shear flows, and capillary breakup of jets. Since the set of problems that are touched on in the article is too broad (it includes, for example, two-layer flows, magnetic convection, viscoelastic flows, etc.) the authors concentrate on the discussion of some classical questions and results and review more recent research.

The article 'Localized Instabilities in Fluids' written by S. Friedlander & A. Lipton-Lifschitz is devoted to recent developments and open problems related to instabilities in ideal fluid flows. The article begins with a broad introduction (the introduction itself and three following sections) in which the authors write down the Euler equations for an inviscid incompressible fluid, exhibit examples of their steady and unsteady exact solutions, give definitions of hydrodynamic stability (spectral, linearized, nonlinear), consider corresponding linearized equations and the spectral problem, and discuss some types of classical ('fast') instabilities. The 'fast' instabilities are associated with discrete unstable eigenvalues for which perturbations grow exponentially with time. At the same time, 'slower' instabilities are associated with the unstable essential spectrum. The growth rate for such instabilities can be not only algebraic ('very slow') but also exponential. The main goal of the article is to describe a method to detect such kind of 'slower' instabilities. This method, being an alternative to the usual spectral and energy methods, is based on the geometrical optics technique that is used to study highly localized short-wave perturbations. It is shown how this method allows the linear stability problem to be reduced to the consideration of a characteristic system of ODEs for determining the wave vector and the velocity amplitude. This seems to be the main advantage of the geometrical optics approach because the reduced problem for ODEs is, of course, more tractable than the original stability problem. As was proved by one of the authors, the formal WKB solutions are close to the actual solutions of the linearized equations and, hence, the flow is linearly unstable if the amplitude is unbounded in time. The growth rate of the unbounded amplitude can be algebraic or exponential. Flows with a hyperbolic stagnation point are considered in the article as a main example when the positive exponential growth rate occurs. A very important and interesting question under discussion in the article is the relation between linear and nonlinear instabilities. The authors describe results when the linear 'fast' instability implies instability in the full nonlinear Euler equation. They note however that it is still unclear whether 'slower' localized exponential instabilities also imply nonlinear instability. This question remains to be an open and challenging problem.

The article 'Dynamo Theory' by A. D. Gilbert contains a brief review of the dynamo theory and its applications to the Solar and Earth dynamos. It is focused on fundamental mechanisms of magnetic field generation and may serve as an introduction to this area of research for researchers and graduate students. In the final section of the paper, the author offers a list of interesting and important open problems which, without any doubt, will attract the attention of both mathematicians and physicists. An extensive list of the relevant references is given at the end of the paper.

F. Dias & G. Ioss's article, entitled 'Water Waves as a Spatial Dynamical System', focuses on the mathematical analysis of two-dimensional travelling waves in the potential flow of one or several layers of perfect fluid, with a free surface and interfaces. Both finite and infinite depth cases are considered. Latest results on the existence of travelling waves in stratified fluids and on three-dimensional travelling waves are also discussed. The main goal of the review is to show the advantages of dynamical systems methods for obtaining results on the spatial behaviour of travelling waves near the basic unperturbed free surface state. These methods are connected with different interesting mathematical subjects such as, for example, elliptic partial differential equations in unbounded domains or the theory of reversible systems in infinite dimensions. A large bibliography on the subject is given.

The last article of the volume 'Shock Waves in General Relativity' written by J. Groah, J. Smoller & B. Temple surveys the authors' results on shock waves in General Relativity. The main goal of this article is to show how shock waves are naturally incorporated into Einstein's theory of General Relativity and to explain the main advantages of shock-wave cosmology. The authors discuss regularity properties for the Einstein equations and note that the question of whether general Lipschitz continuous solutions of these equations can always be smoothed by coordinate transformation is still an open problem. Assuming spherical symmetry the authors prove that the Einstein equations for a perfect fluid in standard Schwarzschild coordinates are weakly equivalent to a system of hyperbolic conservation laws with source terms. From the mathematical point of view, it is quite natural to introduce shock waves for this system. The main important application of the theory of shock waves in General Relativity seems to be a new cosmological model, different from the usual Big-Bang scenario. This model provides a scenario by which the Big-Bang begins with a shock wave explosion. It is shown that the shock position at the present time is one Hubble length from the centre of the explosion. The great advantage of this model, in comparison with the classical Big-Bang model, is that it removes the singularity (infinite pressure) in the core for times after some initial time. This motivates the authors' idea that the expansion of our Universe might be accounted for by an event similar to a classical explosion.

The book represents an excellent attempt to present the subject area of mathematical fluid dynamics to a readership in fluid dynamics, applied mathematics, and general mathematics. It is addressed to researchers and to graduate students.

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Fundamentals of Computational Fluid Dynamics. By TAPAN K. SENGUPTA. Universities Press, Hyderabad, 2004. 350 pp. ISBN 81 7371 478 9. 750 Indian Rupees (paperback). Distributed by Orient Longman Ltd.

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CFD (computational fluid dynamics) has clearly become a central tool in fluid mechanics and the number of textbooks published on this topic demonstrates the complexity of this field. How to teach CFD and what a good textbook should contain are also issues of major importance in many research centres and universities. The description of how a modern CFD code works on unstructured meshes and parallel computers could fill whole books just to present the numerical technology. Such a presentation would not even address numerical analysis issues such as why the chosen method works, if it is stable and which other methods could replace it. Therefore, writing a textbook on CFD today is a difficult and challenging task which must either focus on very limited aspects or only present very general concepts. This book by T. K. Sengupta *Fundamentals of Computational Fluid Dynamics* actually mixes these two approaches:

Firstly, it is a general introduction to CFD, following the usual line of presentation found in most CFD textbooks: equations of motion, classification of PDEs (partial differential equations) found in fluid mechanics, discretization methods, solution of parabolic PDEs, solution of elliptic PDEs, solution of hyperbolic PDEs, meshing techniques, finite volume and finite element methods, examples of solutions for Navier–Stokes examples. These chapters do not provide any really new information but present it in a reasonably compact way which allows easy reading.

Secondly, in addition to this standard and general CFD course, the author uses approximately one third of the book to address much more detailed and modern issues which all relate clearly to his own field of research: the precision of high-order schemes in terms of wave propagation and their applications to DNS (direct numerical simulation) and LES (large-eddy simulation). One important question is the capacity of numerical schemes to propagate waves at the right speed (dispersion effects) and with the right amplitude (dissipation). Even though dispersion and dissipation have been discussed in many places before, it is clear that these properties were not essential in most present industrial codes which use RANS (Reynolds-averaged Navier–Stokes) turbulence models: RANS turbulence models make the flow so viscous that no wave can propagate into it, independently of the numerical scheme. This situation is changing rapidly: with the development of DNS and the explosion of LES for practical applications, CFD users have re-discovered that waves indeed exist in fluids and that capturing these waves numerically is a difficult task.

These two different levels of CFD description are an important characteristic of the book: while the sections devoted to the general description (chapters 1 to 9, 12 and 13) are fairly easy to read and could be an adequate textbook for beginners, the second part (chapters 10 and 11) which addresses spectral analysis and high-order schemes performance for dispersion and dissipation is more advanced and difficult to read. These two chapters should be read by all experts involved in the development of high-order schemes for DNS or LES. They contain an excellent summary of the author's work together with parallel recent studies on these topics. The classical book by R. Vichnevetsky & J. B. Bowles (*Fourier Analysis of Numerical Approximations of Hyperbolic Equations*, SIAM, 1982) on spectral analysis of numerical schemes is probably better organized and should be read first but the present book contains multiple new developments which are certainly important today because they include DNS and LES applications which were not considered by Vichnevetsky & Bowles.

The size of the book (340 pages) is reasonable and will not frighten beginners. It also allows easy searches. However, this limited size implies that more information is found in books like the monograph by C. Hirsch (*Numerical computation of Internal and External Flows*, Wiley, 1988). For example, nothing is said on boundary conditions (which are also a critical aspect of LES and DNS). More generally, the whole text is limited to structured meshes while most modern algorithms use unstructured meshes. The presentation of the book is good even though many typos could be corrected.

In conclusion, this book will be useful for beginners and for students who are looking for a compact description of CFD but also for CFD experts who are developing high-order schemes for DNS and LES on structured meshes.

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