Applied Probability Trust (22 February 2017)

ONLINE APPENDIX

MARSHALL-OLKIN DISTRIBUTIONS, SUBORDINATORS, EFFICIENT SIMULATION, AND APPLICATIONS TO CREDIT RISK

YUNPENG SUN,*

RAFAEL MENDOZA-ARRIAGA,* *The University of Texas at Austin*

VADIM LINETSKY,* *Northwestern University*

Abstract

This document contains complementary material of section 6 of the article *"Marshall-Olkin Distributions, Subordinators, Efficient Simulation, and Applications to Credit Risk"*. In particular, we provide the time-inhomogeneous extensions of Theorems 1 and 2 of the main document. These results are given in Theorems 1 and 2 where Lévy subordinators are replaced with additive subordinators. We also present additional extensions to the MO distribution which incorporate stochastic dynamics and extend the traditional default intensity factor model.

Keywords: Marshall-Olkin multivariate exponential distribution, Lévy process, subordinator, additive subordinator, simulation, dependent lifetimes, failure, default, reliability, credit risk.

2010 Mathematics Subject Classification: Primary 60G51

Secondary 60J28

* Postal address: 11 E 26th St., Suite 2005, New York, NY 10010.

* Postal address: Information, Risk, & Operations Management Department (IROM), McCombs School of Business. The University of Texas at Austin, CBA 5.202, B6500, 1 University Station, Austin, TX 78712, E-mail: rafael.mendoza-arriaga@mccombs.utexas.edu.

* Postal address: Department of Industrial Engineering and Management Sciences, McCormick School of Engineering and Applied Sciences, Northwestern University, 2145 Sheridan Road, Evanston, IL 60208, E-mail: linetsky@iems.northwestern.edu.

Appendix A. Additive Subordinator Construction

We now consider a time-inhomogeneous extension of Marshall-Olkin's fatal shock construction. As in section 4, consider a set of n components subject to failure, and let $\Upsilon = \{1, 2, \cdots, n\}$ be the index set indexing the n components. With each non-empty subset $\Theta = \{i_1, \cdots, i_k\} \subseteq \Upsilon$ with cardinality $|\Theta| = k \leq n$ we now associate a time-inhomogeneous Poisson process $\tilde{N}_t^\Theta$ with the time-dependent arrival rate $\lambda^\Theta(t)$, a non-negative Borel function satisfying $\int_0^t \lambda^\Theta(u)du < \infty$ for all $t \geq 0$. A fatal shock of type Θ arrives at the first jump time $X_\Theta = \inf\{t \geq 0 : \tilde{N}_t^\Theta > 0\}$ of the inhomogeneous Poisson process $\tilde{N}_t^\Theta$, which results in the simultaneous failure of all components with indexes in the set Θ. We set $\lambda^\emptyset \equiv 0$ so that $\tilde{N}_t^\emptyset \equiv 0$ and $X_\emptyset \equiv \infty$. Let τ_i denote the failure time of the ith component (default time of the ith obligor in credit risk applications). Then $\tau_i = \min\{X_\Theta : i \in \Theta\}$, for $i = 1, \cdots, n$. The joint distribution of lifetimes $(\tau_1, \cdots, \tau_n)$ is called the *time-inhomogeneous Marshall-Olkin multivariate exponential distribution* with time dependent parameters $\{\Lambda_t^\Theta, \Theta \subseteq \Upsilon\}$. From Eq. (4.1) it follows that the joint survival function of lifetimes is given by:

$$\mathbb{P}(\tau_1 > t_1, \cdots, \tau_n > t_n) = \exp\Big(-\sum_{\Theta \subseteq \Upsilon} \Lambda_{t_\Theta}^\Theta\Big), \text{ with } t_\Theta := \max\{\mathbf{1}_\Theta(1)t_1, \cdots, \mathbf{1}_\Theta(n)t_n\},$$

where $\mathbf{1}_\Theta(i) = 1\,(0)$ if $i \in \Theta$ $(i \notin \Theta)$, $t_i \geq 0$, and $\Lambda_t^\Theta = \int_0^t \lambda^\Theta(u)du$.

Moreover, as in section 4, it follows that for any non-empty subset $\Theta \subseteq \Upsilon$ the probability that all components with indexes in Θ survive until time $t > 0$ is: $\mathbb{P}(\tau_\Theta > t) = e^{-\int_0^t g^\Theta(u)du}$, where τ_Θ is the time of the first failure in the set of components with indexes in the set Θ, $\tau_\Theta := \tau_{i_1} \wedge \cdots \wedge \tau_{i_k}$, and

$$g^\Theta(t) = \sum_{\Xi \subseteq \Upsilon:\, \Xi \cap \Theta \neq \emptyset} \lambda^\Xi(t) \equiv \sum_{\Xi \subseteq \Upsilon} \lambda^\Xi(t) - \sum_{\Xi \subseteq \Theta^c} \lambda^\Xi(t), \quad t \geq 0, \tag{A.1}$$

and, in particular, $g^{\{i\}}(t) = \sum_{\Theta \subseteq \Upsilon : i \in \Theta} \lambda^\Theta(t)$. Clearly, $g^\Theta(t)$ is a non-negative function satisfying $\int_0^t g^\Theta(u)du < \infty$, for all $t \geq 0$. A time-inhomogeneous extension of the fatal shock construction leads to an extension of the Marshall-Olkin distribution $MO_n(\lambda(t))$ parameterized by $2^n - 1$ non-negative time-dependent intensities $\lambda^\Theta(t)$ or, equivalently, by the $2^n - 1$ non-negative functions $g^\Theta(t)$ related to the intensity parameters $\lambda^\Theta(t)$

by Eq.(A.1). It can be inverted to express λ in terms of g in a similar way as in Lemma 4.1.

Lemma 1. *The intensity parameters $\lambda(t)$ can be expressed in terms of the parameters $g(t)$ for all non-empty subsets Θ by:*

$$\lambda^{\Theta}(t) = \sum_{\Xi \subseteq \Theta} (-1)^{|\Theta|-|\Xi|+1} g^{\Xi^c}(t), \tag{A.2}$$

where $|\Xi|$ and Ξ^c denote the cardinality and the complement of the set Ξ in Υ, respectively, and the sum in Eq.(A.2) is over all subsets of Θ, including the empty set.

It is straight forward to extend our subordination construction of MO distributions to an additive subordinator construction of $MO_n(\lambda(t))$ distributions.

Theorem 1. (Additive subordinator construction) *Let $\mathcal{E}_i$ be n unit-mean independent exponential random variables and $\mathcal{T}$ be an n-dimensional additive subordinator independent of all $\mathcal{E}_i$. Define random lifetimes τ_i, $i = 1, \cdots, n$, by:*

$$\tau_i := \inf\{t \geq 0 : \mathcal{T}_t^i \geq \mathcal{E}_i\}, \quad i = 1, \cdots, n.$$

Then the random vector $(\tau_1, \cdots, \tau_n)$ has the $MO_n(\lambda, t)$ distribution with parameters $\lambda^{\Theta}(t)$ given by Eq.(A.2) with

$$g^{\Theta}(t) = \psi(\mathbf{1}_{\Theta}(1), \cdots, \mathbf{1}_{\Theta}(n), t), \quad \textit{and} \quad \int_0^t g^{\Theta}(u) du < \infty, \ \forall t \geq 0,$$

where ψ is the Laplace exponent of the additive subordinator (see Eq. (6.1)) and $\mathbf{1}_{\Theta}(i) = 1\,(0)$ if $i \in \Theta$ ($i \notin \Theta$).

The proof is entirely similar to the proof of Theorem 1 (of the main document) by replacing $\phi_{\Theta} = \phi(\mathbf{1}_{\Theta}(1), \cdots, \mathbf{1}_{\Theta}(n))$ with time-dependent $\psi_{\Theta}(t) = \psi(\mathbf{1}_{\Theta}(1), \cdots, \mathbf{1}_{\Theta}(n), t)$ and integrating with respect to time. The necessary and sufficient conditions on time-dependent intensities $\lambda(t)$ also parallels Theorem 2 (of the main document).

Theorem 2. *The distribution $MO_n(\lambda(t))$ admits an additive subordinator construction if and only if its parameters satisfy the following condition. Let $U = \{u \geq 0 : \lambda^{\Theta}(u) = 0\}$ for some $\Theta \subseteq \Upsilon$, then $\lambda^{\Xi}(u) = 0$ for all $u \in U$ and for all Ξ such that $\Theta \subseteq \Xi$.*

The proof parallels the proof of Theorem 2 (of the main document).

Appendix B. Further Extensions

We conclude with a brief discussion of the extension of the (additive) subordinator model that combines diffusive intensities with (additive) subordinators via time changes. This more general framework starts with a vector of independent one-dimensional non-negative diffusions, such as CIR processes. As in the standard diffusion intensity framework, it then constructs hazard processes of individual obligors as integrals of linear combinations of the diffusive intensities. It then applies a multi-variate time change given by a multi-dimensional (additive) subordinator to the vector of hazard processes. The resulting model combines the local diffusive behavior of intensity models that generate diffusive dynamics of market spreads over time with the global correlation structure of the multi-variate (additive) subordinator model presented in this paper. Research in this direction has been initiated in [7] for one-dimensional models and in [8] for multi-dimensional models.

Stochastic dynamics: In the previous section we have shown that by using additive subordinators one can generalize the MO shock model to the case of time-dependent parameters. We have also observed that our construction of the MO distribution (both time homogeneous and time inhomogeneous) allows us to generate efficient simulation algorithms even for the hierarchical case (see Remark 2 of the main document). Nonetheless, the MO distribution remains a static one, which may not be sufficiently appropriate for explaining the time-series dynamics as it is required for certain applications, e.g., the mark to market valuation of credit default spreads where volatility plays an important role. However, this limitation can be easily alleviated by introducing absolutely continuous time changes in our formulation. Consider for instance the factor model of equation (4.8),

$$\mathcal{T}^i = \mathcal{S}^{id,i} + \sum_{c=1}^{C} A_{i,c}\mathcal{S}^c, \quad \forall i = 1, \ldots, n,$$

where the time change processes $\mathcal{S}^c$, $c = 1, \ldots, C$, are either a Lévy or Additive subordinators, but where the idiosyncratic time change process $\mathcal{S}^{id,i}$ is specified as an absolute continuous time change that is independent of $\mathcal{S}^c$ and of all $\mathcal{S}^{id,j}$, $\forall j \neq i =$

$1, \ldots, n$. An absolute continuous time change $\mathcal{S}^{id,i}$ is a non-negative and nondecreasing process that starts at zero and that is absolute continuous with respect to the Lebesgue measure, i.e., $\mathcal{S}_t^{id,i} = \int_0^t V_s^i ds$, where V^i is a nonnegative activity rate process that is independent of V^j, $\forall j \neq i = 1, \ldots, n$. A typical example of an absolute continuous time change is given when the activate rate process V^i is specified as a Cox-Ingersoll-Ross (CIR) process (see [3]), which uniquely and strongly solves the SDE,

$$dV_t^i = \kappa_i(\theta_i - V_t^i) + \sigma_i \sqrt{V_t^i} dB_t^i, \quad \text{with} \quad V_0^i = v_i \geq 0; \quad \kappa_i \theta_i > 0, \quad \sigma_i > 0,$$

where θ_i is the long run mean, κ_i is the mean reversion rate, σ_i is the constant volatility, and B^i is a Brownian motion, which is assumed to be independent of B^j, $\forall j \neq i = 1, \ldots, n$. If Feller's condition is satisfied, i.e., $2\theta_i \kappa_i \geq \sigma_i^2$, then the process V^i remains strictly positive when started from $v_i > 0$. It can also be started from $v_i = 0$, in which case it immediately enters the interval $(0, \infty)$ and stays strictly positive for all $t > 0$. In this case the boundary at zero is an entrance boundary. When the Feller condition is not satisfied, $0 < 2\kappa_i \theta_i < \sigma_i^2$, the process can reach zero when started from $v_i > 0$, and zero is an instantaneously reflecting boundary. An appealing property of the CIR specification is that the Laplace transform is known in closed form

$$\mathcal{L}(v_i, t, \lambda) = \mathbb{E}[e^{-\lambda \int_0^t V_s^i ds}] = \mathbb{E}[e^{-\lambda \mathcal{S}_t^{id,i}}] = e^{-A(t,\lambda) - B(t,\lambda) v_i}, \quad \lambda > 0,$$

where the functions A, B are well known solutions to the associated Riccati ODE (for details see [3]). Moreover, paths of the random change process $\mathcal{S}^{id,i}$ can be simulated using well known algorithms such as those found in [1], [6] and [2]. Therefore, one can complement the MO fatal shock model by introducing stochastic dynamics via absolute continuous time changes in order to enrich to the credit default model.

A generalization of the traditional default intensity factor model using the MO fatal shock model: Following the setup proposed above, it is easy to see that we can further generalize the credit model to a factor model that combines the traditional factor model with (jump-) diffusion intensities (see, e.g., [5] and [4]) and the MO fatal shock model as follows,

$$\mathcal{T}^i = \underbrace{\mathcal{S}^{id,i} + \sum_{j=1}^{J} \alpha_{i,j} \mathcal{U}^j}_{\text{Trad. factor model}} + \underbrace{\sum_{c=1}^{C} A_{i,c} \mathcal{S}^c}_{\text{MO fatal shock}}, \quad \forall i = 1, \ldots, n,$$

where $\mathcal{U}^j$, $j = 1, \ldots, J$, are a set of absolute continuous time changes representing, for example, macroeconomic factors, or sector/industry related factors that are common to all or a subset of firms. Each $\alpha_{i,j} \geq 0$ represents the contribution of the absolute continuous time change $\mathcal{U}^j$ to the default intensity of the ith firm. In this case, one can see that the traditional factor model is complemented by the MO factor model, where the latter is introduced into the credit default model in order to capture default clustering events.

References

[1] Broadie, M. and Kaya, O. (2006). Exact simulation of stochastic volatility and other affine jump diffusion processes. *Operations Research* **54,** 217–231.

[2] Chan, J. H. and Joshi, M. S. (2013). Fast and Accurate Long Stepping Simulation of the Heston Stochastic Volatility Model. *Journal of Computational Finance* **16,** 47–97.

[3] Cox, J. C., Ingersoll, J. E. and Ross, S. A. (1985). A Theory of the Term Structure of Interest Rates. *Econometrica* **53,** 385–407.

[4] Duffie, D., Filipovic, D. and Schachermayer, W. (2003). Affine Processes and Applications in Finance. *The Annals of Applied Probability* **13,** 984–1053.

[5] Duffie, D. and Singleton, K. J. (1999). Simulating Correlated Defaults. *Working paper.*

[6] Glasserman, P. and Kim, K.-K. (2011). Gamma Expansion of the Heston Stochastic Volatility Model. *Finance and Stochastics* **15,** 267–296.

[7] Mendoza-Arriaga, R. and Linetsky, V. (2014). Time-Changed CIR Default Intensities with Two-Sided Mean-Reverting Jumps. *The Annals of Applied Probability* **24,** 811–856.

[8] Mendoza-Arriaga, R. and Linetsky, V. (2016). Multivariate Subordination of Markov Processes with Financial Applications. *Mathematical Finance* **26,** 699–747.